Results in $G$ – Metric Spaces

Fixed Points

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Abstract

In this paper, the concept of $F$ – contraction mapping on a $G$-metric space is extended with a consideration on local $F$ – contraction. As a result, two fixed point theorems were proved for $F$ – contraction on a closed ball in a complete $G$-metric space.

Keywords: $G$ – metric spaces, Local fixed points, $F$ – contractions.

1. Introduction and Preliminaries

Bapure Dhage in his PhD thesis [1992] introduced a new class of generalized metric spaces, named $D$ - metric spaces. Mustafa and Sims proved that most of the claims concerning the fundamental structures on $D$ - metric spaces are incorrect and introduced an appropriate notion of $D$ - metric space, named $G$-metric spaces. In fact, Mustafa, Sims and other authors introduced many fixed point results for self mappings in $G$ - metric spaces under certain conditions.

Actually, the method is used in the study of fixed points in metric spaces, and symmetric spaces. In this paper, a general fixed point theorem for pairs of non weakly compatible mappings in $G$ - metric space is proved. In the case of a single mapping some results. In 2012, Wardowski introduced a new concept for contraction mappings as called F-contraction by considering a class of real valued functions.

Let $\mathcal{M}$ be a nonempty set and $Y: \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be a function satisfying the following condition:

1. $Y(q, u, v) = 0$ if and only if $q = u = v$,
2. $0 < Y(q, q, u), \forall q, u \in \mathcal{M}$ with $q \neq u$,
3. $Y(q, q, u) \leq Y(q, u, v), \forall q, u, v \in \mathcal{M}$ with $u \neq v$,
4. $Y(q, u, v) = Y(q, v, u) = \ldots, \text{(symmetry in all three variables)},$
5. $Y(q, u, v) \leq Y(q, a, a) + Y(a, u, v), \forall q, u, v, a \in \mathcal{M}$.

Then the function $Y$ is called generalized metric on $\mathcal{M}$ [1] and the pair $(\mathcal{M}, Y)$ is called a $G$-metric space.

A $G$ -metric space $\mathcal{M}$ is called a symmetric [2] if $\forall q, u, v \in \mathcal{M}$

\[ Y(q, u, u) = Y(q, q, u) \]

Many results and examples about $Y$-metric space and its generalization one can found in [2-10].

Proposition 1 [5]: Let $(\mathcal{M}, Y)$ be a $G$-metric space, then the following statements are equivalent:

1. $(\mathcal{M}, Y)$ is symmetric.
2. $Y(q, u, u) \leq Y(q, u, a)$ for all $q, u, a \in \mathcal{M}$,
3. $Y(q, u, v) \leq Y(q, u, a) + Y(v, u, b)$ for all $q, u, v, a, b \in \mathcal{M}$.
The $Y$-ball with center $r_0$ and radius $\epsilon > 0$ is $B_Y(r_0, \epsilon)$ [10] is:

$$B_Y(r_0, \epsilon) = \{ s \in \mathcal{M} : Y(r_0, s, s) < \epsilon \}.$$ 

The sequence $\{r_n\}$ in a $G-$metric space $(\mathcal{M}, Y)$ is said to be
1. $Y-$convergent to $r$ if $\exists k \in N, \epsilon > 0$ for all $m, n \geq k$ such that $Y(r_n, r_n, r_m) < \epsilon$.
2. $Y-$Cauchy if $\exists k \in N, \epsilon > 0$ for all $m, n, l \geq k$ such that $Y(r_n, r_m, r_l) < \epsilon$.

A $G-$metric space $(\mathcal{M}, Y)$ is complete if every $Y-$Cauchy sequence $(\mathcal{M}, Y)$ is $Y-$convergent in $(\mathcal{M}, Y)$ [1].

**Proposition 2** [11]: Let $(\mathcal{M}, Y)$ be a $G$-metric space the following statements are equivalent
1. $\{r_n\}$ is $Y-$convergent to $r$, if and only if $Y(r_n, r_n, r) \to 0$ as $n \to \infty$,
2. Is $Y(r_n, r, r) \to 0$ as $n \to \infty$ if and only if $Y(r_n, r_m, r) \to 0$ as $m, n \to \infty$.

**Proposition 3** [6]: Let $\{q_n\}$ and $\{u_n\}$ be a sequence in a $G-$metric space $(\mathcal{M}, Y)$ if $\{r_n\}$ converges to $q$ and $\{u_n\}$ converge to $u$. Then $Y(q_n, q_n, u_n)$ converges to $Y(q, q, u)$.

The self-mapping $f$ on a $G$-metric space $(\mathcal{M}, Y)$ is $Y-$continuous at $r \in \mathcal{M}$ [9] iff every sequence $\{r_n\}_{n=1}^{\infty} \subset \mathcal{M}$, with $r_n \to r$, we have $f r_n \to f r$.

A mapping $f : \mathcal{M} \to \mathcal{M}$ is said to be $F$-contraction if there exists $\tau > 0$ such that for all $q, u, v \in \mathcal{M}$,

$$Y(f q, f u, f v) > 0,$$

$$\tau + F(Y(f q, f u, f v)) \leq F(Y(q, u, v)) \text{ for all } q, u, v \in \mathcal{M}$$

(1)

Let $D$ be the class of all functions $F : R^+ \to R$ is a mapping satisfying the following conditions:

(D1) $F$ is strictly increasing, i.e., for all $q, u, v \in R^+$ such that $q < u < v$, $F(q) < F(u) < F(v)$,

(D2) For each sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0, \infty)$, $\lim_{n \to \infty} \alpha_n = 0$ iff $\lim_{n \to \infty} F(\alpha_n) = -\infty$.

(D3) $\exists k \in [0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha_n) = 0$.

Every $F$-contraction is contractive (byD1) and then every $F-$contraction is $Y$-continuous. Clearly, (1) and (D1) implies that every $F$-contraction mapping is $Y$-continuous, since for all $q, u, v \in \mathcal{M}$, with $f q \neq f u \neq f v$, $F(f q, f u, f v) \leq F(Y(q, u, v))$.

For illustration, we give the following example.

**Example 4**

a- Consider $F_1 : (0, \infty) \to R$ as $F_1(\alpha) = \ln \alpha$. It is clear that $F_1 \in D$. Then each self mappings $f$ on a $G$-metric space $(\mathcal{M}, Y)$ is an $F_1$-contraction $\exists$ for all $q, u, v \in \mathcal{M}$, $f q \neq f u \neq f v$

$$Y(q, u, v) \leq e^{-\tau} Y(q, u, v)$$

Then for $q, u, v \in \mathcal{M}$ such that $f q \neq f u \neq f v$ the inequality $Y(q, u, v) \leq e^{-\tau} Y(q, u, v)$ holds.

Therefore, $f$ is a contraction with $h = e^{-\tau}$.

b- Let $F_2 : (0, \infty) \to R$ be $F_2(\alpha) = \alpha + \ln \alpha$. It is clear that $F_2 \in D$. Then each self mappings $f$ on a $G$-metric space $(\mathcal{M}, Y)$ satisfying (1.1) is an $F_2$-contraction such that
2. Main Results

Throughout the following $\mathcal{M}$ is a complete $G -$ metric space  $w. r. t.$ distance function $Y$. We can prove the following theorem

**Theorem 5**

Let $f: \mathcal{M} \to \mathcal{M}$ be $Y -$ continuous self-mapping, $\epsilon > 0$ and $q_0 \in \mathcal{M}$. Suppose that $\exists \, h \in [0,1), \tau > 0$, and $F \in D$. If for all $q, u, v \in \overline{B(q_0, \epsilon)} \subset \mathcal{M}$ with $Y(fq, fu, fv) > 0$ we have

$$\tau + F(Y(fq, fu, fv)) \leq F(hY(q, u, v)), \quad (2)$$

and

$$Y(q_0, f q_0, f q_0) < (1 - k) \epsilon. \quad (3)$$

Then $\exists! \, r^* \in \overline{B(q_0, \epsilon)} \ni r^* = fr^*.$

**Proof:** Suppose $q_1 \in \mathcal{M}$ such that $q_1 = f q_0, \, q_2 = f q_1$. Continuing in this way, we get $q_{n+1} = f q_n, \, \forall n \geq 0.$

Implies that $\{q_n\}$ is non-increasing sequence.

First, to prove $q_n \in \overline{B(q_0, \epsilon)}, \forall n \in \mathbb{N}$, by using mathematical induction. From (3), we get

$$Y(q_0, q_1, q_1) = Y(q_0, f q_0, f q_0) \leq (1 - k) \epsilon < \epsilon. \quad (4)$$

Hence, $q_1 \in \overline{B(q_0, \epsilon)}.$ Suppose $q_{2, \ldots, i} \in \overline{B(q_0, \epsilon)}$ for some $i \in \mathbb{N}$. Then from (2) we obtain

$$F \left( Y(q_i, q_{i+1}, q_{i+1}) \right) = F \left( Y(fq_{i-1}, f q_i, f q_i) \right) \leq F(hY(q_{i-1}, q_i, q_i)) - \tau$$

Since $F$ is strictly increasing, we get

$$\left( Y(q_i, q_{i+1}, q_{i+1}) \right) < hY(q_{i-1}, q_i, q_i) \quad \text{(5)}$$

Now,

$$Y(q_0, q_{i+1}, q_{i+1}) \leq Y(q_0, q_1, q_1) + \cdots + Y(q_i, q_{i+1}, q_{i+1})$$

$$< Y(q_0, q_1, q_1) \left[ 1 + k + \cdots + k^i \right]$$

$$\leq (1 - k) \epsilon \frac{1 - k^{i+1}}{1 - k}$$

$$< \epsilon.$$

Thus

$$q_{i+1} \in \overline{B(q_0, \epsilon)}, \quad \text{Hence} \quad r_n \in \overline{B(q_0, \epsilon)} \quad \forall n \in \mathbb{N}.$$\n
Continuing, we have

$$F \left( Y(q_n, q_{n+1}, r_{n+1}) \right) \leq F \left( Y(q_0, q_1, q_1) \right) - n\tau$$

This implies that

$$F \left( Y(q_n, q_{n+1}, q_{n+1}) \right) \leq F \left( Y(q_0, q_1, q_1) \right) - n\tau \quad \text{(6)}$$

From (6) we get

$$\lim_{n \to \infty} F \left( Y(q_n, q_{n+1}, q_{n+1}) \right) = -\infty.$$ \n
Since $F \in D$. We get

$$\lim_{n \to \infty} Y(q_n, q_{n+1}, q_{n+1}) = 0 \quad \text{(7)}$$

From (D3) there exists $p \in (0,1)$ such that

$$\lim_{n \to \infty} \left( Y(q_n, q_{n+1}, q_{n+1}) \right)^p F \left( Y(q_n, q_{n+1}, q_{n+1}) \right) = 0 \quad \text{(8)}$$
From (6) we have
\[
(Y(q_n, q_{n+1}, q_{n+1}))^F (Y(q_n, q_{n+1}, q_{n+1})) - F(Y(q_0, q_1, q_1)) 
\leq - (Y(q_n, q_{n+1}, q_{n+1}))^n \leq 0.
\] (9)

By (7), (8) and letting \( n \to \infty \), in (9) we get
\[
\lim_{n \to \infty} (n(Y(q_m, q_{n+1}, q_{n+1}))^p = 0.
\] (10)

we observe that from (10), then \( \exists n_1 \in \mathbb{N} \) \( \ni \) \( n \leq n_1 \) we have
\[
Y(q_n, q_{n+1}, q_{n+1}) \leq \frac{1}{n^p}, \quad \forall \ n \geq n_1
\] (11)

Now, \( m, n \in \mathbb{N} \) \( \ni \) \( m = n \) \( \geq n_1 \). Then, by properties of \( Y \) and (11) we obtain
\[
Y(q_n, q_m, q_m) \leq Y(q_n, q_{n+1}, q_{n+1}) + Y(q_{n+1}, q_{n+2}, q_{n+2}) + \ldots + Y(q_{m-1}, q_m, q_m)
\]
\[
= \sum_{j=n}^{m-1} Y(q_j, q_{j+1}, q_{j+1})
\]
\[
\leq \sum_{j=n}^{\infty} Y(q_j, q_{j+1}, q_{j+1})
\]
\[
\leq \sum_{j=n}^{\infty} \frac{1}{j^p}
\] (12)

The series \( \sum_{j=n}^{\infty} \frac{1}{j^p} \) is \( Y \) - convergent.

as \( n \to \infty \), from (12) we get \( \{q_n\} \) is a \( Y \)-Cauchy sequence since
\[
\lim_{n,m \to \infty} Y(q_n, q_m, q_m) = 0.
\]

By completeness of \( \mathcal{M} \), \( \exists r^* \in \overline{B(r_0, \epsilon)} \) \( \ni \) \( q_n \to r^* \) as \( n \to \infty \). Since \( f \) is \( Y \) - continuous.

Then \( q_{n+1} = f q_n \to f r^* \) as \( n \to \infty \), that is, \( r^* = f r^* \).

Hence \( r^* \) is a fixed point of \( f \). To prove uniqueness, let \( q, u \in \overline{B_l(q_0, \epsilon)} \) and \( q \neq u \) be any two fixed point of \( f \).

Then from (2) we have
\[
\tau + F(Y(fq, fu, fu)) \leq F(hY(q, u, u)),
\]
we obtain,
\[
\tau + F(Y(fq, fu, fu)) \leq F(Y(q, u, u)).
\]
which is contradiction, so, \( q = u \).

For more illustration we give the following example.

**Example 6**

Let \( \mathcal{M} = \mathbb{R}^+ \) and \( Y(q, u, v) = |q - u| + |u - v| + |q - v| \). Then \( (\mathcal{M}, Y) \) is a complete G-metric space. Define the mapping \( f: \mathcal{M} \to \mathcal{M} \) by,
\[
f(q) = \begin{cases} 
\frac{q}{4}, & r \in [0,1] \\
\frac{1}{2}, & r \in (1, \infty).
\end{cases}
\]

\( r_0 = 1, \ \epsilon = 3, \overline{B(r_0, \epsilon)} = \left[ -\frac{1}{2}, \frac{3}{2} \right] \).

If \( F(\alpha) = \ln \alpha, \ \alpha > 0 \) and \( \tau > 0 \), then \( Y(1, f1, f1) = \frac{9}{4} < \epsilon \).

If \( q, s, t \in \overline{B(r_0, \epsilon)} \) then
\[
\frac{1}{4} (|q - u| + |u - v| + |q - v|) < \frac{1}{2} (|q - u| + |u - v| + |q - v|)
\]
So,
\[
Y(fq, fu, fv) < hY(q, u, v)
\]

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Hence
\[ \tau + F \left( Y(f q, f u, f v) \right) = \tau + \ln \left( Y(f q, f u, f v) \right) \]
\[ \leq \ln h \left( Y(q, u, v) \right) \]
\[ = F(h Y(q, u, v)) \]

If \( q, u, v \in (1, \infty) \) then
\[ \left| (q - \frac{1}{2}) - \left( u - \frac{1}{2} \right) \right| + \left| (u - \frac{1}{2}) - \left( v - \frac{1}{2} \right) \right| + \left| (v - \frac{1}{2}) - \left( q - \frac{1}{2} \right) \right| = |r - u| + |u - v| + |v - q| \]
\[ \tau + |f q - f u| + |f u - f v| + |f q - f v| > |q - u| + |u - v| + |v - q| \]
So, \( \tau + F \left( Y(f q, f u, f v) \right) > F \left( Y(q, u, v) \right) \).

Then the contraction does not hold on \( \mathcal{M} \).

Now, we present two properties of \( f : \mathcal{M} \to \mathcal{M} \). We say that \( f \) satisfies the condition:

**I** - \( \omega(f q, f u, f v) \geq 1, \forall q, u, v \in \mathcal{M} \) whenever \( \omega : \mathcal{M}^3 \to \mathbb{R}_+ \), \( \omega(q, u, v) \geq 1 \).

**II** - for given a sequence \( \{q_n\} \subset \mathcal{M} \) with \( q_n \to q \in \mathcal{M} \) as \( n \to \infty \), if
\[ \omega(q_n, q_{n+1}, q_{n+1}) \geq \varphi(q_n, q_{n+1}, q_{n+1}), \forall n \in N \Rightarrow f q_n \to f q \], where \( \omega, \varphi : \mathcal{M}^3 \to \mathbb{R}_+ \) are two functions.

If \( \varphi(q, u, v) = 1 \) then (II) reduces (I).

Let \( \Delta \psi = \{ \psi : R^+ \to R^+ : \forall t_1, t_2, t_3, t_4 \in R^+, t_1t_2t_3t_4 = 0, \exists \tau > 0 \exists \psi(t_1, t_2, t_3, t_4) = \tau \}. \)

**Definition 7**

Let \( f \) be a self-mapping on a \( G \)-metric space \( (\mathcal{M}, Y) \) and \( r_0 \in \mathcal{M} \) with \( \varepsilon > 0 \). Suppose that
\[ \omega : \mathcal{M}^3 \to (0, +\infty), \varphi : \mathcal{M}^3 \to R^+ \] are two functions. We say that \( f \) is called \( \omega \)-\( \varphi \)-\( \psi \)-F-contraction on a closed ball if for all \( q, u, v \in B(q_0, \varepsilon) \subseteq \mathcal{M} \), with
\[ \omega(q, f q, f q), (u, f u, f u), (v, f v, f v)] \leq \varphi(q, u, v) \]
and
\[ Y(f q, f u, f v) > 0, \]
we have
\[ \psi[Y(q, f q, f q), Y(u, f u, f u), Y(v, f v, f v)] + F(Y(f q, f u, f v))] \leq F(h Y(q, u, v)), \] (13)
and
\[ Y(q_0, f q_0, f q_0) \leq (1 - k)\varepsilon, \] (14)
where \( 0 \leq k < 1, \psi \in \Delta \psi \) and \( F \in D \).

**Definition 8**

Let \( f : \mathcal{M} \to \mathcal{M} \) be a self-mapping and \( \omega, \varphi : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0, +\infty) \) be two functions. \( f \) is called that is \( \omega \)-admissible mapping with respect to \( \varphi \) if \( q, u, v \in \mathcal{M} \), \( \varphi(q, u, v) \leq \omega(q, u, v) \) implies that \( \psi[f q, f u, f v) \leq \omega(f q, f u, f v) \) and \( Y(f q, f u, f v) > 0 \), we have
\[ \psi[Y(q, f q, f q), Y(u, f u, f u), Y(v, f v, f v), Y(q, f q, f q), Y(u, f u, f u), Y(v, f v, f v)] + F(Y(f q, f u, f v))] \leq F(M(q, u, v)) \] (15)
where \( M(q, u, v) = \max\{ Y(q, u, v), Y(q, f q, f q), Y(u, f u, f u), Y(v, f v, f v), (Y(q, f u, f v) + Y(u, f q, f v) + Y(v, f q, f u)) \} \)
and
\[ \sum_{j=0}^{N} Y(r_0, f r_0, f r_0) \leq \forall j \in N \text{ and } \varepsilon > 0. \] (16)

\[ \psi \in \Delta \psi \text{ and } \mathcal{F} \in \mathcal{D}. \]

**Theorem 9**

Let \( f: \mathcal{M} \to \mathcal{M} \) be \( \omega - \phi \)-\( \psi \mathcal{F} \) contraction mapping on a closed ball where

(i) \( f \) is an \( \omega \)-admissible mapping with respect to \( \phi \),
(ii) there exists \( r_0 \in \mathcal{M} \) such that \( \omega(r_0, f r_0, f r_0) \geq \phi(r_0, f r_0, f r_0) \),
(iii) \( f \) is an \( \omega \)-\( \phi \)-continuous.

Then there exists a point \( r_0 \) in \( B(r_0, \varepsilon) \) such that \( f \) \( r_0 = r \)

**Proof:** Let \( r_0 \) in \( \mathcal{M} \) such that \( \omega(r_0, f r_0, f r_0) \geq \phi(r_0, f r_0, f r_0) \). For \( r_0 \in \mathcal{M} \), let us construct a sequence \( \{ r_n \}_{n=1}^{\infty} \) such that

\[ r_1 = f \ r_0, \ r_2 = f \ r_1 = f^2 \ r_0 \ 	ext{continuing this way}, r_{n+1} = f \ r_n = f^{n+1} \ r_0, \ \forall \ n \in \mathbb{N}. \]

Since, \( f \) is an \( \omega \)-admissible mapping with respect to \( \phi \), then

\[ \omega(r_0, r_1, r_1) = \omega(r_0, f r_0, f r_0) \geq \phi(r_0, f r_0, f r_0) = \phi(r_0, r_1, r_1). \]

Continuous we get

\[ \phi(r_{n-1}, f r_{n-1}, f r_n) = \phi(r_{n-1}, r_n, r_n) \leq \omega(r_{n-1}, r_n, r_n), \ \forall \ n \in \mathbb{N} \] (17)

If \( \exists n \in \mathbb{N} \ \exists Y(r_n, f r_n, f r_n) = 0 \), there is nothing to prove.

So, suppose that \( r_n \neq r_{n+1} \) with

\[ Y(f r_{n-1}, f r_n, f r_n) = Y(f r_{n-1}, f r_n, f r_n) > 0, \ \forall \ n \in \mathbb{N}. \]

First, we see that \( r_n \in B(r_0, \varepsilon), \ \forall \ n \in \mathbb{N}. \)

Since \( f \) be a \( \omega - \phi \)-\( \psi \mathcal{F} \) contraction mapping on a closed ball, we get

\[ Y(r_0, r_1, r_1) = Y(r_0, f r_0, f r_0) \leq (1 - k)\varepsilon < \varepsilon \] (18)

Thus

\[ r_1 \in B(r_0, \varepsilon). \] Suppose \( r_2, ..., r_j \in B(r_0, \varepsilon) \) for some \( j \in \mathbb{N} \), such that

\[ \psi(Y(r_{j-1}, f r_{j-1}, f r_{j-1}), Y(r_j, f r_j, f r_j), Y(r_{j-1}, f r_{j-1}, f r_{j-1}), Y(r_j, f r_{j-1}, f r_{j-1})) \]

\[ + F(Y(f r_{j-1}, f r_j, f r_j)) \leq F(h Y(r_{j-1}, r_j, r_j)). \]

This implies,

\[ \psi(Y(r_{j-1}, r_j, r_j), Y(r_j, r_{j+1}, r_{j+1}), Y(r_{j-1}, r_{j+1}, r_{j+1}), 0) \]

\[ + F(Y(f r_{j-1}, f r_j, f r_j)) \leq F(h Y(r_{j-1}, r_j, r_j)). \]

By definition of \( \psi \),

\[ (Y(r_{j-1}, r_j, r_j), Y(r_j, r_{j+1}, r_{j+1}), Y(r_{j-1}, r_{j+1}, r_{j+1}), 0) = 0, \]

So, \( \exists \tau > 0 \) such that,

\[ \psi(Y(r_{j-1}, r_j, r_j), Y(r_j, r_{j+1}, r_{j+1}), Y(r_{j-1}, r_{j+1}, r_{j+1}), 0) = \tau. \]

Therefore,

\[ F(Y(r_j, r_{j+1}, r_{j+1}) F(Y(r_j, r_{j+1}, r_{j+1})) = F(h Y(r_{j-1}, r_j, r_j)) - \tau \] (19)

To complete, we follow the same steps of the theorem (9), since \( \mathcal{M} \) is complete \( G \) - metric space there exists \( r \in B(r_0, \varepsilon) \) such that \( r_n \to r \) as \( n \to \infty \), \( f \) is an \( \omega \)-\( \phi \)-continuous and

\[ \phi(r_{n-1}, r_n, r_n) \leq \omega(r_{n-1}, r_n, r_n), \ \forall \ n \in \mathbb{N}. \]

Then
\[ r_{n+1} = f(r_n) \rightarrow f(r) \text{ as } n \rightarrow \infty. \]

That is, \( r = f(r) \) hence \( r \) is a fixed point of \( f \).

To illustrate theorem 9, we give the following example

**Example 10**

Let \( \mathcal{M} = \mathbb{R}^+ \) and \( \gamma \) be \( G \)-metric on \( \mathcal{M} \) as in Example (6) Define \( f: \mathcal{M} \rightarrow \mathcal{M} \), \( \omega: \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow [-\infty) \cup (0, +\infty), \phi: \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+ \), \( \psi: (\mathbb{R}^+)^{t} \rightarrow \mathbb{R}^+ \), and \( F: \mathbb{R} \rightarrow \mathbb{R}^+ \) by

\[
 f(r) = \begin{cases} 
 \sqrt{r}, & r \in [0,1], \\
 2r, & r \in (1, \infty) 
\end{cases}, \quad \omega(r, u, v) = \begin{cases} 
 1 & r \in [0,1] \\
 \frac{1}{2}, & \text{otherwise}
\end{cases}, \\
 \phi(r, u, v) = \frac{1}{3} \quad \text{for all } r, u, v \in \mathcal{M}, \\
 \psi(t_1, t_2, t_3, t_4) = \tau > 0 
\]

and \( F(q) = \ln q \) with \( q > 0 \).

\[
 r_0 = \frac{1}{3}, \quad \epsilon = 1, \quad B(r_0, \epsilon) = \left[ \frac{-1}{6}, \frac{5}{6} \right] 
\]

then

\[
 Y\left( \frac{1}{3}, f\frac{1}{3}, f\frac{1}{3} \right) = 0.732 < \epsilon. 
\]

If \( r, u, v \in B(r_0, \epsilon) \) then \( \omega(r, u, v) = e^{r+u+v} > \frac{1}{3} = \phi(r, u, v) \).

On the other hand, \( f(r) \in B(r_0, \epsilon) \), \( \forall r \in B(r_0, \epsilon) \).

Then, \( \omega(\gamma f r, f u, f v) \geq \phi(r, f r, f r) \) with \( Y(\gamma f r, f u, f v) = |\sqrt{r} - \sqrt{u}| + |\sqrt{u} - \sqrt{v}| + |\sqrt{v} - \sqrt{r}| > 0. \)

Clearly \( \omega(0, f 1, f 1) \geq \phi(0, f 1, f 1) \), then we have

\[
 Y(\gamma f r, f u, f v) = \left| \frac{r - u}{\sqrt{r} + \sqrt{u}} \right| + \left| \frac{r - v}{\sqrt{r} + \sqrt{v}} \right| + \left| \frac{u - v}{\sqrt{u} + \sqrt{v}} \right| < h(|r - u| + |u - v| + |v - r|) 
\]

Consequently

\[
 \tau + F(Y(f r, f u, f v)) = \tau + \ln Y(f r, f u, f v) \leq \ln h Y(r, u, v) = F(h Y(r, u, v)) 
\]

If \( r \notin B(r_0, \epsilon) \) or \( u \notin B(r_0, \epsilon) \) or \( v \notin B(r_0, \epsilon) \) then

\[
 \omega(r, u, v) = \frac{1}{5} \geq \frac{1}{3} = \phi(r, u, v) 
\]

\[
 2(|r - u| + |u - v| + |v - r|) > |r - u| + |u - v| + |r - v| \\
 |f r - f u| + |f u - f v| + |f v - f r| > |r - u| + |u - v| + |r - v| \\
 \tau + F(Y(f r, f u, f v)) \geq F(Y(r, u, v)) 
\]

The contraction does not hold.

3. **Conclusion and Open Problem**

This research focus on introducing new idea of F-contraction on a closed ball which is different from F-contraction given in [4]. Therefore a generalization of results is very useful so far as it requires the F-contraction mapping only on a closed ball rather the whole space. This new idea however guides the researcher towards further investigations and applications.
same time, it will be interesting to apply these concepts in various spaces. In future, we suggest study the results in [8] to verify the extent achieved in the setting of $F –$ contraction mappings in modular spaces.

References