Quasi-Fully Cancellation Fuzzy Modules

Hatem Yahya Khalaf  
Department of Mathematics/ College of Education(Ibn-Al-Haitham),  
University of Baghdad  

Hadi Ghali Rashed  
Al-Rusafa the First Education/ Ministry of Education  

Received in 5/December /2016 Accepted in: 28/December/2016  

Abstract  
In this paper it was presented the idea quasi-fully cancellation fuzzy modules and we will denote it by Q-FCF(M), condition universalistic idea quasi-fully cancellation modules It .has been circulated to this idea quasi-max fully cancellation fuzzy modules and we will denote it by Q-MFCF(M). Lot of results and properties have been studied in this research.  

Key word : Q-FCF(M), Q-MFCF(M), Direct Sum Q-FCF(M).
Introduction

Let R be a commutative ring with identity. Let X be a fuzzy module of an R-module M and we will denote it by (X be F(M)). X is called Q-FCF(M) if for every fuzzy ideal I of R and two every fuzzy submodules A and B of X, if IA = IB, then A+(F-annI) = B+(F-annI) where F-annI is a fuzzy submodule of X and define by F-annI = \{x_t \in X: Ix_t = 0\} \forall t \in (0,1)

And X is called Q-MFCF(M) if for every maximal fuzzy ideal of R and for every two fuzzy submodules A and B of X such that IA = IB implies A+(F-annI) = B+(F-annI) see (2.1).

Clearly, every Q-FCF(M) is Q-MFCF(M) see (Remark(2.3)(1)) and every fully cancellation fuzzy module is Q-FCF(M). See (Remark(1.3)(1)) but the converse is not true ingenarl see (Remark(1.6)).

And if X is multiplication and naturally cancellation fuzzy module or (fuzzy Principle ideal), then X is Q-FCF(M). See Proposition (1.4) and Proposition (1.5).

In this chapter, we will study in details the concept of Q-FCF(M). This chapter consists of three parts. In part one we give some basic propositions of Q-MFCF(M).

It turns out that a fuzzy module X is quasi-fully cancellation (quasi-max fully cancellation) if and only if IH \subseteq IK, then H \subseteq K+(F-annI), where H and K are fuzzy submodules of X and I be a fuzzy ideal (fuzzy maximal ideal) of R. Equivalently, I(h_t) \subseteq L_t, then h_t \subseteq L+(F-annI), where h_t \subseteq X if and only if (IH:I) = H+(F-annI) where L is a fuzzy submodule. And h_t is a fuzzy singleton of R, \forall t \in (0,1). see Proposition (1.7) and Proposition (2.9). And fully cancellation fuzzy module is equivalent quasi-fully cancellation in the class X is torsion free fuzzy module over a fuzzy integral domain R see Proposition (1.4).

Part two is devoted to study the relation between max-fully cancellation fuzzy module and Q-MFCF(M) but the converse is not true see Remark (2.7) and Example (2.8). Part three is study the concepts the direct sum of Q-FCF(M) which is mentioned in chapter one section five.

And naturally cancellation fuzzy module X is equivalent to quasi-fully cancellation if X is multiplication fuzzy module.

§ 1. Quasi-Fully Cancellation Fuzzy Modules

In this part we give the concept of Q-FCF(M) this concept is generalization of concept quasi-fully cancellation modules[1], and we give a some basic results and properties of this concept. Also, relationships between the class of Q-FCF(M) and other types of modules are established.

"Recall that an R-module M is called quasi-fully cancellation produle. If for every ideal I of R and for every two submodules A, B of M. Such that IA = IB implies A+ann_MI = B+ann_MI (where ann_MI = \{m \in M, Im = 0\}.[1]"

Here, we introduce the principle definition of our work.

Definition 1.1:

Let X beF(M). X is called Q-FCF(M) for every fuzzy ideal I of R and for every fuzzy submodules A and B of X, if IA = IB, then A+F-annX = B+F-annX where F-annX is a fuzzy submodule of X. And definition by \{x_t \subseteq X: Ix_t = 0\} \forall t \in (0,1).

Proposition 1.2:

Let X beF(M), such that (F-annX)_t = F-annX_t. Then X is a Q-FCF(M) if and only if X_t is a quasi-fully cancellation module, \forall t \in (0,1).
Proof:
Let X be F(M). Let N and K be two submodules of M, and let J be an ideal of R. such that JN=JK.
Now, define: A:M→[0,1], B:M→[0,1] by
\[ A(x)=\begin{cases} t & \text{if } x \in N \forall t \in (0,1) \\ 0 & \text{otherwise} \end{cases} \]
\[ B(x)=\begin{cases} t & \text{if } x \in K \forall t \in (0,1) \text{ and} \\ 0 & \text{otherwise} \end{cases} \]
Define : I:R→[0,1] by
\[ I(x)=\begin{cases} t & \text{if } x \in J \forall t \in (0,1) \\ 0 & \text{otherwise} \end{cases} \]
It is clear that A and B are fuzzy submodules of X and I is a fuzzy ideal of R.
Also , \( A_t = N, B_t = K \) and \( I_t = J \); \( t \in (0,1) \), JN=JK, then \( I_t A_t = I_t B_t \)
Therefore \( (I A)_t = (I B)_t \); \( t \in (0,1) \).
Thus IA=IB.
But X is a Q-FCF(R). Then we get \( A + F - \text{ann} I = B + F - \text{ann} I \)
Hence \( (A + F - \text{ann} I)_t = (B + F - \text{ann} I)_t \); \( t \in (0,1) \)
But \( (F - \text{ann} I)_t = F - \text{ann} I \); \( t \in (0,1) \) see[5,Proposition.(2.2)]
So, \( A_t + F - \text{ann} I_t = B_t + F - \text{ann} I_t \).
Therefore \( N + F - \text{ann} J = K + F - \text{ann} J \).
Thus \( X_t \) is quasi-fully cancellation module.
Another side, let A,B be two fuzzy submodules of X and let I be a fuzzy ideal of R. such that IA=IB. To prove \( A + F - \text{ann} I = B + F - \text{ann} I \).
Now, since IA=IB, then \( (IA)_t = (IB)_t \); \( t \in (0,1) \), so , \( I_t A_t = I_t B_t \)
But \( X_t \) is a quasi-fully cancellation module.
Then \( A_t + F - \text{ann} I_t = B_t + F - \text{ann} I_t \).
But \( (F - \text{ann} I)_t = F - \text{ann} I \); \( t \in (0,1) \) which implies
\( A_t + (F - \text{ann} I)_t = B_t + (F - \text{ann} I)_t \).
So \( (A + F - \text{ann} I)_t = (B + F - \text{ann} I)_t \), by [2, Remark (1.1.7)].
Thus \( A + F - \text{ann} X = B + F - \text{ann} X \).
Therefore X is Q-FCF(M).

Remarks and Examples 1.3:
(1) Every fully cancellation fuzzy module is Q-FCF(M). But the converse is not true in general by the following example:
Let M=\( \mathbb{Z}_4 \) is a Z-module.
Let \( X : M \rightarrow [0,1] \) define by \( X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases} \)
Let A: \( \mathbb{Z} \rightarrow [0,1] \) define by \( A(x) = \begin{cases} t & \text{if } x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \)
Let B: \( \mathbb{Z}_4 \rightarrow [0,1] \) define by \( B(x) = \begin{cases} t & \text{if } x \in \mathbb{Z}_4 \\ 0 & \text{otherwise} \end{cases} \)
Define I: \( \mathbb{Z}_4 \rightarrow [0,1] \) define by \( I(x) = \begin{cases} t & \text{if } x \in \mathbb{Z}_4 \\ 0 & \text{otherwise} \end{cases} \)
It is clear that A and B are fuzzy submodules of X and I is a fuzzy ideal of R.

Mathematics | 194
M=Z_4=X, is not fully cancellation module by [3, Remark and Examples (2.3)(2)].

Since (4), (Z_4)=0, but (Z_4)≠Z_4.

Implies that X is not fully cancellation fuzzy module by [4, Proposition (1.2,2)].

Hence A_1=(4), A_2=Z_4 and B_1=Z_4.

So I_1A_1=I_2B_1 (since (4), (Z_4)=(4).Z_4=(0)). Also, annI_1=ann (4)=Z_4

Then A_2+annI_1=(Z_4)+Z_4=Z_4

Also , B_1+annI_1=Z_4+Z_4=Z_4

Then X=M is quasi-fully cancellation module, and by proposition (1.2).

X is Q-FCF(M).

(2) Any fuzzy module of a Z-module Z is Q-FCF(M).

**Proof:**

By[4, Remark and Examples (1.2.3)(1)] we get X is fully cancellation fuzzy module. And by (1) we interduce X is Q-FCF(M).

(3) Every fuzzy submodule of Q-FCF(M) is quasi-fully cancellation.

**Proof:**

Let X be a Q-FCF(M) of an R-module M. let N,K be two submodules of M and J be an ideal of R. let C be a fuzzy submodule of X.

To prove C is Q-FCF(M).

Define: C: M→[0,1] by C(x)={
1 if x ∈ M
\[
0 \text{ otherwise}
\]

Define A: N→[0,1] by A(x)={
\[
t \text{ if x ∈ N otherwise}
\]

Define B: K→[0,1] by B(x)={
\[
t \text{ if x ∈ K otherwise}
\]

Define I: J→[0,1] by I(x)={
\[
t \text{ if x ∈ J otherwise}
\]

It is clear that A,B are fuzzy submodules of C and I is a fuzzy ideal of R.

Also, I_1=J , A_2=N , B_2=K , C_2=M.

Since X is Q-FCF(M).

Then X is quasi-fully cancellation module by Proposition (1.2).

Since C is a fuzzy submodule of X.

Then C is a submodule of X and by [ 3 , Remark and Examples (2.2)], we get C is quasi-fully cancellation module.

Therefore C is Q-FCF(M). by Proposition(1.2)
(4) Let $X_1$ and $X_2$ be two fuzzy modules of an $R$-module $M_1$, $M_2$ respectively such that $M_1 \cong M_2$. Then $X_1$ is $Q$-FCF($M$) if and only if $X_2$ is $Q$-FCF($M$).

**Proof:**

$(\Rightarrow)$ Let $X_1 : M_1 \rightarrow [0,1]$ define by $X_1(x) = \begin{cases} 1 & \text{if } x \in M_1 \\ 0 & \text{otherwise} \end{cases}$

Let $X_2 : M_2 \rightarrow [0,1]$ define by $X_2(x) = \begin{cases} 1 & \text{if } x \in M_2 \\ 0 & \text{otherwise} \end{cases}$

It is clear that $X_1$ and $X_2$ are fuzzy modules of $M_1$ and $M_2$ respectively.

Since $(X_1)_t = M_1$, $(X_2)_t = M_2$ and $M_1 \cong M_2$, $\forall t \in (0,1]$, then $M_2$ is quasi-fully cancellation module by [3, Remark and Examples (2.2), (5)].

Then $X_2$ is $Q$-FCF($M$) by Proposition (1.2).

Conversely: it is clear.

**Proposition 1.4:**

Let $X$ be a multiplication and naturally cancellation fuzzy module of an $R$-module $M$ then $X$ is $Q$-FCF($M$).

**Proof:**

Since $X$ is multiplication and naturally cancellation fuzzy module, then by [4, Theorem (1.4.3)]. We obtain $X$ is fully cancellation fuzzy module and by Remark and Examples((1.3) (1)).

$X$ is $Q$-FCF($M$).

**Proposition 1.5:**

Let $X$ be a fuzzy torsion free module over a fuzzy integral domain $R$. If $X$ is quasi-fully cancellation modules, then $X$ is fully cancellation.

**Proof:**

Let $X$ be a fuzzy torsion free module and $R$ be a fuzzy integral domain.

Suppose that $IA=IB$ where $A, B$ are two fuzzy submodules of $X$ and $I$ be a fuzzy ideal of $R$.

Since $X$ is quasi-fully cancellation, then $A+(F-\text{ann}xI)=B+(F-\text{ann}xI)$.

Now, let $x_t \subseteq F-\text{ann}xI$, then $Ix_t=0_I$, $\forall t \in (0,1]$.

And hence $r_\ell x_t=0_I$ for each fuzzy singleton $r_\ell$ of $I$, $\forall \ell \in (0,1]$, $r_\ell \neq 0_I$ (since $R$ is integral domain). Then $I \neq 0_I$.

Therefore $x_t=0_I$ (since $X$ is a fuzzy torsion free).

Then $F-\text{ann}xI=0_I$. Thus $A=B$ and hence $X$ is a fully cancellation fuzzy module.
Proposition 1.6:

Let $X$ be $F(M)$ and let $R$ be a fuzzy principle ideal ring. Then $X$ is a quasi-fully cancellation module.

Proof:

Let $A$ and $B$ be two fuzzy submodules of $X$.

Let $I$ be a fuzzy ideal of a fuzzy principle ideal ring $R$.

Suppose that $IA=IB$.

We show that $A+(F\text{-}\text{ann}_xI)=B+(F\text{-}\text{ann}_xI)$

Since $R$ is a fuzzy principle ideal ring, then $I=(r_{\ell})$, where $r_{\ell}$ be a fuzzy singleton of $R$, $\forall \ell \in (0,1]$.

Then $(r_{\ell}) A=(r_{\ell}) B$ and hence $r_{\ell}.a_r=r_{\ell}.b_s$ where $a_r \subseteq A$ and $b_s \subseteq B$ $\forall \tau, s \in (0,1]$.

Now, $r_{\ell}.a_r-r_{\ell}.b_s=0_1$.

Then $(ra-rb) =0\lambda \leq 0_1$ where $\lambda =\min\{\ell_1,\ell_2\}$. Therefore $r_{\ell}(a_r-b_s)=0_1$.

Thus $a_r-b_s \subseteq F\text{-}\text{ann}_xI$. But $a_r=b_s+a_r-b_s \subseteq B+(F\text{-}\text{ann}_xI)$.

Then $A \subseteq B+(F\text{-}\text{ann}_xI)$

Hence $A+(F\text{-}\text{ann}_xI) \subseteq B+(F\text{-}\text{ann}_xI)$

Similarly:

$B+(F\text{-}\text{ann}_xI) \subseteq A+(F\text{-}\text{ann}_xI)$

Thus $A+(F\text{-}\text{ann}_xI)= B+(F\text{-}\text{ann}_xI)$

And hence $X$ is $Q$-FCF(M).

Remark 1.7:

The converse of Remark and examples( (1.3)(1)) is not true in general by the following example:

Let $M=Z_{p\infty}$ and $R=Z$ define by $X:M \rightarrow [0,1]$ such that

$X(x) = \begin{cases} 1 & \text{if } x \in m \\ 0 & \text{otherwise} \end{cases}$

It is clear that $\forall \tau \in (0,1]$, $X\tau=M$ and $M$ is not fully cancellation module

$[3, \text{Examples } (2.5)(2)]$.

Thus $X$ is not fully cancellation fuzzy module by Proposition ( 1.2) .

Proposition 1.8:

Let $X$ be $F(M)$ and let $H,K$ and $L$ are fuzzy submodules of $X$. let $I$ be a fuzzy ideal of $R$.

Then the following statements are equivalent:-

1- $X$ is a $Q$-FCF(M).
2- If \( IH \subseteq IK \), then \( H \subseteq K + (F \text{-ann}_x I) \).

3- \( I(h_t) \subseteq IL \), then \( h_t \subseteq L + (F \text{-ann}_x I) \), where \( h_t \subseteq X \). \( \forall t \in (0,1] \).

4- \( (IH;_x I) = H + (F \text{-ann}_x I) \).

**Proof:**

(1) \( \Rightarrow \) (2) let \( X \) be a \( Q \)-FCF(M) and let \( IH \subseteq IK \).

Then \( IK \subseteq IH + IK = I(H + K) \) by [5, proposition (2.6)].

Hence \( K + (F \text{-ann}_x I) = (H + K) + (F \text{-ann}_x I) \) (since \( X \) is \( Q \)-FCF(M)).

Therefore \( H \subseteq K + (F \text{-ann}_x I) \).

(2) \( \Rightarrow \) (3) let \( I(h_t) \subseteq IL \), where \( h_t \subseteq H \) and by (2) we have \( (h_t) \subseteq L + (F \text{-ann}_x I) \).

Thus \( h_t \subseteq L + (F \text{-ann}_x I) \).

(3) \( \Rightarrow \) (4) let \( x_t \subseteq (IH;_x I) \), \( \forall t \in (0,1] \), then \( Ix_t \subseteq IH \) and by (3) \( x_t \subseteq H + (F \text{-ann}_x I) \).

Therefore \( (IH;_x I) \subseteq H + (F \text{-ann}_x I) \).

Conversely, let \( \alpha_{\ell} \subseteq H + (F \text{-ann}_x I) \), \( \forall \ell \in (0,1] \).

Then \( \alpha_{\ell} = h_t + m_s \), where \( h_t \subseteq H \) and \( m_s \subseteq (F \text{-ann}_x I) \), \( \forall s \in (0,1] \).

Thus \( I \alpha_{\ell} = I h_t + I m_s \). But \( I m_s = 0 \).

Therefore \( I \alpha_{\ell} = I h_t \subseteq IH \).

Then \( \alpha_{\ell} \subseteq (IH;_x I) \).

Thus \( H + (F \text{-ann}_x I) \subseteq (IH;_x I) \), and hence \( (IH;_x I) = H + (F \text{-ann}_x I) \).

(4) \( \Rightarrow \) (1) let \( IH = IK \) we want to prove \( X \) is \( Q \)-FCF(M).

i.e To prove \( H + (F \text{-ann}_x I) = K + (F \text{-ann}_x I) \)

\( K \subseteq (IH;_x I) \) and by (4) we get \( (IH;_x I) = H + (F \text{-ann}_x I) \).

Hence \( K \subseteq H + (F \text{-ann}_x I) \).

Therefore \( K + (F \text{-ann}_x I) \subseteq H + (F \text{-ann}_x I) \).

Similarly:

\( H \subseteq (IK;_x I) \) and by (4), we obtain \( (IK;_x I) = K + (F \text{-ann}_x I) \), then \( H \subseteq K + (F \text{-ann}_x I) \).

Therefore \( H + (F \text{-ann}_x I) = K + (F \text{-ann}_x I) \).

Thus \( X \) is \( Q \)-FCF(M).

**Proposition 1.9:**

Let \( X \) be \( F(M) \). Then \( X \) is \( Q \)-FCF(M) if and only if \( ((A + (F \text{-ann}_x I);B) = (IA;_R IB) \) where \( I \) be a fuzzy ideal of \( R \) and \( A, B \) are two fuzzy submodules of \( X \).

**Proof:** (\( \Rightarrow \))

Let \( x_t \subseteq ((A + (F \text{-ann}_x I);B) \), \( \forall t \in (0,1] \).

**Mathematics** | 198
Then $x_t B \subseteq A + (F - \text{ann}_x I)$ and hence $x_t b \subseteq A + (F - \text{ann}_x I)$ for and $b \subseteq B, \forall s (0,1]$.

Thus $x_t IB \subseteq IA$. Then $x_t \subseteq (IA : R IB)$.

Therefore $((A + (F - \text{ann}_x I)) : B) \subseteq (IA : R IB)$.

Now, let $x_t \subseteq (IA : R IB)$, then $x_t IB \subseteq IA$ and by Proposition (1.7), we get $x_t B \subseteq A + (F - \text{ann}_x I)$ and hence $x_t \subseteq ((A + (F - \text{ann}_x I)) : B)$.

Thus $(IA : R IB) \subseteq ((A + (F - \text{ann}_x I)) : B)$.

Therefore $((A + (F - \text{ann}_x I)) : B) = (IA : R IB)$.

On the other side: Suppose $IB \subseteq IA$, where $I$ be a fuzzy ideal of $R$ and $A, B$ are two fuzzy submodules of $X$. Then $(IA : R IB) = R$.

But $((A + (F - \text{ann}_x I)) : B) = (IA : R IB)$.

Then $((A + (F - \text{ann}_x I)) : B) = R$, it follows that $B \subseteq A + (F - \text{ann}_I)$.

Thus $X$ is $Q$-FCF $(M)$.

§ 2. Quasi-Max Fully Cancellation Fuzzy Modules

As we have mentioned in section one, that every fully cancellation fuzzy module is $Q$-FCF $(M)$ and the converse is not to be true in general.

In this section, we introduce the concept of $Q$-MFCF $(M)$ and to show that every max-fully cancellation fuzzy module is $Q$-FCF $(M)$ but the converse is not true. Moreover, we prove that in the class of faithful fuzzy module, the two concepts max-fully cancellation fuzzy module and $Q$-MFCF $(M)$ are equivalent.

"Recall that an $R$-module $M$ is called quasi-max fully cancellation module if for every maximal ideal $I$ of $R$ and for every two submodules $N$ and $K$ of $M$ such that $IN = IK$ implies $N + \text{ann}_M I = K + \text{ann}_M I [6]$.

We shall fuzzify this concepts as follows:

Definition 2.1:

Let $X$ be $F(M)$. $X$ is called $Q$-MFCF $(M)$ if for every maximal fuzzy ideal of $R$ and for every two fuzzy submodules $A$ and $B$ of $X$ such that $IA = IB$ implies that $A + (F - \text{ann}_x I) = B + (F - \text{ann}_x I)$.

Next, we have the following Proposition.

Proposition 2.2:

Let $X$ be $F(M)$ and let $I$ be a maximal fuzzy ideal of $R$. such that $(F - \text{ann}_I) = F - \text{ann}_I$, then $X$ is a $Q$-MFCF $(M)$ if and only if $X_t$ is a quasi-max fully cancellation module $\forall t (0,1]$.

Proof:

It is similar of proof of Proposition (1.2) only we take $I$ maximal fuzzy ideal.
Remarks and examples 2.3:-

(1) Every $Q$-FCF($M$) is a $Q$-MFCF($M$).

**Proof:** it is clear.

(2) A fuzzy module $X$ of an $Z_6$-module $Z_6$ is $Q$-MFCF($M$).

**Proof:**

Let $M=Z_6$ and $X:M\rightarrow[0,1]$ such that $X(x)=\begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define: $I: \overline{2} \rightarrow[0,1]$ by $I(r)=\begin{cases} t & \text{if } r \in \overline{2} \\ 0 & \text{otherwise} \end{cases}$, $\forall t \in (0,1]$.

Define: $A: \overline{2} \rightarrow[0,1]$ by $A(x)=\begin{cases} t & \text{if } x \in \overline{2} \\ 0 & \text{otherwise} \end{cases}$

Define: $B:Z_6\rightarrow[0,1]$ by $B(x)=\begin{cases} t & \text{if } t \in Z_6 \\ 0 & \text{otherwise} \end{cases}$

It is clear that $A_t=(\overline{2})$, $B_t=Z_6$ and $I_t=\overline{2}$ is a maximal ideal and $X_t=M$, $\forall t \in (0,1]$.

Now, $I_tA_t=(\overline{2})(\overline{2})=\overline{2}$, $Z_6=IB_t=(\overline{2})$

Similarly if $A_t=(0)$, $B_t=(\overline{3})$ since $I_tA_t=(\overline{2})(\overline{0})=\overline{2}$, $B_t=I\overline{t}=(\overline{0})$ where $(0)$, $(\overline{2})$, $(\overline{3})$ are submodules of $M=Z_6$

Then we obtain $(\overline{0})+\text{ann}_M(\overline{2})=\overline{3}+\text{ann}_M(\overline{2})=\overline{3}$

Thus $M$ is $Q$-MFCF($M$). Therefore $X$ is $Q$-MFCF($M$) by Proposition (2.2).

(3) The fuzzy module $X$ of an $Z_4$-module $Z_4$ is a $Q$-FCF($M$).

**Proof:**

By Remark and Examples ((1.3)(2)) . We have $Z_4$ is quasi max fully cancellation module and by Proposition (2.2) we get the result.

(4) Let $X_1$ and $X_2$ be a fuzzy modules of an $R$-module $M_1,M_2$ respectively . if $M_1$ is a $Q$-MFC($M$)and $M_1\cong M_2$ , then $X_1$ is a $Q$-MFCF($M$) if and only if $X_2$ is a $Q$-MFCF($M$).

**Proof:**

$(\Rightarrow)$ let $X_1:M_1\rightarrow[0,1]$ define by $X_1(x)=\begin{cases} 1 & \text{if } x \in M_1 \\ 0 & \text{otherwise} \end{cases}$

Let $X_2:M_2\rightarrow[0,1]$ define by $X_2(x)=\begin{cases} 1 & \text{if } x \in M_2 \\ 0 & \text{otherwise} \end{cases}$

Clear that $(X_1)_t=M_1$ ,is a quasi-max fully cancellation module, then $X_1$ is a $Q$-MFCF($M$) by Proposition (2.7).

But $M_1\cong M_2$ , then $M_1$ is a quasi-max fully cancellation module by [6, Remark and Examples (1.3)(4)]

Therefore $X_2$ is a $Q$-MFCF($M$).
Conversely: it is clear.

(5) Let $X$ be $F(M)$ and let $C$ be a fuzzy submodule of $X$, then $C$ is $Q$-MFCF($M$).

Proof:

The prove is similar of Remarks and Examples (1.3)(4) only we take the ideal of a ring $R$ in maximal fuzzy ideal.

The following Lemmas are needed to prove next Proposition.

**Lemma 2.4:**

Let $X$ be $F(M)$ and let $A, B$ and $C$ are fuzzy submodules of $X$ such that $C \subseteq B$. Then $C+(B \cap A) = (C+A) \cap B$.

Proof:

First to show that $C+(B \cap A) \subseteq B \cap (C + A)$, (since $C \subseteq B$ and $B \cap A \subseteq B$).

Then $C+(B \cap A) \subseteq B$ by [2].

Further $C \subseteq C+A$, $B \cap A \subseteq C+A$.

Thus $C+(B \cap A) \subseteq C+A$.

Therefore $C+(B \cap A) \subseteq B \cap (C+A)$.

Conversely: to show that $B \cap (C + A) \subseteq C+ (B \cap A)$.

Let $b_t \in B \cap (C + A)$, $\forall t \in (0,1]$.

Then $b_t = c_\ell + a_s$ for some fuzzy singletons $c_\ell \subseteq C$, $a_s \subseteq A$ $\forall \ell, s \in (0,1]$.

Hence $b_t = (c+a)_t$ where $t = \min\{\ell, s\}$.

Then $b_t = c + a$ by [7, Definition (1.1.3)(3)].

Hence $a = b_t - c_\ell$ and $a_s = b_t - c_\ell$.

Thus $a_s \subseteq B$ (since $b_t \subseteq B$ and $c_\ell \subseteq C \subseteq B$).

Hence $a_s \subseteq B \cap A$ and so $b_t = c_\ell + a_s \subseteq C+ (B \cap A)$.

Therefore $C+(B \cap A) = (C+A) \cap B$.

**Lemma 2.5:**

Let $A$ be a fuzzy submodule of a fuzzy module $X$ of an $R$-module $M$, let $I$ be any fuzzy ideal of $R$, then $F-\text{ann}_A I = F-\text{ann}_A X \cap A$.

Proof:

Since $F-\text{ann}_A I$ is a fuzzy submodule of a fuzzy submodule $A$, then $F-\text{ann}_A I$ is a fuzzy submodule of $X$. and $F-\text{ann}_A I \cap A \subseteq F-\text{ann}_A I$.

Conversely: to show that $F-\text{ann}_A I \subseteq F-\text{ann}_A I \cap A$.

**Mathematics | 201**
Let \( x_t \in FannA \) , where \( x_t \subseteq A \), \( \forall t \in (0,1] \).

Then \( x_tI = 0_1 \)
\[
0_1 \cap A = x_tI \cap A = FannA \cap A
\]
But \( x_t \subseteq A \subseteq X \implies x_t \subseteq X \).

Thus \( FannA \cap A = 0_1 = x_tI = FannA \)

Now, we have the following Proposition.

**Proposition 2.6:**

Let \( A \) be a fuzzy submodule of a \( Q\text{-MFCF}(M) \) \( X \), then \( A \) is a \( Q\text{-MFCF}(M) \).

**Proof:**

Let \( B, C \) are two fuzzy submodules of a fuzzy submodule \( A \) and let \( I \) be a maximal fuzzy ideal of \( R \) such that \( IB = IC \), then \( B, C \) are fuzzy submodules of \( X \).

Since \( X \) is a \( Q\text{-MFCF}(M) \)

Then \( B + (FannA) = C + (FannA) \).

But \( FannA = FannA \cap A \) by Lemma (2.5).

Hence \( B + FannA = B + ((FannA) \cap A) \)
\[
= (B + (FannA)) \cap A \quad \text{by Lemma (2.4)}.
\]
\[
= (C + (FannA)) \cap A
\]
\[
= (C + ((FannA) \cap A)) \quad \text{by Lemma (2.4)}
\]
\[
= (C + (FannA)) \quad \text{by Lemma (2.5)}
\]

Thus \( A \) is a \( Q\text{-MFCF}(M) \).

**Remark 2.7:**

Every max-fully cancellation fuzzy module is \( Q\text{-MFCF}(M) \).

**Proof:**

It is clear.

The converse of Remark (2.7) is not true in general for example:-

Let \( M = \mathbb{Z}_6 \), \( R = \mathbb{Z}_6 \)

Let \( X : M \rightarrow [0,1] \) define by \( X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases} \)

\( X_t = M \ \forall t \in (0,1] \), then \( X \) is a \( Q\text{-MFCF}(M) \) by Remark((1.3)(1))

and it is not max-fully cancellation fuzzy module since

Let \( I: (\overline{2}) \rightarrow [0,1] \) define by \( I(r) = \begin{cases} t & \text{if } r \in (\overline{2}) \\ 0 & \text{otherwise} \end{cases} \), \( \forall t \in (0,1] \).

And \( A: (\overline{0}) \rightarrow [0,1] \) define by \( A(x) = \begin{cases} t & \text{if } x \in (\overline{0}) \\ 0 & \text{otherwise} \end{cases} \)
Also,  B:([3] → [0,1] define by B(x) = \begin{cases} t & \text{if } x \in (3) \\ 0 & \text{otherwise} \end{cases}, \forall t \in (0,1].

It is clear that A,B are fuzzy suodules of X and I is a fuzzy ideal of R.

IA=IB ⇔ I_A=I_B,

(2). (0) = (2). (3) = (0), but (0) ≠ (3)

Then A ≠ B. Thus X is not max-fully cancellation fuzzy module.

The convers of Remark (2.7) is true under the following conditions.

**Proposition 2.8:**

Let X be F(M) and let F-ann_x I=0, be (F-faithful) for every non-empty fuzzy ideal I of R. Then every Q-MFCF(M) is max-fully cancellation fuzzy module.

**Proof:**

Let I be a maximal fuzzy ideal of R. and let A,B be two fuzzy submodules of X such that IA=IB

Since X is Q-MFCF(M), then A+(F-ann_x I)=B+(F-ann_x I). But (F-ann_x I)=0, Thus A=B

Then X is max-fully cancellation fuzzy module.

The following is characterization of Q-MFCF(M).

**Theorem 2.9:**

Let X be F(M), let A,B be two fuzzy submodules of X and let I be a maximal fuzzy ideal of R. Then the following statements are equivalent :

1. X is a Q-MFCF(M).
2. If IA⊆ IB , then A⊆ B+(F-ann_x I).
3. If I(x_i) ⊆ IB, then x_i⊆ B+(F-ann_x I), where x_i∈ X.
4. (IA\cap I)= B+(F-ann_x I).

**Proof:**

Compare this proof with the proof of Proposition (1.7).

The next result gives another characterization for Q-MFCF(M).

**Proposition 2.10:**

Let X be F(M). Then for any maximal fuzzy ideal I of R the following statements are equivalent :

1. X is Q-MFCF(M).
2. For every fuzzy submodules A,B of X then ((A+(F-ann_x I)):B)=(IA:IB).
Proof:
It is similar proof of Proposition (1.8).

§3. The Direct Sum of Quasi-Fully Cancellation Fuzzy Modules and Its Generalization
In this part we study the direct sum of two Q-FCF(M) and we prove some results about it.

Also, we study its generalizations.

First, we give the following lemma, which is needed in the next our Proposition.

Lemma 3.1:
Let X be F(M), M=M₁⊕M₂ where M₁, M₂ are submodules of M, if X=A₁⊕A₂, where A₁, A₂ are fuzzy submodules of X, then F-annₓI=F-annₐ₁I⊕F-annₐ₂I where I is a fuzzy ideal of R.

Proof:
We must prove that F-annₓI ⊆ F-annₐ₁I⊕F-annₐ₂I.

Let xₓS F-annₓI, then Iₓ₁=0₁ and xₓS X, ∀tє (0,1].
Since xₓS X =⇒ xₓS X, M=M₁⊕M₂.
Then x=x₁+x₂ for some x₁є M₁, x₂є M₂.

Define I: J→{0,1} by
I(x) = {t if xє J, 0 otherwise, ∀tє(0,1].

Be a fuzzy ideal of R.

It is clear tat I=J.

Then J.x=J(x₁+x₂)=0 (since + on M is a direct sum) it follows

That: J.x₁=Jx₂=0.

(Jx₁)₁=(Jx₂)₁=0₁, ∀tє(0,1].
I(x₁)=I(x₂)=0₁ (since J=I₁).
I(x₁)=I(x₂)=0₁.

Thus (x₁₁)₁S F-annₐ₁I and (x₂₁)₁S F-annₐ₂I.
Therefore xₓ=(x₁₁)₁+(x₂₁)₁S F-annₐ₁I⊕F-annₐ₂I.

Conversely: let xₓS F-annₐ₁I⊕F-annₐ₂I.
Then xₓS ((x₁₁)₁, (x₂₁)₁) where (x₁₁)₁ S F-annₐ₁I and (x₂₁)₁ S F-annₐ₂I ∀ℓ, s є (0,1].
Thus I(x₁₁)=0₁ and I(x₂₁)=0₁. Therefore I(x₁₁)+I(x₂₁)=0₁,
and hence I((x₁₁)+ (x₂₁))=0₁
Then [J(x₁+x₂)]₁=0₁, ∀tє(0,1] where λ=min {t,s,ℓ} implies, J(x₁+x₂)=0.
Therefore Jx=0 and hence I(x₁)=0₁.

Mathematics | 204
Hence $x_i \subseteq \text{F-ann}_x I$. Therefore $\text{F-ann}_x I = \text{F-ann}_{A_1} I \oplus \text{F-ann}_{A_2} I$

**Proposition 3.2:**
Let $X$ be $\text{F}(M)$ and let $X = A_1 \oplus A_2$, where $A_1$ and $A_2$ are two fuzzy submodules of $X$, such that $\text{F-ann}_{A_1} I \oplus \text{F-ann}_{A_2} I = \lambda_R$ where $\lambda_R(x) = 1; \forall x \in R$. Then $A_1$ and $A_2$ are Q-FCF(M).

**Proof:**
$(\Rightarrow)$ Let $I$ be a non-empty fuzzy ideal of $R$ and let $A$ and $B$ be fuzzy submodules of $X$. Suppose that $IA = IB$ we show that $A + (\text{F-ann}_x I) = B + (\text{F-ann}_x I)$

Since $(\text{F-ann}_{A_1}) + (\text{F-ann}_{A_2}) = \lambda_R$, then by [4, Lemma (1.5.5)] we get, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ for some fuzzy submodules $A_1, A_2$ of $A$ and for some fuzzy submodules $B_1, B_2$ of $B$.

Now, $I(A_1 \oplus A_2) = I(B_1 \oplus B_2)$.

Hence $(IA_1, IA_2) = (IB_1, IB_2)$. [7, Proposition (3.2.4)]

Which implies that, $IA_1 = IB_1$ and $IA_2 = IB_2$.

But $A_1$ and $A_2$ are Q-FCF(M).

Thus $A_1 + (\text{F-ann}_{A_1} I) = B_1 + (\text{F-ann}_{A_1} I)$ and $A_2 + (\text{F-ann}_{A_2} I) = B_2 + (\text{F-ann}_{A_2} I)$

It follows that, $A_1 + A_2 + (\text{F-ann}_{A_1} I) + (\text{F-ann}_{A_2} I) = B_1 + B_2 + (\text{F-ann}_{A_1} I) + (\text{F-ann}_{A_2} I)$

Then we have: $A + (\text{F-ann}_x I) = B + (\text{F-ann}_x I)$

Therefore $X$ is Q-FCF(M).

$(\Leftarrow)$ it is clear by used Remarks and Examples ((1.3) (4)).

We end this section by the following result.

**Proposition 3.3:**
Let $X$ be $\text{F}(M)$ and let $X = A_1 \oplus A_2$ where $A_1$ and $A_2$ are two fuzzy submodules of $X$, such that $\text{F-ann}_{A_1} + \text{F-ann}_{A_2} = \lambda_R$ where $\lambda_R(x) = 1; \forall x \in R$. Then $A_1$ and $A_2$ are Q-MFCF(M) if and only if $X$ is Q-MFCF(M).

**Proof:**
First side, let $A$ and $B$ be two fuzzy submodules of $X$ and let $I$ be a maximal fuzzy ideal of $R$.

Since $\text{F-ann}_{A_1} + \text{F-ann}_{A_2} = \lambda_R$, where $\lambda_R(x) = 1; \forall x \in R$. Then by [4, Lemma (1.5.5)] we get, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ and by similar procedure as in the Proposition (3.2) . the required result can be obtained.

Another side, it is clear by Remarks and Examples( (2.3)(5)).
References


الموديولات الحذف شبه النامة الضبابية

حاتم حبي خلف
قسم الرياضيات/ كلية التربية للعلوم الصرفة ( ابن الهيثم ) / جامعة بغداد
هادي غالي راشد
تربية الرصافة الأولى / وزارة التربية


الخلاصة

في هذا البحث تم دراسة فكرة الموديولات الحذف شبه النامة الضبابية وقد تم إعطائهما الرمز Q-FCF(M) وهي أعمم لحالة الموديولات الحذف شبه النامة الإعتيادية وقد تم تعميم هذه الفكرة إلى موديولات شبه النامة العظمى Q-MFCF(M) . نتائج عديدة وخصائص كثيرة تم دراستها في بحثنا.

الكلمات المفتاحية: الموديولات الحذف شبه النامة الضبابية ، الموديولات الحذف شبه النامة الإعتيادية ، الموديولات الحذف شبه النامة عظمى الضبابية .