T-Abso and T-Abso Quasi Primary Fuzzy Submodules

Wafaa H. Hanoon
Department of Computer Science, College of Education for Girls, University of Kufa, Iraq.
wafaah.hannon@uokufa.edu.iq

Hatem Y. Khalaf
Department of Mathematics, College of Education for Pure Science Ibn- Haitham University of Baghdad, Baghdad, Iraq.
dr.hatamyahya@yahoo.com

Article history: Received 4 November 2018, Accepted 24 November 2018, Publish January 2019

Abstract
Let $\mathcal{M}$ be a unitary $R$-module and $R$ is a commutative ring with identity. Our aim in this paper is to study the concepts T-ABSO fuzzy ideals, T-ABSO fuzzy submodules and T-ABSO quasi primary fuzzy submodules, also we discuss these concepts in the class of multiplication fuzzy modules and relationships between these concepts. Many new basic properties and characterizations on these concepts are given.

Keywords: T-ABSO fuzzy ideal, T-ABSO fuzzy submodule, Quasi- prime fuzzy submodule, T-ABSO primary fuzzy submodule, T-ABSO quasi primary fuzzy submodule, Multiplication fuzzy module.

1. Introduction
In this paper all ring is commutative with identity and all modules are unitary. Deniz S. et al in [1] presented the concept of 2-absorbing fuzzy ideal which is a generalization of prime fuzzy ideal. Prime submodule which play an important turn in the module theory over a commutative ring. A prime submodule $N$ of an $R$-module $\mathcal{M}$, $N \neq \mathcal{M}$, with property $a \in R$, $x \in \mathcal{M}$, $ax \in N$ implies that $\in N$ or $a \in (N: \mathcal{M})$ [2]. This concept was generalized to concept of prime fuzzy submodule which was presented by Rabi [3]. In 1999, Abdul-Razakm, presented and studied quasi-prime submodule let $N < \mathcal{M}$, $N$ be called a quasi-prime if for $a, b \in R$, $m \in \mathcal{M}$, $abm \in N$, implies either $am \in N$ or $bm \in N$ [4]. In 2001, Hatam generalized it to fuzzy quasi-prime submodules [5]. Darani, et al in [6] presented the definition of 2-absorbing submodule. Let $N < \mathcal{M}$, $N$ be called 2-absorbing submodule of $\mathcal{M}$ if whenever $r, b \in R$, $x \in \mathcal{M}$ and $rbx \in N$, then $rx \in N$ or $bx \in N$ or $rb \in (N: \mathcal{M})$. Hatam and wafaa expanded this concept that is: if $X$ be a fuzzy module of an $R$-module $\mathcal{M}$. A proper fuzzy submodule $A$ of $X$ is called T-ABSO fuzzy submodule if whenever $a_s, b_l$ be fuzzy singletons of $R$, and $x_v \subseteq X$, $s, l, v \in L$, such that $a_s b_l x_v \subseteq A$, then either $a_s b_l \subseteq (A: \mathcal{M})$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$ [7]. McCasland and Moore presented the concept of $\mathcal{M}$-radical of $N$ such: Let $N$ be a proper module of a nonzero $R$-module $\mathcal{M}$, then the $\mathcal{M}$-radical of $N$, denoted by $\mathcal{M}$-rad $N$ is defined to be the intersection of all prime module including $N$, see [8]. Mostafanasab et al, were presented the connotation of 2-absorbing primary submodule. So, A proper submodule $N$ of an $R$-module $\mathcal{M}$ is called 2-absorbing primary submodule of $\mathcal{M}$ if whenever $a$, $b \in R$ and $m \in \mathcal{M}$ and $abm \in N$, then $am \in \mathcal{M}$-rad $N$ or $bm \in \mathcal{M}$-rad $N$ or $ab \in (N: \mathcal{M})$. 
Ibn Al-Haitham Jour. for Pure & Appl. Sci.                                                   IHJPAS
https://doi.org/10.30526/32.1.1930
Vol. 32 (1) 2019

[9]. Rabí and Hassan in 2008 were presented the concept of quasi primary fuzzy submodule. A proper fuzzy submodule \( A \) of fuzzy module \( X \) is said to be quasi primary fuzzy submodule if \((A:B)\) is a primary fuzzy ideal of \( R \) for each fuzzy submodule \( B \) of \( X \) such that \( A \subset B \) [10]. Suat K. et al, studied and presented the connotation of 2-absorbing quasi primary submodule, i.e., A proper submodule \( N \) of \( \tilde{M} \) is said to be 2-absorbing quasi primary submodule if the condition \( abq \in N \) implies either \( ab \in N:R \tilde{M} \) or \( aq \in \tilde{M}:R \text{-rad}(N) \) or \( bq \in \tilde{M}:R \text{-rad}(N) \) for every \( a, b \in R \) and \( q \in \tilde{M} \) [11]. This paper is composed of two sections.

In section (1) we present the definition of T-ABSO fuzzy ideals and we give some characterizations of this definition for ideals. Also many properties and outcomes of this concept are given. In section (2) we present the definition of T-ABSO fuzzy submodules, many basic properties and outcomes are studied. In section (3) we present the concept of T-ABSO quasi primary fuzzy submodules and we study the relationships this concept with among T-ABSO fuzzy submodules and T-ABSO primary fuzzy submodules. Several important results have been demonstrated. Note that we denote to fuzzy module, submodule.

2. T-ABSO F. Ideals

In this section, we introduce the concepts of T-ABSO and T-ABSO primary ideals. Some concepts and propositions which are needed in the next section.

**Definition 1.** [1]

Let \( \hat{H} \) be a non-constant F. ideal of \( R \). Then \( \hat{H} \) is called T-ABSO F. ideal if for any F. points \( a_s, b_l, r_k \) of \( R \), \( a_s b_l r_k \in \hat{H} \) implies that either \( a_s b_l \in \hat{H} \) or \( a_s r_k \in \hat{H} \) or \( b_l r_k \in \hat{H} \).

The following proposition characterize T-ABSO F. ideal in terms of its level ideal.

**Lemma 2.** [1]

Let \( A \) be F. ideal of \( R \). If \( A \) is T-ABSO F. ideal, then \( A_v \) is T-ABSO ideal of \( R \), \( \forall \ v \in L \),

Recall that Let \( \hat{H} \) be any F. ideal of \( R \). Then the radical F. of \( \hat{H} \), denoted by \( \sqrt{\hat{H}} \), is defined by:

\[ \sqrt{\hat{H}} = \bigcap \{U : U \text{ is a prime F. ideal of } R \text{ containing } \hat{H} \} \] [12].

Now, we give these propositions which are used in the next section.

**Proposition 3.**

Suppose that \( R \) be a ring and \( \hat{H} \) is T-ABSO F. ideal of \( R \). Then \( \sqrt{\hat{H}} \) is T-ABSO F. ideal of \( R \) and \( a_v^2 \subseteq \hat{H} \) for each F. singleton \( a_v \subseteq \sqrt{\hat{H}} \), \( \forall \ v \in L \).

**Proof.** Let \( \hat{H} \) be T-ABSO F. ideal and \( a_v \subseteq \sqrt{\hat{H}} \), hence \( a \in \sqrt{\hat{H}}_v \). Then \( a^2 \in \hat{H}_v \). So that \( \sqrt{\hat{H}}(a^2) \geq v \). Thus \( (a_v)^2 \subseteq \hat{H} \). Since \( (a_v)^2 = a_v^2 \), so that \( a_v^2 \subseteq \hat{H} \).

Now, let \( a_s, b_l, r_k \) be F. singletons of \( R \) such that \( a_s b_l r_k \subseteq \sqrt{\hat{H}} \). Then \( (a_s b_l r_k)^2 = a_s^2 b_l^2 r_k^2 \subseteq \hat{H} \). Since \( \hat{H} \) is T-ABSO F. ideal, then either \( a_s^2 b_l^2 \subseteq \hat{H} \) or \( a_s^2 r_k^2 \subseteq \hat{H} \) or \( b_l^2 r_k^2 \subseteq \hat{H} \), since \( (a_s b_l)^2 = a_s^2 b_l^2 \), \( (a_s r_k)^2 = a_s^2 r_k^2 \), \( (b_l r_k)^2 = b_l^2 r_k^2 \) hence either \( (a_s b_l)^2 \subseteq \hat{H} \) or \( (a_s r_k)^2 \subseteq \hat{H} \) or \( (b_l r_k)^2 \subseteq \hat{H} \). So that either \( a_s b_l \subseteq \sqrt{\hat{H}} \) or \( a_s r_k \subseteq \sqrt{\hat{H}} \) or \( b_l r_k \subseteq \sqrt{\hat{H}} \). Thus \( \sqrt{\hat{H}} \) is T-ABSO F. ideal of \( R \).
Lemma 4.

Let Ĥ ⊆ P be F. ideal of a ring R, where P is a prime F. ideal. Then the following expressions are equivalent:
1. P is a minimal prime F. ideal of Ĥ;
2. For each F. singleton \( a_v \subseteq P \), there exists F. singleton \( b_l \) of R\P and a non-negative integer \( n \) such that \( b_l a^n_v \subseteq \hat{H}, \forall v, l \in L. \)

Proof. (1) ⇒ (2) Let P be a minimal prime F. ideal of Ĥ and \( a_v \subseteq P \), suppose that for every F. singleton \( b_l \) of R\P, \( b_l a^n_v \not\subseteq \hat{H}, \forall n \in N. \) In particular, \( a_v \not\subseteq \hat{H}, \forall n \in N. \) Let \( A = \{1, a_v, a^n_v, \ldots\} \) and \( B = \{K: K \text{ is F. ideal of } R \text{ such that } K \cap A = \emptyset, \hat{H} \subseteq K \subseteq P\}. \) Then \( B \not= \emptyset, \) since \( \hat{H} \subseteq B, \) it is obvious B is partially ordered by inclusion. By [13], B has a maximal F. ideal say \( \bar{U}. \) Then \( \bar{U} \) is a prime F. ideal by [12], such that \( \hat{H} \subseteq \bar{U} \subseteq P. \) Since \( P \) is a minimal prime F. ideal of Ĥ, so \( \bar{U} = P \), this is a contradiction to \( a_v \subseteq P = \bar{U}, \) hence \( b_l a^n_v \subseteq \hat{H}. \)

(2) ⇒ (1) Suppose that for each F. singleton \( a_v \subseteq P, \) there exists F. singleton \( b_l \subseteq R \setminus P \) and \( n \in N \) such that \( b_l a^n_v \subseteq \hat{H}. \) Let \( K \) be a prime F. ideal of R such that \( \hat{H} \subseteq K \subseteq P. \) We claim that \( P \subseteq K. \) Since \( a_v \subseteq P, \) then there exists F. singleton \( b_l \subseteq R \setminus P \) and \( n \in N \) such that \( b_l a^n_v \subseteq \hat{H} \subseteq K. \) Since \( K \) is a prime F. ideal, then either \( b_l \subseteq K \) or \( a_v \subseteq K. \) Hence \( a_v \subseteq K \) as \( b_l \subseteq R \setminus P. \) So that \( \hat{H} \subseteq K \), then \( \hat{H} = K; \) that is \( P \) is a minimal prime F. ideal of Ĥ.

Proposition 5.

Suppose that Ĥ is T-ABSO F. ideal of a ring R. Then there are at most two prime F ideals of R that are minimal over Ĥ.

Proof. Assume that \( K = \{P_i: P_i \text{ is a prime F. ideal of } R \text{ which is minimal over } \hat{H}\}. \) Let \( K \) have at least three prime F. ideals. Let \( P_1, P_2 \in K \) be two different prime F. ideals. Then there exists F. singleton \( a_s \subseteq P_1 \setminus P_2 \) and there exists F singleton \( b_t \subseteq P_2 \setminus P_1. \)

We show that \( a_s b_t \subseteq \hat{H}. \) By lemma (4), there exist F. singletons \( x_v \not\subseteq P_1 \) and \( y_h \not\subseteq P_2, \) such that \( x_v a^n_s \subseteq \hat{H} \) and \( y_h b^m_t \subseteq \hat{H} \) for some \( n, m \geq 1. \) Since Ĥ is T-ABSO F. ideal of R, we have \( x_v a^n_s \subseteq \hat{H} \) and \( y_h b^m_t \subseteq \hat{H}. \) Since \( a_s, b_t \not\subseteq P_1 \cap P_2 \) and \( x_v a^n_s, y_h b^m_t \subseteq \hat{H} \subseteq P_1 \cap P_2 \), we get \( x_v \subseteq P_2 \setminus P_1 \) and \( y_h \subseteq P_1 \setminus P_2 \), thus \( x_v, y_h \not\subseteq P_1 \cap P_2. \) Since \( x_v a_s \subseteq \hat{H} \) and \( y_h b_t \subseteq \hat{H}, \) we have \( (x_v + y_h) a_s b_t \subseteq \hat{H}. \) Observe that \( (x_v + y_h) \not\subseteq P_1 \) and \( (x_v + y_h) \not\subseteq P_2. \) Since \( (x_v + y_h) a_s \not\subseteq P_2 \) and \( (x_v + y_h) b_t \not\subseteq P_1, \) we conclude that neither \( (x_v + y_h) a_s \subseteq \hat{H} \) nor \( (x_v + y_h) b_t \subseteq \hat{H} \) and hence \( a_s b_t \subseteq \hat{H}. \) Now, suppose there exists \( P_3 \subseteq K \) such that \( P_3 \) is neither \( P_1, \) nor \( P_2. \) Then we can choose \( r_k \subseteq P_1 \setminus (P_2 \cup P_3), \) \( c_n \subseteq P_2 \setminus (P_1 \cup P_3) \) and \( d_m \subseteq P_3 \setminus (P_1 \cup P_2). \) By the same way we show that \( r_k c_n \not\subseteq \hat{H}. \) Since Ĥ \( \subseteq P_1 \cap P_2 \cap P_3 \) and \( r_k c_n \not\subseteq \hat{H}, \) we get either \( r_k \subseteq P_3 \) or \( c_n \subseteq P_3 \) this is a discrepancy. Hence \( K \) has at most two prime F. ideals of R.

Proposition 6

Let \( \hat{H} \) be T-ABSO F. ideal of R. Then one of the following expressions must hold
1. \( \sqrt{\hat{H}} = P \) is a prime F. ideal of R such that \( P^2 \subseteq \hat{H} \)
2. \( \sqrt{\hat{H}} = P_1 \cap P_2, \) \( P_1 P_2 \subseteq \hat{H}, \) and \( (\sqrt{\hat{H}})^2 \subseteq \hat{H} \) where \( P_1, P_2 \) are the only distinct prime F. ideals of R that are minimal over \( \hat{H}. \)
**Proof.** By proposition (5), we get either $\sqrt{\bar{H}} = P$ is a prime F. ideal of $R$ or $\sqrt{\bar{H}} = P_1 \cap P_2$, where $P_1, P_2$ are the only distinct prime F. ideals of $R$ that are minimal over $\bar{H}$. Assume that $\sqrt{\bar{H}} = P$ is prime F. ideal of $R$. Let F. singletons $a_s, b_l \subseteq P$. By proposition (3), we have $a_s^2, b_l^2 \subseteq \bar{H}$. So that $a_s(a_s + b_l)b_l \subseteq \bar{H}$. Since $\bar{H}$ is T-ABSO F. ideal, we get $a_s(a_s + b_l) = a_s^2 + a_s b_l \subseteq \bar{H}$ or $(a_s + b_l)b_l = a_s b_l + b_l^2 \subseteq \bar{H}$ or $a_s b_l \subseteq \bar{H}$. From each case implies that $a_s b_l \subseteq \bar{H}$, and so $P^2 \subseteq \bar{H}$.

Suppose that $\sqrt{\bar{H}} = P_1 \cap P_2$, where $P_1, P_2$ are the only distinct prime F. ideals of $R$ that are minimal over $\bar{H}$. Let F singletons $a_s, b_l \subseteq \sqrt{\bar{H}}$. By the same way of above, we have $a_s b_l \subseteq \bar{H}$ and hence $(\sqrt{\bar{H}})^2 \subseteq \bar{H}$. Now, we show that, $P_1 P_2 \subseteq \bar{H}$. By proposition (3), we have $x_v^2 \subseteq \bar{H}$ for each F. singleton $x_v \subseteq \sqrt{\bar{H}}$. Let be F. singleton $y_h \subseteq P_1 \setminus P_2$ and $r_k \subseteq P_2 \setminus P_1$. By the proof of proposition (5), we have $y_h r_k \subseteq \bar{H}$. Let F singletons $c_n \subseteq \sqrt{\bar{H}}$ and $d_m \subseteq P_2 \setminus P_1$, choose F. singleton $f_u \subseteq P_1 \setminus P_2$. Then $f_u d_m \subseteq \bar{H}$ by the proof of proposition (5) and $(c_n + f_u) \subseteq P_1 \setminus P_2$. Thus $c_n d_m + f_u d_m = (c_n + f_u) d_m \subseteq \bar{H}$. So that $c_n d_m \subseteq \bar{H}$. By the same method we show that if $c_n \subseteq \sqrt{\bar{H}}$ and $d_m \subseteq P_1 \setminus P_2$.Thus $P_1 P_2 \subseteq \bar{H}$.

**Proposition 7**

Let $\bar{H}$ be T-ABSO F. ideal of $R$ such that $\sqrt{\bar{H}} = P$ is a prime F. ideal, of $R$ and suppose that $
abla \bar{H} \neq P$. For each F. singleton $a_v \subseteq P \setminus \bar{H}$, let $A_{a_v} = \{b_l \subseteq R: b_l a_v \subseteq \bar{H}\}, \forall \; v, l \in L$. Then $A_{a_v}$ is a prime F. ideal of $R$. Furthermore, either $A_{b_l} \subseteq A_{a_v}$ or $A_{a_v} \subseteq A_{b_l}$ for each F. singletons $a_v, b_l \subseteq P \setminus \bar{H}$.

**Proof.** Let $a_v \subseteq P \setminus \bar{H}$. Since $P^2 \subseteq \bar{H}$ by proposition (6), we have $P \subseteq A_{a_v}$. Assume that $P \neq A_{a_v}$ and $b_l r_k \subseteq A_{a_v}$ for some F. singleton $b_l, r_k$ of $R$. Since $P \subseteq A_{a_v}$, we may suppose that $b_l \not\subseteq P$ and $r_k \not\subseteq P$, hence $b_l r_k \not\subseteq \bar{H}$. Since $b_l r_k \subseteq A_{a_v}$ we have $b_l r_k a_v \subseteq \bar{H}$. Since $\bar{H}$ is T-ABSO F. ideal of $R$ and $b_l r_k \not\subseteq \bar{H}$, we have either $b_l a_v \subseteq \bar{H}$ or $r_k a_v \subseteq \bar{H}$, thus either $b_l \subseteq A_{a_v}$ or $r_k \subseteq A_{a_v}$. Hence $A_{a_v}$ is a prime F. ideal of $R$. Now, let $a_v, b_l \subseteq P \setminus \bar{H}$ for F. singletons $a_v, b_l$ of $R$ and assume that F. singleton $r_k \subseteq A_{a_v} \setminus A_{b_l}$. Since $P \subseteq A_{b_l}$, so $r_k \subseteq A_{a_v} \setminus P$. We show that $A_{b_l} \subseteq A_{a_v}$. Let F. singleton $x_s$ of $R$ such that $x_s \subseteq A_{b_l}$. Since $P \subseteq A_{a_v}$, we may suppose that $x_s \subseteq A_{b_l}$. Since $r_k \not\subseteq P$ and $x_s \not\subseteq P$, we have $r_k x_s \not\subseteq \bar{H}$. Since $r_k (a_v + b_l) x_s \subseteq \bar{H}$ and $r_k x_s, r_k b_l \not\subseteq \bar{H}$, we have $(a_v + b_l) x_s \subseteq \bar{H}$. Hence $a_v x_s \subseteq \bar{H}$ since $(a_v + b_l) x_s \subseteq \bar{H}$ and $x_s b_l \subseteq \bar{H}$. Hence $x_s \subseteq A_{a_v}$. So that $A_{b_l} \subseteq A_{a_v}$.

**Proposition 8.** Assume that $\bar{H}$ is F. ideal of $R$ such that $\bar{H} \neq \sqrt{\bar{H}}$ and $\sqrt{\bar{H}}$ is a prime F. ideal of $R$. Then the following expressions are equivalent:
1- $\bar{H}$ is T-ABSO F. ideal of $R$;
2- $A_{a_v} = \{b_l \subseteq R: b_l a_v \subseteq \bar{H}\}, \forall \; v, l \in L$, is a prime F. ideal of $R$ for each F. singleton $a_v \subseteq \sqrt{\bar{H}} \setminus \bar{H}$.

**Proof.**
Proof. (1) ⇒ (2) This is obvious by proposition (7).
(2)⇒(1) Assume that \( a_v b_l r_k \subseteq \hat{H} \) for F. singletons \( a_v, b_l, r_k \) of R.
Since \( \sqrt{\hat{H}} \) is a prime F. ideal of R, we may suppose that \( a_v \subseteq \sqrt{\hat{H}} \).
If \( a_v \subseteq \hat{H} \), then \( a_v b_l \subseteq \hat{H} \). Thus suppose that \( a_v \subseteq \sqrt{\hat{H}} \setminus \hat{H} \). Hence \( b_l r_k \subseteq A_{a_v} \). But \( A_{a_v} \) is a prime F. ideal of R, then by proposition (7), either \( b_l a_v \subseteq \hat{H} \) or \( r_k a_v \subseteq \hat{H} \). Thus \( \hat{H} \) is T-ABSO F. ideal of R.

Proposition 9.
Assume that \( \hat{H} \) is a non-constant proper F. ideal of a ring R. Then the following expressions are equivalent:
1- \( \hat{H} \) is T-ABSO F. ideal of R;
2- If \( \emptyset K T \subseteq \hat{H} \) for F. ideals \( \emptyset, K, T \) of R, \( \emptyset K \subseteq \hat{H} \) or \( K T \subseteq \hat{H} \) or \( \emptyset T \subseteq \hat{H} \).

Proof. (1)⇒(2) Assume that \( \emptyset K T \subseteq \hat{H} \) for F. ideals \( \emptyset, K, T \) of R. By proposition (5), we have \( \sqrt{\hat{H}} \) is a prime F. ideal of R or \( \sqrt{\hat{H}} = P_1 \cap P_2 \) where \( P_1, P_2 \) are non-constant distinct prime F. ideals of R that are minimal over \( \hat{H} \). If \( \hat{H} = \sqrt{\hat{H}} \), then it is readily showed that, \( \emptyset K \subseteq \hat{H} \) or \( K T \subseteq \hat{H} \) or \( \emptyset T \subseteq \hat{H} \). Thus suppose that \( \hat{H} \neq \sqrt{\hat{H}} \). We see the following:
(1) Assume that \( \sqrt{\hat{H}} \) is a prime F. ideal of R. Then we perhaps suppose that \( \emptyset \subseteq \sqrt{\hat{H}} \) and \( \emptyset \not\subseteq \hat{H} \). Let F. singleton \( a_v \) of R such that \( a_v \subseteq \emptyset \setminus \hat{H} \). Since \( a_v K T \subseteq \hat{H} \), we have \( K T \subseteq A_{a_v} \) where \( A_{a_v} = \{ b_l \subseteq R : b_l a_v \subseteq \hat{H} \} \). Since \( A_{a_v} \) is a prime F. ideal of R by proposition (8), we have either \( K \subseteq A_{a_v} \) or \( T \subseteq A_{a_v} \). If \( K \subseteq A_{a_v} \) and \( T \subseteq A_{x_s} \) for each F. singleton \( x_s \subseteq \emptyset \setminus \hat{H} \), then \( \emptyset K \subseteq \hat{H} \) (and \( \emptyset T \subseteq \hat{H} \)) and we are finished. Hence suppose that \( K \subseteq A_{r_k} \) and \( T \not\subseteq A_{r_k} \) for some F. singleton \( r_k \subseteq \emptyset \setminus \hat{H} \). Since \( \{ A_{w_h} : w_h \subseteq \emptyset \setminus \hat{H} \} \), is a set of prime F. ideals of R that are linearly ordered by proposition (7), since \( K \subseteq A_{r_k} \) and \( T \not\subseteq A_{r_k} \), we have \( K \subseteq A_{z_n} \) for some F. singleton \( z_n \subseteq \emptyset \setminus \hat{H} \). Thus \( \emptyset K \subseteq \hat{H} \).
(2) Assume that \( \sqrt{\hat{H}} = P_1 \cap P_2 \) where \( P_1, P_2 \) are non-constant distinct prime F. ideals of R that are minimal over \( \hat{H} \). We suppose that \( \emptyset \subseteq P \). If either \( K \subseteq P_2 \) or \( T \subseteq P_2 \), then either \( \emptyset K \subseteq \hat{H} \) or \( \emptyset T \subseteq \hat{H} \) because \( P_1 P_2 \subseteq \hat{H} \) by proposition (6). Hence suppose that \( \emptyset \subseteq \sqrt{\hat{H}} \) and \( \emptyset \not\subseteq \hat{H} \). By the same way in (1) and by proposition (7), we are finished from this proof.

(2)⇒(1) it is trivial. Now, we give the concept of T-ABSO quasi primary F. ideal as follows:

Definition 10.
A proper F. ideal \( \hat{H} \) of R is called T-ABSO quasi primary F. ideal of R if \( \sqrt{\hat{H}} \) is T-ABSO F. ideal of R.

Proposition 11.
A proper F. ideal \( \hat{H} \) of R is T-ABSO quasi primary F. of R iff whenever for each F. singleton \( a_s, b_l, r_h \) of R, \( s, l, h \in L \), such that \( a_s b_l r_h \subseteq \hat{H} \), then \( a_s b_l \subseteq \sqrt{\hat{H}} \) or \( a_s r_h \subseteq \sqrt{\hat{H}} \) or \( b_l r_h \subseteq \sqrt{\hat{H}} \).
Proof. ($\iff$) Suppose that $\hat{H}$ is a proper F. ideal of $R$ and whenever for each F. singleton $a_s, b_l, r_h$ of $R$, such that $a_s b_l r_h \subseteq \hat{H}$, then $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_h \subseteq \sqrt{\hat{H}}$ or $b_l r_h \subseteq \sqrt{\hat{H}}$. Let $a_s b_l r_h \subseteq \sqrt{\hat{H}}$, $a_s r_h \subseteq \sqrt{\hat{H}}$ and $b_l r_h \subseteq \sqrt{\hat{H}}$. Since $a_s b_l r_h \subseteq \sqrt{\hat{H}}$, then there exists $n \in Z^+$ such that $(a_s b_l r_h)^n = a_s^n b_l^n r_h^n \subseteq \hat{H}$. Since $a_s^n r_h^n \not\subseteq \hat{H}$ and $b_l^n r_h^n \not\subseteq \hat{H}$, then we have $a_s^n b_l^n = (a_s b_l)^n \subseteq \hat{H}$. So that $a_s b_l \subseteq \sqrt{\hat{H}}$. Thus $\sqrt{\hat{H}}$ is T-ABSO F. ideal of $R$ and so that $\hat{H}$ is T-ABSO quasi primary F. ideal of $R$.

($\Rightarrow$) Let $\hat{H}$ be T-ABSO quasi primary F. ideal of $R$ and for each F. singleton $a_s, b_l, r_h$ of $R$, such that $a_s b_l r_h \subseteq \hat{H}$ or $a_s r_h \subseteq \hat{H}$ or $b_l r_h \subseteq \hat{H}$. The proposition specificities T-ABSO quasi primary F. ideal in terms of its level ideal is given as follow

**Proposition 12.**

A F. ideal $\hat{H}$ of $R$ is T-ABSO quasi primary F. iff the level ideal $\hat{H}_v$ is T-ABSO quasi primary ideal of $R, \forall v \in L$.

**Proof.** ($\Rightarrow$) Let $abr \in \hat{H}_v$ for each $a, b, r \in R$ then $\hat{H}(abr) \succeq v$ hence $(abr)_v \subseteq \hat{H}_v$. So that $a_s b_l r_h \subseteq \hat{H}_v$ where $v = \min\{s, l, k\}$. Since $\hat{H}$ is T-ABSO quasi primary F., then either $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_h \subseteq \sqrt{\hat{H}}$ or $b_l r_h \subseteq \sqrt{\hat{H}}$ hence either $(ab)_v \subseteq \sqrt{\hat{H}}$ or $(br)_v \subseteq \sqrt{\hat{H}}$ or $(ar)_v \subseteq \sqrt{\hat{H}}$ and so $ab \in \sqrt{\hat{H}_v}$ or $ar \in \sqrt{\hat{H}_v}$ or $br \in \sqrt{\hat{H}_v}$. Thus $\hat{H}_v$ is T-ABSO quasi primary ideal of $R$.

($\Leftarrow$) Let $a_s b_l r_h \subseteq \hat{H}_v$ for F. singletons $a_s, b_l, r_h$ of $R, \forall s, l, k \in L$. Hence $(abr)_v \subseteq A$, where $v = \min\{s, l, k\}$, so that $\hat{H}(abr) \succeq v$ and $abr \in \hat{H}_v$. But $\hat{H}_v$ is T-ABSO quasi primary ideal then either $ab \in \sqrt{\hat{H}_v}$ or $ar \in \sqrt{\hat{H}_v}$ or $br \in \sqrt{\hat{H}_v}$, hence either $(ab)_v \subseteq \sqrt{\hat{H}}$ or $(ar)_v \subseteq \sqrt{\hat{H}}$ or $(br)_v \subseteq \sqrt{\hat{H}}$. So that either $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_h \subseteq \sqrt{\hat{H}}$ or $b_l r_h \subseteq \sqrt{\hat{H}}$. Thus $\hat{H}$ is T-ABSO quasi primary F. ideal of $R$. The following theorem gives a characterisation of T-ABSO quasi primary F. ideal.

**Theorem 13.**

Let $\hat{H}$ be a proper F. ideal of $R$. Then $\hat{H}$ is T-ABSO quasi primary F. ideal iff whenever $\hat{U} K T \subseteq \hat{H}$ for some F. ideals $\hat{U}, K, T$ of $R$, then $\hat{U} K \subseteq \sqrt{\hat{H}}$ or $\hat{U} T \subseteq \sqrt{\hat{H}}$ or $K T \subseteq \sqrt{\hat{H}}$. **Proof.**

($\Leftarrow$) Assume that $\hat{U} K T \subseteq \hat{H}$ for some F. ideals $\hat{U}, K, T$ of $R$, then $\hat{U} K \subseteq \sqrt{\hat{H}}$ or $\hat{U} T \subseteq \sqrt{\hat{H}}$ or $K T \subseteq \sqrt{\hat{H}}$ and let $a_s b_l r_h \subseteq \hat{H}$ for F. singleton $a_s, b_l, r_h$ of $R$. Hence $< a_s > < b_l > < r_h > \subseteq \hat{H}$ and so that $< a_s > < b_l > \subseteq \sqrt{\hat{H}}$ or $< a_s > < r_h > \subseteq \sqrt{\hat{H}}$ or $< b_l > < r_h > \subseteq \sqrt{\hat{H}}$. Then $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_h \subseteq \sqrt{\hat{H}}$ or $b_l r_h \subseteq \sqrt{\hat{H}}$. By proposition (11), then $\hat{H}$ is T-ABSO quasi primary F. ideal of $R$.

($\Rightarrow$) Assume that $\hat{H}$ is T-ABSO quasi primary F. ideal of $R$ and $\hat{U} K T \subseteq \hat{H}$ for some F. ideals $\hat{U}, K, T$ of $R$, then $\hat{U} K T \subseteq \sqrt{\hat{H}}$. Since $\sqrt{\hat{H}}$ is T-ABSO F. ideal of $R$, then $\hat{U} K \subseteq \sqrt{\hat{H}}$ or $\hat{U} T \subseteq \sqrt{\hat{H}}$ or $K T \subseteq \sqrt{\hat{H}}$ by proposition (9).
3. T-ABSO F. Subm.

In this section we present the concept of T-ABSO F. subm. and we introduce many basic properties and results about this concept.

Definition 14.
Let $X$ be F. M. of an R-M. $\mathcal{M}$. A proper F. subm. $A$ of $X$ is called T-ABSO F. subm. if whenever $a_s$, $b_l$ be F. singletons of $R$, and $x_v \subseteq X$, $\forall s, l, v \in L$ such that $a_s b_l x_v \subseteq A$, then either $a_s b_l \subseteq (A; R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$, see [7].

The proposition specificities T-ABSO F. subm. in terms of its level subm. is given as follow:

Proposition 15.
Let $A$ be T-ABSO F. subm. of F. M. $X$ of an R-M. $\mathcal{M}$, iff the level subm. $A_v$ is T-ABSO subm. of $X_v$, for all $v \in L$, see [7].

Remarks and Examples
1. The intersection of two distinct prime F. subms. of F. M. $X$ of an R-M. $\mathcal{M}$ is T-ABSO F. subm.
   
   Proof. Let $A$ and $B$ be two distinct prime F. subms. of $X$. Suppose that F. singletons $a_s$, $b_l$ of $R$, $x_v \subseteq X$ such that $a_s b_l x_v \subseteq A \cap B$, but $a_s x_v \not\subseteq A \cap B$ and $b_l x_v \not\subseteq A \cap B$. Then $a_s x_v \not\subseteq A$, $b_l x_v \not\subseteq A$, $a_s x_v \not\subseteq B$ and $b_l x_v \not\subseteq B$ these are impossible since $A$ and $B$ are prime F. subms. So suppose that $a_s x_v \not\subseteq A$ and $b_l x_v \not\subseteq B$. Since $a_s b_l x_v \subseteq A$ and $a_s b_l x_v \subseteq B$, then $b_l \subseteq (A; R X)$ and $a_s \subseteq (B; R X)$. So that $a_s b_l \subseteq (A; R X) \cap (B; R X) = (A \cap B; R X)$. Thus $A \cap B$ is T-ABSO F. subm. of $X$. (2). Every prime F. subm. is T-ABSO F. subm.
   
   Proof. Let $A$ be a prime F. subm. of F. M. $X$ of an R-M. $\mathcal{M}$. Let $a_s b_l x_k \subseteq A$ for F. singletons $a_s$, $b_l$ of $R$ and $x_k \subseteq X$. Then $(ab)_v \subseteq A$ where $v = \min\{s, l, k\}$. Since $A$ is a proper subm. of $X$ then $A_v$ is a proper subm. of $X_v$, hence $A_v$ is prime subm. of $X_v$. So that $A_v$ is T-ABSO subm. (see [14]), hence $ab \in (A; R X)_v$, then either $(ab)_v \subseteq (A; R X)$ or $(ax)_v \subseteq A$ or $(bx)_v \subseteq A$. So either $ab \in (A_v; R X_v)$ or $ax \in A_v$ or $bx \in A_v$.
   
   Since $(A_v; R X_v) = (A; R X)_v$ by [5]. So that Then either $a_s b_l \subseteq (A; R X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Thus $A$ is T-ABSO F. subm. of $X$. However, the converse incorrect in general, for example:

   Let $X: Z_{24} \rightarrow L$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z_{24} \\ 0 & \text{o.w.} \end{cases}$
   
   It is obvious that $X$ is F. M. of $Z_{24}$ as $Z$-M.

   Let $A: Z_{24} \rightarrow L$ such that $A(y) = \begin{cases} v & \text{if } y \in (\bar{6}) \forall v \in L \\ 0 & \text{o.w.} \end{cases}$
   
   It is obvious that $A$ is F. subm. of $X$. Now $A_v = (\bar{6})$ is not prime subm. of $Z_{24}$, since $2, 3 \notin (\bar{6})$ but $3 \notin (\bar{6})$ and $2 \notin (\bar{6})$ and $Z_{24} = 6Z$.

   (3) It obvious every quasi-prime F. subm. is T-ABSO F. subm. However T-ABSO F. subm. may not be quasi-prime F. subm. for example:

   Let $X: Z \rightarrow L$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z \\ 0 & \text{o.w.} \end{cases}$
It is obvious that $X$ is F. M. of $Z$.

Let $A: Z \rightarrow L$ such that $A(v) = \begin{cases} v & \text{if } y \in 6Z, \\ 0 & \text{o.w.} \end{cases} \forall v \in L$

It is obvious that $A$ is F. subm. of $X$.

$A_v=6Z$ is T-ABSO subm. of $Z$, since if $x, y, z \in Z$ and $x\neq y\neq z \in 6Z=\langle v \rangle$ then at least one of $x, y$ and $z$ is even or one of them is 6. Then either $x\neq y \in A_v$ or $x\neq z \in A_v$, or $y\neq z \in A_v$. But $6Z=A_v$ is not quasi-prime, since $2.3.1 \notin 6Z$ and $2.1\notin 6Z$. So that $A$ is T-ABSO F. subm., but $A$ is not quasi-prime F. subm. (4) Let $A, B$ be two F. subms. of $M. X$ of an R-M. $M$, and $B \subseteq A$. If $A$ is T-ABSO F. subm. of $X$, then it is not necessary that $B$ is a T-ABSO F. subm., for example:

Let $X: Z_{24} \rightarrow L$ such that $X(v) = \begin{cases} 1 & \text{if } y \in Z_{24}, \\ 0 & \text{o.w.} \end{cases} \forall v \in L$

It is obvious that $X$ is F. M. of $Z$.

Let $A: Z_{24} \rightarrow L$ such that $A(v) = \begin{cases} v & \text{if } y \in (2), \\ 0 & \text{o.w.} \end{cases} \forall v \in L$

And $B: Z_{24} \rightarrow L$ such that $B(v) = \begin{cases} v & \text{if } y \in (12), \\ 0 & \text{o.w.} \end{cases} \forall v \in L$

It is obvious that $A$ and $B$ are F. subms. of $X$.

Now, $A_v=(2)$ and $B_v=(12)$ where $B_v \subset A_v$, since $A_v=(2)$ is maximal subm. of $Z_{24}$ as $Z$-M., then $A_v$ is prime subm. by[15]. Implies that $A_v$ is T-ABSO subm. by[14]. But $2.2.3 \in B_v$, $2.3 \notin B_v$ and $2.2=4 \notin B_v$. Thus $B_v$ is not T-A BSO subm. Of $Z_{24}$ as $Z$-M. hence $B$ is not T-ABSO F. subm. (5) Let $A$ and $B$ be F. subms. of $M. X$ of an R-M. $M$ and $A \subseteq B$. If $A$ is T-ABSO F. subm. of $X$, then $A$ is T-ABSO F. subm. of $B$. Proof. If $B=X$, then don’t need to prove. Let $a_s b_t x_k \subseteq A$ for F. singletons $a_s, b_t \subseteq R$ and $x_k \subseteq B$, implies $(abx)_v \subseteq A$ hence $v = \min\{s, l, k\}$ $abx \in A_v$, where $a, b \in R, x \in B_v$. Since $A \subseteq B$ implies where $A_x \subseteq B_v$. So that $A$ is T-ABSO F. subm. of $X$, then $A_v$ is T-ABSO subm. Of $X_v$. Hence $A_v$ is T-ABSO subm. Of $B_v$ by[14], so that either $ab \in (A_v \cap B_v)$ or $(ab)_v \subseteq A$ or $ax \in A_v$ or $bx \in A_v$, then $(ab)_v \subseteq A$ or $(ax)_v \subseteq A$ or $(bx)_v \subseteq A$. Thus $A$ is T-ABSO F. subm. of $B$. (6) The sum BSO F. subm. is not necessary T-ABSO F. subm., for example:

Let $X: Z \rightarrow L$ such that $X(v) = \begin{cases} 1 & \text{if } y \in Z, \\ 0 & \text{o.w.} \end{cases} \forall v \in L$

It is obvious that $X$ is F. M. of $Z$.

Let $A: Z \rightarrow L$ such that $A(v) = \begin{cases} v & \text{if } y \in 2Z, \\ 0 & \text{o.w.} \end{cases} \forall v \in L$

It is obvious that $A$ is F. subm. of $X$.

Let $B: Z \rightarrow L$ such that $B(v) = \begin{cases} v & \text{if } y \in 3Z, \\ 0 & \text{o.w.} \end{cases} \forall v \in L$

It is obvious that $B$ is F. subm. of $X$. Now, $A_v=2Z, B_v=3Z$ where $A_v$ and $B_v$ be T-ABSO subms. of $Z$-M. $Z$, but $A_v + B_v = Z = X_v$ is not T-ABSO subm., implies that $A+B=X$ is not T-ABSO F. subm. (7) Let $A$ and $B$ be two F. subms. of $M. X$ of an R-M. $M$. If $A$ is T-ABSO F. subm. then it is not necessary that $B$ is T-ABSO F. subm., for example:

Let $X: Z \rightarrow L$ such that $X(v) = \begin{cases} 1 & \text{if } y \in Z, \\ 0 & \text{o.w.} \end{cases} \forall v \in L$
It is obvious that $X$ is F. M. of Z-M. $Z$.

Let $A: Z \rightarrow L$ such that $A(v) = \begin{cases} 1 & \text{if } v \in 12Z \\ 0 & \text{otherwise} \end{cases}$ for all $v \in L$.

Let $B: Z \rightarrow L$ such that $B(v) = \begin{cases} 1 & \text{if } v \in 10Z \\ 0 & \text{otherwise} \end{cases}$ for all $v \in L$.

It is obvious that $A$ and $B$ are F. subms. of $X$. Now, $A_v = 2Z$, $B_v = 20Z$ where $A_v$ is T-ABSO subm. of $Z$ as Z-M., but $2Z \subseteq 20Z$ and $B_v = 20Z$ is not T-ABSO subm. of $Z$ as Z-M. since $2.2.5 \in B_v = 20Z$, but $2.5 \notin B_v = 20Z$ and $2.2 \notin B_v = 20Z$. Thus $A \subseteq B$ where $A$ is T-ABSO F. subm. of $X$ and $B$ is not T-ABSO F. subm. of $X$. (8) The intersection of two T-ABSO F. subms. need not be T-ABSO F. subm., for example:

Let $X: Z \rightarrow L$ such that $X(v) = \begin{cases} 1 & \text{if } v \in Z \\ 0 & \text{otherwise} \end{cases}$.

It is obvious that $X$ is F. M. of Z-M. $Z$.

Let $A: Z \rightarrow L$ such that $A(v) = \begin{cases} 1 & \text{if } v \in 12Z \\ 0 & \text{otherwise} \end{cases}$.

Let $B: Z \rightarrow L$ such that $B(v) = \begin{cases} 1 & \text{if } v \in 10Z \\ 0 & \text{otherwise} \end{cases}$.

It is obvious that $A$ and $B$ are F. subms. of $X$. $A_v = 12Z$, $B_v = 10Z$ are T-ABSO subms. in the $Z$ as Z-M. But $A_v \cap B_v = 12Z \cap 10Z = 120Z$ which is not T-ABSO since $2.6.10 \in 120Z$, but $2.10 \notin 120Z$ and $6.10 \notin 120Z$ and $2.6 \notin 120Z$. Hence $A$ and $B$ subms., but $A \cap B$ is not T-ABSO F. subm (9) Let $A$ be T-ABSO are two T-ABSO F. subm. of F. M. $X$ of an R-M. $M$. Then for each $B \subseteq X$, either $B \subseteq A$ or $B \cap A$ is T-ABSO F. subm. of $B$.

**Proof.** Assume that $B \subseteq A$ then $B \cap A \subseteq B$. Let $a_s, b_l$ be F. singletons of $R$ and $x_k \subseteq B$, such that $a_s b_l x_k \subseteq B \cap A$, implies $a_s b_l x_k \subseteq A$. Since $A$ is T-ABSO F. subm., thus either $a_s b_l \subseteq (A; R X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Then either $a_s b_l \subseteq (B \cap A; R)$ or $a_s x_k \subseteq B \cap A$ or $b_l x_k \subseteq B \cap A$. Thus $B \cap A$ is T-ABSO F. subm. of $B$.

**Proposition 17.**

Let $f: \hat{M}_1 \rightarrow \hat{M}_2$ be an epimorphism, where $X_1, X_2$ are F. M. of R-M. $\hat{M}_1$ and $\hat{M}_2$ resp. If $B$ is T-ABSO F. subm. of $X_2$, then $f^{-1}(B)$ is T-ABSO F. subm. of $X_1$. **Proof.** Since $B$ is F. subm. of $X_2$, then $f^{-1}(B)$ is F. subm. of $X_1$, since $f$ is epimorphism. Let $a_s b_l x_k \subseteq f^{-1}(B)$ for F. singletons $a_s, b_l$ of $R$ and $x_k \subseteq X_1$. Then $a_s b_l f(x_k) \subseteq B$ and since $B$ is T-ABSO F. subm., then either $a_s f(x_k) \subseteq B$ or $b_l f(x_k) \subseteq B$ or $a_s b_l \subseteq (B; R X_2)$. Hence either $a_s x_k \subseteq f^{-1}(B)$ or $b_l x_k \subseteq f^{-1}(B)$ or $a_s b_l x_k \subseteq f^{-1}(B)$ and $f(x_k) \subseteq f^{-1}(B)$ or $a_s b_l X_2 \subseteq B$. But $f(X_1) \subseteq X_2$, so that $a_s b_l f(X_1) \subseteq B$, hence $a_s b_l X_1 \subseteq f^{-1}(B)$, implies $a_s b_l \subseteq (f^{-1}(B); R X_1)$ Thus $f^{-1}(B)$ is T-ABSO F. subm. of $X_1$.

**Proposition 18.**

Let $f: \hat{M}_1 \rightarrow \hat{M}_2$ be an epimorphism, and $X_1, X_2$ are F. M. of R-M. $\hat{M}_1$ and $\hat{M}_2$ resp. Let $A \subseteq X_1$ such that $F-ker f \subseteq A$. Then $A$ is T-ABSO F. subm. of $X_1$ iff $f(A)$ is T-ABSO F. subm. of $X_2$.

**Proof.** ($\Rightarrow$) Let $a_s, b_l$ be F. singletons of $R$ and $y_h \subseteq X_2$ where $y_h = f(x_k)$ for some F. singleton $x_k \subseteq X_1$, such that $a_s b_l y_h \subseteq f(A)$. Hence $a_s b_l f(x_k) \subseteq f(A)$ $a_s b_l f(x_k) \subseteq f(A)$ since $f$ is onto. Then $a_s b_l f(x_k) = f(z_n)$ for some F. singleton $z_n \subseteq A$. So that $f(a_s b_l x_k) = f(z_n)$. Hence $a_s b_l x_k \subseteq f^{-1}(f(A))$ and $a_s b_l x_k \subseteq (A; R X_2)$ and $a_s b_l x_k \subseteq f^{-1}(B)$ and $f(x_k) \subseteq f^{-1}(B)$ and $a_s b_l x_k \subseteq f^{-1}(B)$, implies $a_s b_l \subseteq (f^{-1}(B); R X_1)$ Thus $f^{-1}(B)$ is T-ABSO F. subm. of $X_1$. 

118
f(z_n), hence f(a_s b_l x_k) - f(z_n) = 0_1; that is f(a_s b_l x_k - z_n) = 0_1, implies a_s b_l x_k - z_n \subseteq F - kerf \subseteq A.

So that a_s b_l x_k \subseteq A. Since A is T-ABSO F. subm., then either a_s b_l \subseteq (A; R X_1) or a_s x_k \subseteq A or b_l x_k \subseteq A. Hence either a_s b_l x_k \subseteq A \rightarrow (a_s b_l x_k) \subseteq f(A) or f(a_s x_k) \subseteq f(A) or f(b_l x_k) \subseteq f(A), implies either a_s b_l f(x_k) \subseteq f(A) or a_s f(x_k) \subseteq f(A) or b_l f(x_k) \subseteq f(A). Then either a_s b_l \subseteq (f(A); R X_2) or a_s y_h \subseteq f(A) or b_l y_h \subseteq f(A). Thus f(A) is T-ABSO F. subm.

(\Rightarrow) Let a_s b_l x_k \subseteq A for F. singletons a_s, b_l of R and x_k \subseteq X_1. Hence f(a_s b_l x_k) \subseteq f(A), implies a_s b_l f(x_k) \subseteq f(A). But f(A) is T-ABSO F. subm., then either a_s b_l \subseteq (f(A); R X_2) or a_s f(x_k) \subseteq f(A) or b_l f(x_k) \subseteq f(A).

If a_s b_l \subseteq (f(A); X_2), then a_s b_l X_2 \subseteq f(A), implies a_s b_l f(x_k) \subseteq f(A) since f is onto. Hence f(a_s b_l x_k) \subseteq f(A), so that a_s b_l X_1 \subseteq A; that is a_s b_l \subseteq (A; R X_1). If a_s f(x_k) \subseteq f(A) then f(a_s x_k) = f(z_n) for some F. singleton z_n \subseteq A, \forall n \in L. Hence f(a_s x_k) = f(z_n) = 0_1, implies a_s x_k - z_n \subseteq F - kerf \subseteq A. So that a_s x_k \subseteq A. If b_l f(x_k) \subseteq f(A), then by the same way above, we have b_l x_k \subseteq A. Therefore, A is T-ABSO F. subm. of X_1.

**Proposition 19.** Let A be a proper F. subm. of F. M. X of an R-M M. Then A is T-ABSO F. subm. of X iff a_s b_l B \subseteq A for F. singletons a_s, b_l of R and B is F. subm. of X implies a_s b_l \subseteq (A; R X) or a_s B \subseteq A or b_l B \subseteq A.

**Proof.** (\Rightarrow) Let A be T-ABSO F. subm. and a_s b_l B \subseteq A. Assume that a_s b_l \subseteq (A; X), a_s B \subseteq A and b_l B \subseteq A. Then there exist F. singletons x_v, y_k \subseteq B, such that a_s x_v \subseteq A and b_l y_k \subseteq A. Since a_s b_l x_v \subseteq A and a_s b_l \subseteq (A; R X), a_s x_v \subseteq A, we have b_l x_v \subseteq A. Also since a_s b_l y_k \subseteq A and a_s b_l \subseteq (A; R X), b_l y_k \subseteq A, we have a_s y_k \subseteq A. Now, since a_s b_l (x_v + y_k) \subseteq A and a_s b_l \subseteq (A; R X), we have a_s (x_v + y_k) \subseteq A or b_l (x_v + y_k) \subseteq A. If a_s (x_v + y_k) \subseteq A, then (a_s x_v + a_s y_k) \subseteq A and since a_s y_k \subseteq A, we get a_s x_v \subseteq A, this is a discrepancy. If b_l (x_v + y_k) \subseteq A, then (b_l x_v + b_l y_k) \subseteq A and since b_l x_v \subseteq A, we get b_l y_k \subseteq A this is a discrepancy. Thus either a_s b_l \subseteq (A; R X) or a_s B \subseteq A or b_l B \subseteq A.

(\Leftarrow) It is obvious. The next theorem gives a characterization of T-ABSO F. subm.

**Theorem 20.**

Let A be a proper F. subm. of F. M. X of an R-M M. Then the following expressions are equivalent:

1- A is T-ABSO F. subm. of X;

2- If \( \tilde{H} \cup \tilde{U} \subseteq A \), for some F. ideals \( \tilde{H}, \tilde{U} \) of R and F. subm. B of X, then either \( \tilde{H} B \subseteq A \) or \( \tilde{U} B \subseteq A \) or \( \tilde{H} \cup \tilde{U} \subseteq (A; R X) \).

**Proof.** (1) \( \Rightarrow \) (2) Suppose that A is T-ABSO F. subm. of X and \( \tilde{H} \cup \tilde{U} \subseteq A \) for some F. ideals \( \tilde{H}, \tilde{U} \) of R and some F. subm. B of X. Let \( \tilde{H} \cup \subseteq (A; R X) \), to prove \( \tilde{H} B \subseteq A \) or \( \tilde{U} B \subseteq A \). Assume that \( \tilde{H} B \subseteq A \) and \( \tilde{U} B \subseteq A \), then there exist F. singletons \( a_s \subseteq \tilde{H} \) and \( b_l \subseteq \tilde{U} \), such that \( a_s B \subseteq A \) and \( b_l B \subseteq A \). But \( a_s b_l B \subseteq A \) and neither \( a_s B \subseteq A \) nor \( b_l B \subseteq A \) and A is T-ABSO F. subm., so that \( a_s b_l \subseteq (A; R X) \). Since \( \tilde{H} \cup \subseteq (A; R X) \), then there exist F. singletons \( x_v \subseteq \tilde{H} \) and \( y_k \subseteq \tilde{U} \), such that \( x_v y_k \subseteq (A; R X) \). But \( x_v y_k B \subseteq A \), so that \( x_v B \subseteq A \) or \( y_k B \subseteq A \) by proposition (19).
Now we have the following:

1) If \( x_v B \subseteq A \) and \( y_v B \nsubseteq A \), since \( a_s y_v B \subseteq A \) and \( y_v B \nsubseteq A \), so that \( a_s y_v \subseteq (A;_R X) \) by proposition (19). Since \( x_v B \subseteq A \) and \( a_s B \nsubseteq A \), hence \( (a_s + x_v) y_v B \subseteq A \). On the other hand, \( (a_s + x_v)y_v B \subseteq A \) and neither \( (a_s + x_v) y_v B \subseteq A \) nor \( y_v B \subseteq A \), we get \( (a_s + x_v) y_v (A;_R X) \) by proposition (19). But \( (a_s + x_v) y_v = (a_s y_v + x_v y_v) \subseteq (A;_R X) \) and \( a_s y_v \subseteq (A;_R X) \), we get \( x_v y_v \subseteq (A;_R X) \) this is a discrepancy.

2) If \( y_v B \subseteq A \) and \( x_v B \nsubseteq A \). By the same way of (1), we get a discrepancy.

3) If \( x_v B \subseteq A \) and \( y_v B \subseteq A \). Since \( y_v B \subseteq A \) and \( b_t B \nsubseteq A \), we have \( (a_s + x_v) b_t B \subseteq A \) and neither \( (a_s + x_v) B \subseteq A \) nor \( (a_s + x_v) B \subseteq A \), we have\( (a_s + x_v) b_t \subseteq (A;_R X) \) by proposition (19). Since \( a_s b_t \subseteq (A;_R X) \) and \( a_s b_t + a_s y_v \subseteq (A;_R X) \), we get \( a_s y_v \subseteq (A;_R X) \). Since \( (a_s + x_v) b_t B \subseteq A \) and neither \( b_t B \subseteq A \) nor \( (a_s + x_v) B \subseteq A \), we get \( (a_s + x_v) b_t \subseteq (A;_R X) \) by proposition (19), where \( (a_s + x_v) b_t = (a_s b_t + a_s y_v + x_v b_t + y_v y_k) \subseteq (A;_R X) \). But \( (a_s b_t + a_s y_v + x_v b_t) \subseteq (A;_R X) \), so that \( x_v y_k \subseteq (A;_R X) \) this is a discrepancy. Thus \( H B \subseteq A \) or \( \cup B \subseteq A \).

(2) \( \Rightarrow \) (1) It is obvious.

Theorem 21.

If \( A \) is T-ABSO F. subm. of F. M. \( X \) of an R-M. \( \hat{M} \), then \( (A;_R X) \) is T-ABSO F. ideal of R.

Proof. Let \( a_s b_t r_k \subseteq (A;_R X) \) for F. singletons \( a_s, b_t, r_k \) of R.

If \( a_s r_k \not\subseteq (A;_R X) \) and \( b_t r_k \not\subseteq (A;_R X) \), then there exist F. singletons \( x_v, y_h \subseteq X \setminus A \), such that \( a_s r_k x_v \not\subseteq A \) and \( b_t r_k y_h \not\subseteq A \). Since \( a_s b_t (r_k (x_v + y_h)) \subseteq A \) and \( A \) is T-ABSO F. subm., then either \( a_s b_t \subseteq (A;_R X) \) or \( a_s r_k (x_v + y_h) \subseteq A \) or \( b_t r_k (x_v + y_h) \subseteq A \). If \( a_s r_k (x_v + y_h) \subseteq A \) and since \( a_s r_k x_v \subseteq A \), then we have \( a_s r_k y_h \subseteq A \). So that \( a_s b_t (r_k y_h) \subseteq A \) and \( b_t r_k y_h \subseteq A \), hence \( a_s b_t \subseteq (A;_R X) \). By the same method if \( b_t r_k (x_v + y_h) \subseteq A \), we get \( a_s b_t \subseteq (A;_R X) \).

Thus \( (A;_R X) \) is T-ABSO F. ideal of R.

Theorem 22.

Let \( X \) be multiplication F. M. of an R-M. \( \hat{M} \), and \( A \) is a proper F. subm. of \( X \). If \( (A;_R X) \) is T-ABSO F. ideal of R, then \( A \) is T-ABSO F. subm. of \( X \).

Proof. Let \( a_s b_t x_v \subseteq A \) for F. singletons \( a_s, b_t \) of R and \( x_v \subseteq X \), then \( a_s b_t x_v \subseteq X \setminus A \). But \( x_v \geq \hat{H} X \) for some F. ideal \( \hat{H} \) of R since \( X \) is multiplication F. M., so that \( a_s b_t \hat{H} X \subseteq A \). Thus \( a_s b_t \hat{H} \subseteq (A;_R X) \), so we have that \( a_s \geq b_t \geq \hat{H} \subseteq (A;_R X) \). Since \( (A;_R X) \) is T-ABSO F. ideal of R, we get either \( a_s \geq \hat{H} \subseteq (A;_R X) \) or \( b_t > \hat{H} \subseteq (A;_R X) \) or \( a_s \geq b_t > \subseteq (A;_R X) \) by Proposition (9).

1) If \( a_s > \hat{H} \subseteq (A;_R X) \), then \( a_s \geq \hat{H} X \subseteq A \) and so \( a_s \geq x_v \subseteq A \). Hence \( a_s x_v \subseteq A \)

2) If \( b_t > \hat{H} \subseteq (A;_R X) \), then by the same method \( b_t x_v \subseteq A \).

3) If \( a_s \geq b_t \subseteq (A;_R X) \), then \( a_s b_t \subseteq (A;_R X) \).

By combining theorem (21) and theorem (22), we have the following corollary:
Corollary 23.
Let $A$ be a proper F. subm. of a multiplication F. M. $X$ of an R-M. $\hat{M}$. Then $A$ is T-ABSO F. subm. of $X$ iff $(A;_R X)$ is T-ABSO F. ideal of $R$.

Remark 24.
The condition $X$ is multiplication F. M. can't be deleted from theorem (22). See the following example:

Let $X: \mathbb{Z}_{p^\infty} \to \mathbb{L}$ such that $X(y) = \begin{cases} 1 & \text{if } y \in \mathbb{Z}_{p^\infty} \\ 0 & \text{o.w.} \end{cases}$. It is obvious that $X$ is F. M. of $\mathbb{Z}_{p^\infty}$.

Let $A: \mathbb{Z}_{p^\infty} \to \mathbb{L}$ such that $A(y) = \begin{cases} v & \text{if } y \in (0) \\ 0 & \text{o.w.} \end{cases}$ for all $v \in \mathbb{L}$. It is obvious that $A$ is F. subm. of $X$.

Now, $A(x) = (0)$ is not T-ABSO subm. of $X(x) = \mathbb{Z}_{p^\infty}$, since $p^2 < \frac{1}{p^2} + Z > (0)$ but $p < \frac{1}{p^2} + Z > (0)$ and $p^2 \notin ((0);_Z \mathbb{Z}_{p^\infty}) = 0$. Note $(0)$ is a prime ideal in $Z$, so that $((0);_Z \mathbb{Z}_{p^\infty}) = 0$ is T-ABSO ideal in $Z$; that is $(A;_R X_v)$ is T-ABSO ideal in $Z$, then $(A;_R X)$ is T-ABSO F. ideal in $Z$. Thus $A$ is not T-ABSO F. subm. of $X$, but $(A;_R X)$ is T-ABSO F. ideal in $Z$.

Now, we gave the following theorem is a characterization of T-ABSO F. subm.

Theoerm 25.
Let $A$ be a proper F. subm. of a multiplication F. M. $X$ of $\hat{M}$. Then $A$ is T-ABSO F. subm. of $X$ iff $A_1 A_2 A_3 \subseteq A$ implies that $A_1 A_2 \subseteq A$ or $A_1 A_3 \subseteq A$ or $A_2 A_3 \subseteq A$, where $A_1, A_2, A_3$ are F. subm. of $X$.

Proof. ($\Rightarrow$) Since $X$ is a multiplication F., then $A_1 = \hat{H} X$, $A_2 = \hat{U} X$ and $A_3 = K X$ for some F. ideals $\hat{H}, \hat{U}$ and $K$ of $R$. So that the product of $A_1, A_2$ and $A_3$ as follows: $A_1 A_2 A_3 = \hat{H}\hat{U} K X \subseteq A$. Thus $\hat{H}\hat{U} X \subseteq (A;_R X)$.

Proof. ($\Leftarrow$) Let $\hat{H}\hat{U} B \subseteq A$ for some F. ideals $\hat{H}, \hat{U}$ of $R$ and $B$ is F. subm. of $X$. Since $X$ is a multiplication F. M., then $B = EX$ for some F. ideal $E$ of $R$. Then $\hat{H}\hat{U} E X \subseteq A$. Let $A_1 = \hat{H} X$ and $A_2 = \hat{U} X$, so that $A_1 A_2 B = \hat{H}\hat{U} E X \subseteq A$. So by hypotheses either $A_1 B \subseteq A$ or $A_2 B \subseteq A$ or $A_1 A_2 \subseteq A$, hence $\hat{H} E X \subseteq A$ or $\hat{U} E X \subseteq A$ or $\hat{H}\hat{U} X \subseteq A$. Thus $\hat{H} B \subseteq A$ or $\hat{U} B \subseteq A$ or $\hat{H}\hat{U} \subseteq (A;_R X)$. Therefore, $A$ is T-ABSO F. subm. of $X$ by theorem (20). Now, the definitions of finitely generated F. M. see [17, Definition (2.11)] and faithful F. M. see [3, Definition (3.2.6)]. We give the following proposition.

Proposition 26.
Let $X$ be a finitely generated multiplication F. M. of an R-M. $\hat{M}$. If $\hat{H}$ is T-ABSO F. ideal of $R$ such that $F-\text{ann}X \subseteq \hat{H}$, then $\hat{H} X$ is T-ABSO F. subm. of $X$.

Proof. Let $a_x b_x x_v \subseteq \hat{H} X$, where $a_x, b_x$ be F. singletons of $R$ and $x_v \subseteq X$, hence $a_x b_x l < x_v \subseteq \hat{H} X$. But $X$ is a multiplication F. M., then $< x_v > = \hat{U} X$ for some F. ideal $\hat{U}$ of $R$. Thus
\(a_s b_l X \subseteq \hat{H} X\). So that \(a_s b_l \subseteq \hat{H} + F - \text{ann}X = \hat{H}\) since \(F\text{-ann}X \subseteq \hat{H}\). But \(\hat{H}\) is T-ABSO F. ideal of R, so that either \(a_s b_l \subseteq \hat{H}\) or \(a_s \subseteq \hat{H}\) or \(b_l \subseteq \hat{H}\). Then we have \(a_s b_l X \subseteq \hat{H} X\) or \(a_s X \subseteq \hat{H} X\) or \(b_l X \subseteq \hat{H} X\), so that \(a_s b_l \subseteq (\hat{H} X : R)\) or \(a_s x_v > \subseteq \hat{H} X\) or \(b_l < x_v > \subseteq \hat{H} X\). Hence \(a_s b_l \subseteq (\hat{H} X : R)\) or \(a_s x_v \subseteq \hat{H} X\) or \(b_l x_v \subseteq \hat{H} X\). So that \(\hat{H} X\) is T-ABSO F. subm. of \(X\).

**Corollary 27.**

Let \(X\) be a faithful finitely generated multiplication F. M. of \(\hat{M}\). If \(\hat{H}\) is T-ABSO F. ideal of R, then \(\hat{H} X\) is T-ABSO F. subm. of \(X\).

**Proof.** By proposition (26), it follows immediately.

**Corollary 28.**

Suppose that \(X\) be a faithful finitely generated multiplication F. M. of \(\hat{M}\). Then every proper F. subm. of \(X\) is T-ABSO iff every proper F. ideal of R is T-ABSO.

**Proof.** \((\Leftarrow)\) By corollary (27), it follows immediately.

\((\Rightarrow)\) Let \(\hat{H}\) be a proper F. ideal of R. Then \(A=\hat{H} X\) is a proper subm. of \(X\). Since \(A\) is T-ABSO F. subm., so that \((A : R X)\) is T-ABSO F. ideal by theorem (21). But \(X\) is a multiplication F. M., hence \(A=(A : R X)\) by [5]. Thus \(\hat{H} X= (A : R X)\). Since \(X\) is a faithful finitely generated multiplication F. M., then \(X_v\) is a faithful finitely generated multiplication M. by [16, 17], implies that \(X_v=\hat{M}\) is cancellation R-M. by [18]. Hence \(X\) is a cancellation F. M. by [8]. Therefore \(\hat{H}= (A : R X)\); that is \(\hat{H}\) is T-ABSO F. ideal of R.

Recall that Let \(X\) be F. M. of an R-M. \(\hat{M}\), and let \(A\) be F. subm. of \(X\). \(A\) is called a pure F. subm., if for each F. singleton \(y \subseteq A\), see [19].

**Proposition 29.**

Let \(A\) be a proper pure F. subm. of F. M. X of \(\hat{M}\). If \(0_1\) is T-ABSO F. subm. of \(X\), then \(A\) is T-ABSO F. subm. of \(X\).

**Proof.** Let \(a_s b_l x_v \subseteq A\) where \(a_s, b_l\) F. singletons of R and \(x_v \subseteq X\). Put \(\hat{H}=a_s b_l\), hence \(a_s b_l x_v \subseteq \hat{H} X \cap A\), but \(\hat{H} X \cap A=\hat{H} A\). So \(a_s b_l x_v = a_s b_l y_h\), for some F. singleton \(y_h \subseteq A\), then \(a_s b_l (x_v - y_h) \subseteq 0_1\), but \(0_1\) is T-ABSO F. subm., hence \(a_s (x_v - y_h) \subseteq 0_1\) or \(b_l (x_v - y_h) \subseteq 0_1\) or \(a_s b_l \subseteq F - \text{ann}X \subseteq (A : R X)\).

So we have \(a_s x_v = a_s y_h \subseteq A\) or \(b_l x_v = b_l y_h \subseteq A\) or \(a_s b_l \subseteq (A : R X)\). Therefore \(A\) is T-ABSO F. subm. of \(X\).

Now, we give the concept of a cancellative F. M. as follows:

**Definition 30.** A F. M. X of \(\hat{M}\) is called a cancellative F. if whenever \(a_s x_v = a_s y_k\) for F. singletons \(a_s\) of R and \(x_v, y_k \subseteq X\), \(\forall s, v, k \in L\), then \(x_v = y_k\)

**Proposition 31.**

Let \(X\) be a cancellative F. M. of \(\hat{M}\), and \(A\) be a proper F. subm. of \(X\). Then \(A\) is a pure F. subm. of \(X\) iff \(A\) is T-ABSO F. subm. of \(X\) with \((A : R X) = 0_1\).
Proof. \((\Rightarrow)\) Assume that \(A\) is a pure F. subm. of \(X\) and \(a_s b_t x_v \subseteq A\) such that \(a_s b_t \notin (A;_R X)\) for F. singletons \(a_s, b_t\) of R and \(x_v \subseteq X\). Then \(a_s b_t x_v \subseteq a_s b_t X \cap A = a_s b_t A\), hence \(a_s b_t y_k = a_s b_t y_k\) for some F. singleton \(y_k \subseteq A\). Since \(X\) is a cancellative F. M., then \(b_t x_v = b_t y_k \subseteq A\). Thus \(A\) is T-ABSO F. subm. of \(X\).

Now, assume that F. singleton \(r_k \subseteq (A;_R X)\) with \(r_k \neq 0_1\). Since \(A \neq X\) there exists F. singleton \(x_v \subseteq X \setminus A\) such that \(r_k x_v \subseteq r_k X \cap A = r_k A\), so there exists F. singleton \(y_k \subseteq A\), such that \(r_k x_v = r_k y_k\); hence \(x_v = y_k\); this is a contradiction. So that \((A;_R X) = 0_1\).

\((\Leftarrow)\) Suppose that \(A\) is T-ABSO F. subm. of \(X\). Let \(a_s b_t x_v \subseteq a_s b_t X \cap A\) for F. singletons \(a_s, b_t\) of R and \(x_v \subseteq X\). We may suppose that \(a_s b_t \neq 0_1\). Since \(A\) is T-ABSO F. subm. of \(X\), then either \(a_s x_v \subseteq A\) or \(b_t x_v \subseteq A\). If \(b_t x_v \subseteq A\) and \(b_t\) be F. singleton of R, \(a_s b_t X \subseteq a_s b_t A\). By the same method to prove the case if \(a_s x_v \subseteq A\); that is \(a_s b_t A \subseteq a_s b_t X \cap A\). Thus \(a_s b_t X \cap A = a_s b_t A\). So that \(A\) is a pure F. subm.

4. T-ABSO Quasi Primary F. Subm.

In this section we present the concept of T-ABSO quasi primary F. subm. and study the relationships this concept among T-ABSO F. subm. and T-ABSO primary F. subm. Many basic properties and outcomes are given. Now, we give the following definition:

**Definition 32.**

Let \(A\) be a proper F. subm. of non-empty F. M. \(X\) of an R-M. \(ℳ\). Then the \(X\)-F. radical of \(A\), denoted by \(X-R(A)\) is defined to the intersection of all prime F. subm. including \(A\). We give the pursue lemma which are needed in the next proposition.

**Lemma 33.**

Let \(X\) be a multiplication F. M. of \(ℳ\), let \(A\) be a proper F. subm. of \(X\). Then the following expressions are equivalent:

1. \(A\) is a prime F. subm. of \(X\).
2. \((A;_R X)\) be a prime F. ideal of R.
3. \(A = \tilde{H} X\) for some a prime F. ideal \(\tilde{H}\) of \(\tilde{R}\) with \(F\text{-ann} X \subseteq \tilde{H}\).

**Proof.** (1) \(\Rightarrow\) (2) it follows by [20, proposition (2.5)].

(2) \(\Rightarrow\) (3) Since \(X\) is a multiplication F. M., so that \(A = (A;_R X)X\) by[5]. Put \(\tilde{H} = (A;_R X)\) be a prime F. ideal of R. Now, since \(F\text{-ann} X = (0_1;_R X)\) and \((0_1;_R X) \subseteq (A;_R X) = \tilde{H}\). So that \(F\text{-ann} X \subseteq \tilde{H}\).

(3) \(\Rightarrow\) (1) Let \(a_s x_v \subseteq A\) for F. singleton \(a_s\) of R and \(x_v \subseteq X\), and \(x_v \notin A\) to prove \(a_s \notin (A;_R X)\). By(3), \(A = \tilde{H} X\) for some a prime F. ideal \(\tilde{H}\) of \(R\) with \(F\text{-ann} X \subseteq \tilde{H}\), so that \(F\text{-ann} X\) is a prime F. ideal of \(R\), but \(F\text{-ann} X = (0_1;_R X)\), hence \((0_1;_R X)\) is a prime F. ideal of \(R\). Let \(a_s b_t \subseteq (0_1;_R X)\), for F. singleton \(b_t\) of R, and \(b_t \notin (0_1;_R X)\), then \(a_s \subseteq (0_1;_R X)\). Since \((0_1;_R X) \subseteq (A;_R X), so that\(a_s \subseteq (A;_R X), Thus\( A\) is a prime F. subm. of \(X\).\)

**Lemma 34.**

Let \(X\) be a finitely generated multiplication F. M. of \(ℳ\) and let \(A\) be a F. subm. of \(X\). Then \(X - R(A) = \sqrt{A;_R X} . X\).
Ibn Al-Haitham Jour. for Pure & Appl. Sci.                                          IHJPAS
https://doi.org/10.30526/32.1.1930
Vol. 32 (1) 2019

Proof. If \( X - R(A) = X \), then the result is directly. So that \( X - R(A) \neq X \), if \( B \) is any prime \( F \). subm. of \( X \) which contains \( A \), we get \( (A : R) X \subseteq (B : R) X \). We prove that \( (B : R) X \) is a prime \( F \). ideal. Assume that \( a_s b_t \subseteq (B : R) X \) for \( F \). singleton \( a_s, b_t \) of \( R \), so that \( a_s b_t X \subseteq B \), then either \( b_t X \subseteq B \) or \( b_t x_r \subseteq X / B \) for some \( F \). singleton \( x_r \subseteq X \). But \( B \) is a prime \( F \). subm. and \( a_s (b_t x_r) \subseteq B \), then either \( (b_t x_r) \subseteq B \) or \( a_s \subseteq (B : R) X \). Thus \( a_s \subseteq (B : R) X \) or \( b_t \subseteq (B : R) X \). So that \( (B : R) X \) is a prime \( F \). ideal.

Hence \( \sqrt{A : R} X \subseteq (B : R) X \) by \([13]\), then \( \sqrt{A : R} X \subseteq (B : R) X \) is a prime \( F \). ideal. Assume that \( a_0 b_0 \subseteq (A : R) X \) for \( F \). singleton \( a_0, b_0 \) of \( R \), so that \( a_0 b_0 x_r \subseteq B \), then either \( b_0 x_r \subseteq B \) or \( b_0 x_r \subseteq X / B \) for some \( F \). singleton \( x_r \subseteq X \). But \( B \) is a prime \( F \). subm. and \( a_0 (b_0 x_r) \subseteq B \), then either \( (b_0 x_r) \subseteq B \) or \( a_0 \subseteq (B : R) X \). Thus \( a_0 \subseteq (B : R) X \) or \( b_0 \subseteq (B : R) X \). So that \( (B : R) X \) is a prime \( F \). ideal.

Hence \( \sqrt{A : R} X \subseteq (B : R) X \) by \([13]\), then \( \sqrt{A : R} X \subseteq (B : R) X \). Since \( B \) is an arbitrary prime \( F \). subm. containing \( A \), we get \( \sqrt{A : R} X \subseteq \sqrt{B : R} X \). Hence \( \sqrt{A : R} X \subseteq \sqrt{B : R} X \).

Now, since \( X \) is a multiplication \( F \). M., hence \( X - R(A) = X - (A : R) X \).

We must prove that \( (X - R(A) : R) X \subseteq \sqrt{A : R} X \). Let \( K \) be any prime \( F \). ideal such that \( (A : R) X \subseteq K \). Since \( K \) is a prime \( F \). ideal containing \( F \). ann \( X \subseteq (0 : R) X \), then \( K X \) is a prime \( F \). subm. of \( X \) containing \( A = (A : R) X \) by lemma \([33]\). Thus \( (X - R(A) : R) X \subseteq \sqrt{A : R} X \) by \([13]\), hence \( X - R(A) = (X - R(A) : R) X \subseteq \sqrt{A : R} X . X \) by \([13]\), then \( \sqrt{A : R} X \subseteq \sqrt{B : R} X \).

From \((1) \) and \((2) \), we get \( \sqrt{A : R} X = \sqrt{B : R} X \).

Before the next proposition we give these lemmas and definition which are needed in the proof of the next proposition. We give this definition as follows:

**Definition 35.**

Let \( X \) be an \( R \). M. of an \( R \). M. \( \mathcal{M} \). If \( P \) is a maximal \( F \). ideal of \( R \) then we define \( F \cdot G_p (X) = \{x_r \subseteq X : (1 - a_s) x_r = 0 \} \) for some \( F \). singleton \( a_s \subseteq P, \forall v, s \in L \} \).

It is obvious \( F \cdot G_p (X) \) is a \( F \). subm. of \( X \). \( X \) is called \( P \)-cyclic \( F \). M. if there exist \( F \). singleton \( b_0 \subseteq P \) and \( x_r \subseteq X \) such that \( (1 - b_0) x_r < x_r >, \forall l, v \in L \).

**Lemma 36.**

Let \( R \) be a commutative ring with unity. Then \( F \). M. \( X \) of an \( R \). M. \( \mathcal{M} \) is a multiplication \( F \). M. iff for every maximal \( F \). ideal \( P \) of \( R \) either \( X = F \cdot G_p (X) \) or \( X \) is \( P \)-cyclic \( F \). M.

**Proof.** \((\Rightarrow) \) Assume that \( X \) is a multiplication \( F \). M. Let \( P \) be maximal \( F \). ideal of \( R \). Suppose that \( X = \mathcal{P} X \), let \( F \). singleton \( x_v \subseteq X \), then \( x_v > = \hat{H} X \) for some \( F \). ideal \( \hat{H} \) of \( R \). Hence \( x_v > = \hat{H} X = \mathcal{H} \mathcal{P} X = \mathcal{P} \mathcal{H} X = \mathcal{P} < x_v > \), then \( x_v = a_s x_v \) for some \( F \). singleton \( a_s \subseteq P \). Thus \( (1 - a_s) x_v = 0 \), so that \( x_v \subseteq F \cdot G_p (X) \). It follows that \( X = F \cdot G_p (X) \).

Now, suppose that \( X \neq \mathcal{P} X \) then there exists \( F \). singleton \( x_v \subseteq X \), \( x_v \subseteq \mathcal{P} X \). So that there exists an ideal \( \mathcal{U} \) of \( R \) such that \( x_v > = \mathcal{U} X \). It is obvious that \( \mathcal{U} \subseteq P \) and so \( (1 - b_0) \subseteq \mathcal{U} \) for some \( F \). singleton \( b_0 \subseteq P \). Hence \( (1 - b_0) X \subseteq \mathcal{U} X \). Thus \( X \) is \( P \)-cyclic \( F \). M. \((\Leftarrow) \) Suppose that for each maximal \( F \). ideal \( P \) of \( R \) either \( X = F \cdot G_p (X) \) or \( X \) is \( P \)-cyclic \( F \). M. Let \( A \) be \( F \). subm. of \( X \) and \( \hat{H} = (A : R) X \). It is obvious that \( \hat{H} X \subseteq A \). Suppose that \( F \). singleton \( y_k \subseteq A \) and \( K = \{r_h \subseteq R : r_h y_k \subseteq \hat{H} X \} \). Assume that \( K \neq R \), then there exists a maximal \( F \). ideal \( E \) of \( R \) such that \( K \subseteq E \) by \([13, \text{proposition(1.3.2.4)}]\). If \( X = F - G_E (X) \) then \( (1 - a_s) y_k = 0 \) for some \( F \). singleton \( a_s \subseteq E \), and \( (1 - a_s) \subseteq K \subseteq E \) this is a discrepancy. Thus by
hypothesis there exist F. singletons \( b_1 \subseteq E, z_n \subseteq X \) such that \((1_v - b_1)X \subseteq <z_n>\). It follows that \((1_v - b_1)A \) is F. subm. of \(<z_n>\) and so that \((1_v - b_1)A = D z_n \) where \(D \) is F. ideal \( \{ r_h : r_h z_n \subseteq (1_v - b_1)A \} \) of \(R\). Note that \((1_v - b_1)D X = D (1_v - b_1)X \subseteq D z_n \subseteq A \). So that \((1_v - b_1)D \subseteq H\). Thus for F. singleton \( y_k \subseteq A \), \((1_v - b_1)^2 y_k \subseteq (1_v - b_1)^2 A = (1_v - b_1)D z_n \subseteq H\).

So that \((1_v - b_1)^2 \subseteq K \subseteq E\) this is a discrepancy. Thus \(K = R\) and \(y_k \subseteq HX\). Therefore \(A = HX\) and \(X\) is a multiplication F. M.

**Lemma 37.**

Let \(X\) be a multiplication F. M. of an R-M. \(\hat{M}\), then

\[
\cap_{i \in I} (\hat{H}_i X) = (\cap_{i \in I} (\hat{H}_i + F - annX)) X
\]

for any non-empty collection of F. ideals \(\hat{H}_i(i \in I)\) of \(R\).

**Proof.** Assume that \(X\) is a multiplication F. M. Let \(\hat{H}_i(i \in I)\) be any non-empty collection of F. ideals of \(R\), let \(\cup = \cap_{i \in I} (\hat{H}_i + F - annX)\), then \(\cup X = (\cap_{i \in I} (\hat{H}_i + F - annX)) X\). It is obvious that \(\cup X \subseteq \cap_{i \in I} (\hat{H}_i X)\). Now, let be F. singleton \(x_v \subseteq \cap_{i \in I} (\hat{H}_i X)\) and let \(G = \{ a_s \subseteq R : a_s x_v \subseteq \cup X \}, \forall s, v \in L\) Suppose that \(G \neq R\), then there exists a maximal F. ideal \(P\) of \(R\) such that \(G \subseteq P\), it is obvious that \(x_v \not\subseteq F - G\) \(X\) and hence \(X\) is \(P\)-cyclic F. M. by lemma (36). Then there exist F. singletons \(a_s \subseteq P\) and \(y_k \subseteq X\) such that \((1_v - a_s)X \subseteq <y_k>\). Hence \((1_v - a_s)x_v \subseteq \cap_{i \in I} (\hat{H}_i y_k)\) for each \(i \in I\) there exists F. singleton \(b_{l_i} \subseteq \hat{H}_i\), \(\forall l_i \in L\), such that \((1_v - a_s)x_v = b_{l_i}y_k\). Choose \(j \in \Lambda\), for each \(i \in I\), \(b_{l_j}y_k = b_{l_j}y_k\), so that \((b_{l_j} - b_{l_i})y_k = 0_1\), implies that: \((1_v - a_s)(b_{l_j} - b_{l_i})X = (b_{l_j} - b_{l_i})(1_v - a_s)X \subseteq (b_{l_j} - b_{l_i}) < y_k > = 0_1\), \((1_v - a_s)(b_{l_j} - b_{l_i}) = 0_1\). Thus \((1_v - a_s)b_{l_j} = (1_v - a_s)b_{l_i} \subseteq \hat{H}_i(i \in I)\), then \((1_v - a_s)b_{l_j} \subseteq \cup\). Hence \((1_v - a_s)^2 x_v = (1_v - a_s)b_{l_j} y_k \subseteq \cup X\).

It follows that \((1_v - a_s)^2 \subseteq G \subseteq P\) this is a discrepancy. Thus \(G = R\) and \(x_v \subseteq \cup X\), so that \(\cap_{i \in I} (\hat{H}_i X) \subseteq \cup X\) implies that \(\cap_{i \in I} (\hat{H}_i X) = \cup X\) That is \(\cap_{i \in I} (\hat{H}_i X) = (\cap_{i \in I} (\hat{H}_i + F - annX)) X\). Now, we give the proposition as follows:

**Proposition 38.**

Let \(X\) be a multiplication finitely generated F. M. of an R-M. \(\hat{M}\) and \(A\) be T-ABSO F. subm. of \(X\). Then one of the following satisfy:

1- \(X-R(A) = P\) is a prime F. subm. of \(X\) such that \(P^2 \subseteq A\).

2- \(X-R(A) = P_1 \cap P_2, P_1 \cap P_2 \subseteq A\) and \((X-R(A))^2 \subseteq A\) where \(P_1, P_2\) are the only distinct minimal prime F. subms. of \(A\).

**Proof.** By theorem (21), \((A; R) X\) is T-ABSO F. ideal of \(R\). So that either \(R((A; R) X) = \cup\) is a prime F. ideal of \(R\) such that \(\cup^2 \subseteq (A; R) X\) or \(R((A; R) X) = \cup_1 \cap \cup_2\), \(\cup_1 \cup_2 \subseteq (A; R) X\) and \(R((A; R) X)^2 \subseteq (A; R) X\) where \(\cup_1, \cup_2\) are the only distinct minimal prime F. ideals of \((A; R) X\) by proposition (6), where \(R((A; R) X) = \sqrt{A; R X}\). if the first case satisfies, then since \(X\) is F. multiplication, we have \(X-R(A) = R((A; R) X) X = \cup X\) is a prime F. subm. of \(X\). Put \(\cup X = P\) by lemma (33) and lemma (34), and \(\cup X^2 = \cup^2 X \subseteq (A; R) X = A\). Now, suppose that the latter case satisfies, then by lemma(33), \(\cup_1 X\) and \(\cup_2 X\) are the only distinct minimal prime F. subms.
of $A$ and $X$: $R(A) = R((A; R) X) = (\cup_1 \cup_2) X = \cup_1 X \cap \cup_2 X$ by lemma (37). Moreover $(\cup_1 X)(\cup_2 X) = (\cup_1 \cup_2) X \subseteq (A; R) X = A$ and $(X - R(A))^2 = (R((A; R) X))^2 = (R((A; R) X))^2 X \subseteq (A; R) X = A$.

We give the definition of T-ABSO primary F. subm. as follows:

**Definition 39.** Let $A$ be a proper F. subm. of $F. M.$ of $M$ for all $v \in L$, iff the level subm. $A_v$ is T-ABSO primary subm. of $X_v$.

**Proposition 40.**

Let $A$ be T-ABSO primary F. subm. of $F. M.$ for all $v \in L$, iff the level subm. $A_v$ is T-ABSO primary subm. of $X_v$.

**Proof.** ($\Rightarrow$) Let $abx \in A_v$ for any $a, b \in R$ and $x \subseteq X$, then $A(abx) \supseteq v$, so $(abx)_v \subseteq A$ implies that $a_s b_t x_k \subseteq A$ where $v = \min\{s, l, k\}$. Since $A$ be T-ABSO primary F. subm., so either $a_s x_k \subseteq X - R(A)$ or $b_t x_k \subseteq X - R(A)$ or $a_s b_t \subseteq (A; R) X$.

If $a_s x_k \subseteq X - R(A)$, then $(ax)_v \subseteq X - R(A)$, so $ax \in X_v - R(A_v)$.

If $b_t x_k \subseteq X - R(A)$, then $(bx)_v \subseteq X - R(A)$, so $bx \in X_v - R(A_v)$.

Hence $ab \in (A_v; R) X_v$. Thus $A_v$ is T-ABSO primary subm. of $X_v$.

($\Leftarrow$) Let $a_s b_t x_k \subseteq A$ for F. singletons $a_s, b_t$ of R and $x_k \subseteq X, s, l, k \in L$.

hence $(abx)_v \subseteq A$ where $v = \min\{s, l, k\}$ so that $A(abx) \supseteq v$, implies $abx \in A_v$, but $A_v$ is T-ABSO primary subm. of $X_v$ so either $ax \in X_v - R(A_v)$ or $bx \in X_v - R(A_v)$ or $ab \in (A_v; R) X_v$. Since $(A_v; R) X_v = (A_v; R) X_v$, hence $ab \in (A_v; R) X_v$. Then either $(ax)_v \subseteq X - R(A)$ or $(bx)_v \subseteq X - R(A)$ or $(ab)_v \subseteq (A_v; R) X_v$, implies either $a_s x_k \subseteq X - R(A)$ or $b_t x_k \subseteq X - R(A)$ or $a_s b_t \subseteq (A_v; R) X_v$. Thus $A$ be T-ABSO primary F. subm. of $X$.

**Remark 41.**

Every T-ABSO F. subm. is T-ABSO primary F. subm., but the converse in general incorrect, for example:

Let $X: Z \to L$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z \\ 0 & \text{o.w.} \end{cases}$

It is obvious that $X$ is F. M. of Z-M. Z.

Let $A: Z \to L$ such that $A(v) = \begin{cases} v & \text{if } v \in 12Z \\ 0 & \text{o.w.} \end{cases}$ $\forall v \in L$

It is obvious that $A$ is F. subm. of $X$.

Now, $A_v = 12Z$ and $X_v = Z$ as Z-M. Note that $A_v = 12Z$ is not T-ABSO subm. since $2.2.3 \in 12Z = A_v$ but $2.2 \notin 12Z = A_v$ and $2.3 \notin 12Z = A_v$.

But $X_v - R(A_v) = Z - R(12Z) = 2Z \cap 3Z = 6Z$ where $2Z$ and $3Z$ are prime subms. of $X_v$ containing $A_v$. So that $A_v$ is T-ABSO primary subm. of $X_v$ since $2.3 = 6 \in 6Z$. Thus $A$ is not T-ABSO F. subm., but it is T-ABSO primary F. subm. of $X$. We give the concept of T-ABSO quasi primary F. subm. as follows:
**Definition 42.** A proper F. subm. $A$ of F. M. $X$ of $\mathcal{M}$ is called T-ABSO quasi primary F. subm. if $a_b x \subseteq A$ implies either $a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$ or $a_s x \subseteq X - R(A)$ or $b_l x \subseteq X - R(A)$ for each F. singleton $a_s, b_l$ of R and $x \subseteq X$, $\forall s,l,v \in L$.

The following proposition characterize T-ABSO quasi primary F. subm. in terms of its level subm.

**Proposition 43.** Let $A$ be T-ABSO quasi primary F. subm. of F. M. $X$ of $\mathcal{M}$ iff the level subm. $A_v$ is T-ABSO quasi primary subm. of $X_v$ $\forall v \in L$.

**Proof.** $(\Rightarrow)$ Let $abx \subseteq A_v$ for any $a, b \in R$ and $x \in X_v$, then $A(abx) \supseteq R$, so $(abx)_v \subseteq A$ implies that $a_s b_l \subseteq A$ where $v = \min\{s, l, k\}$. Since $A$ be a T-ABSO quasi primary F. subm., so either $a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$ or $a_s x \subseteq X - R(A)$ or $b_l x \subseteq X - R(A)$.

If $a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$ then $(ab)_v \subseteq \sqrt{A_{\mathcal{R}} X}$, so $ab \in \sqrt{A_{\mathcal{R}} X}$. Hence $cav \in \sqrt{A_{\mathcal{R}} X}$.

If $a_s x \subseteq X - R(A)$, then $(ax)_v \subseteq X - R(A)$, so $ax \in X_v - R(A_v)$.

If $b_l x \subseteq X - R(A)$, then $(bx)_v \subseteq X - R(A)$, so $bx \in X_v - R(A_v)$.

Thus $A_v$ is a T-ABSO quasi primary subm. of $X_v$.

$(\Leftarrow)$ Let $a_s b_l \subseteq A$ for F. singletons $a_s, b_l$ of R and $x_k \subseteq X$, hence $(abx)_v \subseteq A$ where $v = \min\{s, l, k\}$ so that $A(abx) \supseteq R$, implies $ab \in A_v$, but $A_v$ is T-ABSO quasi primary subm. of $X_v$, so either $ab \in \sqrt{A_{\mathcal{R}} X}$ or $ax \in X_v - R(A_v)$ or $bx \in X_v - R(A_v)$. Since $\sqrt{A_{\mathcal{R}} X} = \sqrt{A_{\mathcal{R}} X} v$, hence $ab \in \sqrt{A_{\mathcal{R}} X}$, so either $a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$ or $a_s x_k \subseteq X - R(A)$ or $b_l x_k \subseteq X - R(A)$.

Thus $A$ be T-ABSO quasi primary F. subm. of $X$.

**Theorem 44.** Let $A$ be a proper F. subm. of F. M. $X$ of $\mathcal{M}$. Then the following expressions are equivalent:

1. $A$ is T-ABSO quasi primary F. subm. of $X$;
2. For every F. singleton $a_s, b_l$ of R and $x_k \subseteq X$, hence $(abx)_v \subseteq A$ where $v = \min\{s, l, k\}$ so that $A(abx) \supseteq R$, implies $ab \in A_v$, but $A_v$ is T-ABSO quasi primary subm. of $X_v$, so either $ab \in \sqrt{A_{\mathcal{R}} X}$ or $ax \in X_v - R(A_v)$ or $bx \in X_v - R(A_v)$. Since $\sqrt{A_{\mathcal{R}} X} = \sqrt{A_{\mathcal{R}} X} v$, hence $ab \in \sqrt{A_{\mathcal{R}} X}$, so either $a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$ or $a_s x_k \subseteq X - R(A)$ or $b_l x_k \subseteq X - R(A)$.

3. For every F. singleton $a_s, b_l$ of R and $x_k \subseteq X$, hence $(abx)_v \subseteq A$ where $v = \min\{s, l, k\}$ so that $A(abx) \supseteq R$, implies $ab \in \sqrt{A_{\mathcal{R}} X}$, hence $ab \in \sqrt{A_{\mathcal{R}} X}$, so either $a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$ or $a_s x_k \subseteq X - R(A)$ or $b_l x_k \subseteq X - R(A)$.

**Proof.** $(1) \rightarrow (2)$ Assume that $A$ is T-ABSO quasi primary F. subm. of $X$, let F. singleton $a_s, b_l$ of R.

If $a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$, then $(ab)_n = a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$ for some $n \in Z^+$, hence $(A_X a_s b_l) = X$.

Now, suppose that $a_s b_l \subseteq \sqrt{A_{\mathcal{R}} X}$. Let $x_v \subseteq A(X a_s b_l)$, then $a_s b_l x_v \subseteq A$. Since $A$ is T-ABSO quasi primary F. subm., then $a_s x_v \subseteq X - R(A)$ or $b_l x_v \subseteq X - R(A)$. So that $(A_X a_s b_l) \subseteq (X - R(A):X a_s)$ or $(X - R(A):X b_l)$.

$(2) \rightarrow (3)$ By (2), we have $(A_X a_s b_l) \subseteq (X - R(A):X a_s)$ or $(X - R(A):X b_l)$.

So that $(A_X a_s b_l) \subseteq (X - R(A):X a_s)$ or $(A_X a_s b_l) \subseteq (X - R(A):X b_l)$. 

127
(3)→(1) Let \( a_s b_l x_v \subseteq A \) and \( a_s b_l \not\subseteq \sqrt{A:R X} \) for F. singletons \( a_s, b_l \) of \( R \) and \( x_v \subseteq X \), hence 
\[ (a_s b_l)^n = a_s^n b_l^n \not\subseteq (A:R X) \] for some \( n \in Z^+ \), then \((A:R a_s^n b_l^n) \neq X\). By (3), we have that \( x_v \subseteq (A: x a_s b_l) \subseteq (X - R(A): x a_s) \) or \( x_v \subseteq (A: x a_s b_l) \subseteq (X - R(A): x b_l) \). Thus \( x_v a_s \subseteq X - R(A) \) or \( x_v b_l \subseteq X - R(A) \). So that \( A \) is T-ABSO quasi primary F. subm. of \( X \).

Lemma 45.

Let \( X \) be F. M. of \( M \). Suppose that \( A \) is T-ABSO quasi primary F. subm. of \( X \) and \( a_s b_l B \subseteq A \) for F. singleton \( a_s, b_l \) of \( R \), \( \forall s, l \in L \), and F. subm. \( B \) of \( X \). If \( a_s b_l \not\subseteq \sqrt{A:R X} \), then \( a_s B \subseteq X - R(A) \) or \( b_l B \subseteq X - R(A) \).

Proof. Since \( B \subseteq (A: x a_s b_l) \) and \((A: x a_s^n b_l^n) \neq X\) for some \( n \in Z^+ \), by theorem (44), we get 
\[ B \subseteq (A: x a_s b_l) \subseteq (X - R(A): x a_s) \] or \( B \subseteq (A: x a_s b_l) \subseteq (X - R(A): x b_l) \).

Then \( a_s B \subseteq X - R(A) \) or \( b_l B \subseteq X - R(A) \).

Theorem 46.

Let \( A \) be a proper F. subm. of F. M. \( X \) of \( M \), then the following expressions are equivalent:
1- \( A \) is T-ABSO quasi primary F. subm. of \( X \);
2- For F. singleton \( a_s \) of \( R \), \( \forall s \in L \), F. ideal \( \hat{H} \) of \( R \) and F. subm. \( B \) of \( X \) with \( a_s \hat{H} B \subseteq A \), then either 
\[ a_s \hat{H} \not\subseteq \sqrt{A:R X} \] or \( a_s B \subseteq X - R(A) \) or \( \hat{H} B \subseteq X - R(A) \);
3- For F. ideals \( \hat{H}, \hat{U} \) of \( R \) and F. subm. \( B \) of \( X \) with \( \hat{H} \hat{U} B \subseteq A \), then either \( \hat{H} \hat{U} \not\subseteq \sqrt{A:R X} \) or \( \hat{H} \hat{U} \subseteq X - R(A) \) or \( \hat{U} B \subseteq X - R(A) \).

Proof. (1)→(2) Assume that \( a_s \hat{H} B \subseteq A \) with \( a_s \hat{H} \not\subseteq \sqrt{A:R X} \) and \( \hat{H} B \not\subseteq X - R(A) \). Then there exist F. singletons \( b_l, r_k \in \hat{H} \), such that \( a_s b_l \not\subseteq \sqrt{A:R X} \) and \( r_k B \not\subseteq X - R(A) \). Now, we prove that \( a_s B \subseteq X - R(A) \). Suppose that \( a_s B \not\subseteq X - R(A) \). Since \( a_s b_l B \subseteq A \), by lemma (45), we have \( b_l B \not\subseteq X - R(A) \), hence \( (b_l + r_k) B \not\subseteq X - R(A) \). By using lemma (45), we have \( a_s b_l + a_s r_k = a_s b_l + a_s r_k \not\subseteq \sqrt{A:R X} \), because \( a_s (b_l + r_k) B \subseteq A \). Since \( a_s b_l + a_s r_k \not\subseteq \sqrt{A:R X} \), and \( a_s b_l \not\subseteq \sqrt{A:R X} \), we have \( a_s r_k \not\subseteq \sqrt{A:R X} \). Since \( a_s r_k B \subseteq A \), by lemma (45), we have \( r_k B \subseteq X - R(A) \). Then \( a_s B \subseteq X - R(A) \) or \( a_s B \subseteq X - R(A) \).

(2)→(3) Suppose that \( \hat{H} \hat{U} B \subseteq A \) with \( \hat{H} \hat{U} \not\subseteq \sqrt{A:R X} \) for F. ideals \( \hat{H}, \hat{U} \) of \( R \) and F. subm. \( B \) of \( X \). Hence \( a_s \hat{U} \not\subseteq \sqrt{A:R X} \) for some F. singleton \( a_s \). Now, we prove that \( \hat{H} B \subseteq X - R(A) \) or \( \hat{U} B \subseteq X - R(A) \). Assume that \( \hat{H} B \not\subseteq X - R(A) \) and \( \hat{U} B \not\subseteq X - R(A) \). Since \( a_s \hat{U} B \subseteq A \), by (2), we have \( a_s B \subseteq X - R(A) \), then there exists \( y_h \in \hat{H} \) such that \( y_h B \not\subseteq X - R(A) \). Since the assumption \( \hat{H} B \not\subseteq X - R(A) \). Since \( \hat{y} B \subseteq X - R(A) \) or \( \hat{U} B \subseteq X - R(A) \) or \( \hat{H} B \subseteq X - R(A) \) or \( \hat{U} B \subseteq X - R(A) \).

(3)→(1) Let \( a_s b_l x_v \subseteq A \), for F. singletons \( a_s, b_l \) of \( R \) and \( x_v \subseteq X \). Put \( \hat{H} = \langle a_s \rangle \), \( \hat{U} = \langle b_l \rangle \) and \( B = \langle x_v \rangle \), then \( \hat{H} \hat{U} B \subseteq A \). By (3), we have either \( \hat{H} \hat{U} \not\subseteq \sqrt{A:R X} \) or \( \hat{H} B \subseteq X - R(A) \) or \( \hat{U} B \subseteq X - R(A) \); that is either \( a_s > < b_l > \) or \( x_v > \subseteq X - R(A) \) or \( < a_s > \subseteq X - R(A) \) or \( < b_l > \subseteq X - R(A) \) or \( < x_v > \subseteq X - R(A) \) or \( < a_s > \subseteq X - R(A) \) or \( < b_l > \subseteq X - R(A) \) or \( < x_v > \subseteq X - R(A) \).
\[ R(A) \text{ or } b_t x_v \leq X - R(A). \text{ Hence either } a_s b_t \leq \sqrt{A;R X} \text{ or } a_s x_v \leq X - R(A) \text{ or } b_t x_v \leq X - R(A). \text{ Thus } A \text{ is T-ABSO quasi primary F. subm. of } X. \]

**Theorem 47.**

Let \( X \) be a multiplication F. M. of \( M \), and \( A \) be F. subm. of \( X \). Then the following are satisfied:

1- If \( A \) is a multiplication F. M. and \( (A;R X) \) is T-ABSO quasi primary F. ideal of \( R \), then \( A \) is T-ABSO quasi primary F. subm. of \( X \).

2- If \( X \) is a finitely generated multiplication F. M. and \( A \) is T-ABSO quasi primary F. subm. of \( X \), then \( (A;R X) \) is T-ABSO quasi primary F. ideal of \( R \).

**Proof.** (1) Assume that \( X \) is a multiplication F. M., \( (A;R X) \) is T-ABSO quasi primary F. ideal of \( R \) and \( \hat{H}B \subseteq A \) for F. ideals \( \hat{H}, \ U \) of \( R \) and F. subm. \( B \) of \( X \). Since \( X \) is a multiplication F. M., we have \( B = KX \) for some F. ideal \( K \) of \( R \). Therefore, \( \hat{H} \cap \hat{U} \subseteq \sqrt{A;R X} \) or \( \hat{H} \subseteq \sqrt{A;R X} \subseteq (X - R(A);R X) \) or \( \hat{U} \subseteq \sqrt{A;R X} \subseteq (X - R(A);R X) \). Hence \( \hat{H} \subseteq \sqrt{A;R X} \) or \( \hat{U} \subseteq \sqrt{A;R X} \).

(2) Assume that \( A \) is T-ABSO quasi primary F. subm. of a finitely generated multiplication F. M. \( X \). Let F. singletons \( a_s \), \( b_t \), \( r_k \) of \( R \), such that \( a_s b_t r_k \subseteq (A;R X) \) with \( a_s b_t \not\subseteq \sqrt{A;R X} \). Hence \( a_s b_t (r_k x_v) \subseteq A \) for every F. singleton \( x_v \subseteq X \). Since \( A \) is T-ABSO quasi primary F. subm. of \( X \) and \( a_s b_t \not\subseteq \sqrt{A;R X} \), then we have \( a_s r_k x_v \subseteq X - R(A) \) or \( b_t r_k x_v \subseteq X - R(A) \) for all \( x_v \subseteq X \). Hence \( (X - R(A);X a_s r_k) \cup (X - R(A);X b_t r_k) \subseteq X \), so that \( (X - R(A);X a_s r_k) = X \) or \( (X - R(A);X b_t r_k) = X \). Then we have \( a_s r_k \subseteq (X - R(A);R X) = \sqrt{A;R X} \) or \( b_t r_k \subseteq (X - R(A);R X) = \sqrt{A;R X} \). Thus \( (A;R X) \) is T-ABSO quasi primary F. ideal of \( R \).

**Theorem 48.**

Let \( X \) be a finitely generated multiplication F. M. of \( M \). For any F. subm. \( A \) of \( X \), the following expressions are equivalent:

1- \( A \) is T-ABSO quasi primary F. subm. of \( X \);

2- \( X \)-R(\( A \)) is T-ABSO F. subm. of \( X \).

**Proof.** (1)→(2) Assume that \( A \) is T-ABSO quasi primary F. subm. of \( X \). By theorem (47) and proposition (6), then we have \( \sqrt{A;R X} = \mathbb{U} \) is a prime F. ideal of \( R \) or \( \sqrt{A;R X} = \mathbb{U}_1 \cap \mathbb{U}_2 \) where \( \mathbb{U}_1, \mathbb{U}_2 \) are distinct prime F. ideals minimal over \( (A;R X) \). If \( \sqrt{A;R X} = \mathbb{U} \), hence \( X \)-R(\( A \))=\( \mathbb{U} \) is a prime subm. by lemma (43), so that \( X \)-R(\( A \)) is T-ABSO F. subm. of \( X \).

Now, if \( \sqrt{A;R X} = \mathbb{U}_1 \cap \mathbb{U}_2 \) where \( \mathbb{U}_1, \mathbb{U}_2 \) are distinct prime F. ideals minimal over \( (A;R X) \), then we have \( X \)-R(\( A \))=\( (\mathbb{U}_1 \cap \mathbb{U}_2)X \). Since \( F \)-\( annX=(0;1;R X) \) and \( (0;1;R X) \subseteq (A;R X) \) and \( \mathbb{U}_1, \mathbb{U}_2 \) are distinct prime F. ideals minimal over \( (A;R X) \). So that \( F \)-\( annX \subseteq \mathbb{U}_1, \mathbb{U}_2 \).

Then \( X \)-R(\( A \))=\( ((\mathbb{U}_1 + F - annX) \cap (\mathbb{U}_2 + F - annX))X = \mathbb{U}_1 X \cap \mathbb{U}_2 X \) by lemma (47).
Since $\mathbb{U}_1 X$, $\mathbb{U}_2 X$ are two distinct prime F. subms., so that $X$-R(A) is T-ABSO F. subm. of $X$ by remarks and examples(16)part(1).

$\Rightarrow (1)$ Assume that $X$-R(A) is T-ABSO F. subm. of $X$. Let $a_s b_t x_v \subseteq A$, for F. singletons $a_s$, $b_t$ of $R$ and $x_v \subseteq X$. Since $A \subseteq X$-R(A), then $a_s b_t x_v \subseteq X - R(A)$. But $X$-R(A) is T-ABSO F. subm. of $X$, so that $a_s b_t \subseteq (X - R(A);_R X) = A;_R X$ or $a_s x_v \subseteq X - R(A)$ or $b_t x_v \subseteq X - R(A)$. Thus $A$ is T-ABSO quasi primary F. subm. of $X$. By combining theorem (47) and theorem (48), we get the following corollary is beneficial to determine T-ABSO quasi primary F. subm. of a finitely generated multiplication F. M.

**Corollary 49.**

For any F. subm. $A$ of a finitely generated multiplication F. M. $X$ of $\hat{M}$. Then the following expressions are equivalent:

1. $A$ is T-ABSO quasi primary F. subm. of $X$;
2. $X$-R(A) is T-ABSO F. subm. of $X$;
3. $X$-R(A) is T-ABSO primary F. subm. of $X$;
4. $X$-R(A) is T-ABSO quasi primary F. subm. of $X$;
5. $A;_R X$ is T-ABSO F. ideal of $R$;
6. $A;_R X$ is T-ABSO primary F. ideal of $R$;
7. $A;_R X$ is T-ABSO quasi primary F. ideal of $R$;
8. $(A;_R X)$ is T-ABSO quasi primary F. ideal of $R$.

**4. Conclusions**

Through our research we concluded to the concepts (prime and quasi-prime) F. subm. lead to the concept T-ABSO F. subm. we reached the concept T-ABSO F. subm. one of the most important conclusions is the theorem (20), and explain the relationship if $A$ is T-ABSO F. subm. with $(A;_R X)$ is T-ABSO F. ideal under the class of a multiplication F. M. in corollary (23). Also we concluded the relationship $X - R(A)$ with $A;_R X$ under the class of a multiplication F. M. in lemma (45), and explain the relationships $A$ is T-ABSO quasi primary F. subm. with $(A;_R X)$ is T-ABSO quasi primary F. ideal and $A$ is T-ABSO quasi primary F. subm. with $X - R(A)$ is T-ABSO F. subm. under the class of a multiplication F. M. as in theorem (47), and theorem (48).

**References**


