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Results in $G -$ Metric SpacesFixed Points

Anaam Neamah Faraj
anaamnema1@gmail.com
Salwa Salman Abed
salwaalbundi@yahoo.com
Departmen of Mathmatics, College of Education for pure sciences, Ibn Al Haitham, University of Baghdad, Baghdad, Iraq.

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Abstract

In this paper, the concept of F −contraction mapping on a G-metric space is extended with a consideration on local $F -$ contraction. As a result, two fixed point theorems were proved for $F -$ contraction on a closed ball in a complete G-metric space.

Keywords: $G -$ metric spaces, Local fixed points, $F -$ contractions.

1.Introduction and Preliminaries

Bapure Dhage in his PhD thesis [1992] introduced a new class of generalized metric spaces, named D - metric spaces. Mustafa and Sims proved that most of the claims concerning the fundamental structures on D - metric spaces are incorrect and introduced an appropriate notion of D - metric space, named G-metric spaces. In fact, Mustafa, Sims and other authors introduced many fixed point results for self mappings in G - metric spaces under certain conditions.

Actually, the method is used in the study of fixed points in metric spaces,and symmetric spaces. In this paper, a general fixed point theorem for pairs of non weakly compatible mappings in G - metric space is proved. In the case of a single mapping some results. In 2012, Wardowski introduced a new concept for contraction mappings as called F-contraction by considering a class of real valued functions.

Let $\mathcal{M}$ be a nonempty set and $\Upsilon : \mathcal{M}\times \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be a function satisfying the following condition:

1. $\Upsilon (q,u,v) = 0$ if and only if $q = u = v$,
2. $0 < \Upsilon (q,q,u),\ \forall\, q,u \in \mathcal{M}\ with\ q \neq u,$
3. $\Upsilon (q,q,u) \leq \Upsilon (q,u,v)\ ,\forall\ q,u,v \in \mathcal{M}\ \ with\ u \neq v,$
4. $\Upsilon (q,u,v) = \Upsilon(q,v,u) = \ldots$, (symmetry in all three vairables),
5. $\Upsilon (q,u,v) \leq \Upsilon (q,a,a) + \Upsilon (a,u,v)\ ,\forall\ q,u,v,a \in \mathcal{M}.$

Then the function Υ is called generalized metric on $\mathcal{M}$ [1] and the pair $(\mathcal{M},\Upsilon)$ is called a G-metric space.

A G -metric space $\mathcal{M}$ is called a symmetric [2] if $\forall\, q,u,v \in \mathcal{M}$

$$\Upsilon\ (q,u,u) = \Upsilon\ (q,q,u)$$

Many results and examples about Υ-metric space and its generalization one can found in [2-10].

Proposition 1 [5]: Let $(\mathcal{M},\Upsilon)$ be a G-metric space, then the following statements are equivalent:

1-$(\mathcal{M},\Upsilon)$ is symmetric.

2-$\Upsilon\ (q,u,u) \leq \Upsilon (q,u,a)$ for all $q,u,a \in \mathcal{M},$

3-$\Upsilon\ (q,u,v) \leq \Upsilon (q,u,a) + \Upsilon(v,u,b)$ for all $q,u ,v,a,b \in \mathcal{M}.$

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The Υ-ball with center r_0 and radius $\epsilon > 0$ is $B_\Upsilon(r_0, \epsilon)$ [10] is:

$$B_\Upsilon(r_0,\epsilon) = \{s \in \mathcal{M} : \Upsilon(r_0, s, \text{and } s) < \epsilon\}.$$

The sequence $\{r_n\}$ in a $G-$ metric space $(\mathcal{M}, \Upsilon)$ is said to be

1- $\Upsilon-$ convergent to r if $\exists k \in N, \epsilon > 0$ for $all\ m, n \geq k$ such that $\Upsilon(r, r_n, r_m) < \epsilon$.

2- Υ – Cauchy if $\exists\ k \in N, \epsilon > 0$ for $all\ m, n, l \geq k$ such that $\Upsilon(r_n, r_m, r_l) < \epsilon$.

A G–metric space $(\mathcal{M}, \Upsilon)$ is complete if every Υ -Cauchy sequence $(\mathcal{M}, \Upsilon)$ is Υ- convergent in $(\mathcal{M}, \Upsilon)$ [1].

Proposition 2 [11]: Let $(\mathcal{M}, \Upsilon)$ be a G-metric space the following statements are equivalent

1- $\{r_n\}$ is Υ-convergent to r, if and only if $\Upsilon(r_n, r_n, r) \to 0 \quad as \quad n \to \infty$,

2- Is $\Upsilon(r_n, r, r) \to 0 \quad as \quad n \to \infty$ if and only if $\Upsilon(r_n, r_m, r) \to 0 \quad as\ m, n \to \infty$.

Proposition 3 [6]: Let $\{q_n\}$ and $\{u_n\}$ be a sequence in a G –metric space $(\mathcal{M}, \Upsilon)$ if $\{r_n\}$ converges to q and $\{u_n\}$ converge to u. Then $\Upsilon(q_n, q_n, u_n)$ converges to $\Upsilon(q, q, u)$.

The self- mapping f on a G-metric space $(\mathcal{M}, \Upsilon)$ is Υ - continuous at $r \in \mathcal{M}$ [9] iff every sequence$\{r_n\}_{n=1}^{\infty} \subset \mathcal{M}$, with $r_n \to r$, we have $f\, r_n \overset{\Upsilon}{\to} f\, r$.

A mapping $f : \mathcal{M} \to \mathcal{M}$ is said to be F-contraction if there exists $\tau > 0$ such that for all $q, u, v \in \mathcal{M}$,

$$\Upsilon(fq, fu, fv) > 0,$$

$$\tau + F(\Upsilon(fq, fu, fv)) \leq F(\Upsilon(q, u, v)). \text{ for all } q, u, v \in \mathcal{M} \tag{1}$$

Let D be the class of all functions $F: R^+ \to R$ is a mapping satisfying the following conditions:

(D1) F is strictly increasing, i.e., for all $q, u, v \in R^+$ such that $q < u < v$, $F(q) < F(u) < F(v)$,

(D2) For each sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0, \infty)$, $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

(D3) $\exists k \in [0, 1)$ such that $\lim_{\alpha\to 0+} \alpha^k F(\alpha_n) = 0$.

Every F-contraction is contractive (byD1) and then every F –contraction is Υ-continuous. Clearly, (1) and (D1) implies that every F-contraction mapping is Υ-continuous, since for all $q, u, v \in \mathcal{M}$, with$f\, q \neq fu \neq fv$, $F(f\, q, fu, fv) \leq F(\Upsilon(q, u, v))$.

For illustration, we give the following example.

Example 4

a- Consider $F_1: (0, \infty) \to R$ as $F_1(\alpha) = ln\alpha$. It is clear that$F_1 \in D$. Then each self mappings f on a G-metric space $(\mathcal{M}, \Upsilon)$ is an F_1-contraction $\ni$ for all $q, u, v \in \mathcal{M}$, $fq \neq fu \neq fv$

$$\Upsilon(fq, fu, fv) \leq e^{-\tau}\Upsilon(q, u, v)$$

Then for $q, u, v \in \mathcal{M}$ such that $fq \neq f\, u \neq fv$ the inequality $\Upsilon(fq, fu, fv) \leq e^{-\tau}\Upsilon(q, u, v)$ holds.

Therefore, f is a contraction with $h = e^{-\tau}$.

b- Let $F_2 : (0, \infty) \to R$ be $F_2(\alpha) = \alpha + ln\alpha$. It is clear that $F_2 \in D$. Then each self mappings f on a G-metric space $(\mathcal{M}, \Upsilon)$ satisfying (1.1) is an F_2-contraction such that

$$\frac{\Upsilon(fq,fu,fv)}{\Upsilon(q,u,v)}e^{\Upsilon(q,u,v)-\Upsilon(q,u,v)} \leq e^{-\tau}, \text{ for all} q,u,v \in \mathcal{M}, \; fq \neq f\,u \neq fv$$

2. Main Results

Throughout the following $\mathcal{M}$ is a complete $G-$ metric space $w.r.t.$ distance function Υ. We can prove the following theorem

Theorem 5

Let $f\colon \mathcal{M} \to \mathcal{M}$ be $\Upsilon-$ continuous self-mapping, $\epsilon > 0$ and $q_0 \in \mathcal{M}$. Suppose that $\exists\ h \in [0,1), \tau > 0$, and $F \in D$.If for all $q,u,v \in \overline{B(q_0,\epsilon)} \subset \mathcal{M}$ with $\Upsilon(fq,fu,fv) > 0 \ni$

$$\tau + F\,(\Upsilon\,(fq,fu,fv)) \leq F(h\,\Upsilon\,(q,u,v)), \tag{2}$$

and

$$\Upsilon\,(q_0,fq_0,fq_0) < (1-k)\epsilon. \tag{3}$$

Then $\exists!\ r^*$ in $\overline{B(q_0,\epsilon)} \ni r^* = fr^*$.

Proof: Suppose $q_1 \in \mathcal{M}$ such that $q_1 = fq_0$, $q_2 = fq_1$. Continuing in this way, we get

$$q_{n+1} = fq_n, \quad \forall n \geq 0.$$

Implies that $\{q_n\}$ is non-increasing sequence.

First, to prove $q_n \in \overline{B(q_0,\epsilon)}, \forall n \in \mathbb{N}$, by using mathematical induction. From (3), we get

$$\Upsilon\,(q_0,q_1,q_1) = \Upsilon\,(q_0,fq_0,fq_0) \leq (1-k)\epsilon < \epsilon. \tag{4}$$

Hence, $q_1 \in \overline{B(q_0,\epsilon)}$. Suppose $q_{2,\ldots\ldots}, q_i \in \overline{B(q_0,\epsilon)}$ for some $i \in \mathbb{N}$. Then from (2) we obtain

$$F\left(\Upsilon\,(q_i,q_{i+1},q_{i+1})\right) = F\left(\Upsilon\,(fq_{i-1},fq_i,fq_i)\right) \leq F\left(h\,\Upsilon\,(q_{i-1},q_i,q_i)\right) - \tau$$

Since F is strictly increasing, we get

$$\left(\Upsilon\,(q_i,q_{i+1},q_{i+1})\right) < h\,\Upsilon\,(q_{i-1},q_i,q_i) \tag{5}$$

Now,

$$\begin{aligned}\Upsilon\,(q_0,q_{i+1},q_{i+1}) &\leq \Upsilon\,(q_0,\ q_1,q_1) + \cdots + \Upsilon\,(q_i,q_{i+1},q_{i+1})\\ &< \ \Upsilon\,(q_0,q_1,\ q_1)[1+k+\cdots+k^i]\\ &\leq (1-k)\epsilon\frac{\left(1-k^{i+1}\right)}{1-k}\\ &< \epsilon.\end{aligned}$$

Thus

$$q_{i+1} \in \overline{B(q_0,\epsilon)}, \text{ Hence } r_n \in \overline{B(q_0,\epsilon)} \quad \forall\, n \in \mathbb{N}.$$

Continuing, we have

$$F\left(\Upsilon\,(q_n,q_{n+1},r_{n+1})\right) \leq \ F\left(\Upsilon\,(q_0,q_1,q_1)\right) - n\tau$$

This implies that

$$F\left(\Upsilon\,(q_n,q_{n+1},q_{n+1})\right) \leq F\left(\Upsilon\,(q_0,q_1,q_1)\right) - n\tau \tag{6}$$

From (6) we get

$$\lim_{n\to\infty} F\left(\Upsilon\,(q_n,q_{n+1},q_{n+1})\right) = -\infty\,.$$

Since$F \in D$. We get

$$\lim_{n\to\infty} \Upsilon\,(q_n,q_{n+1},q_{n+1}) = 0 \tag{7}$$

From (D3) there exists $p \in (0,1)$ such that

$$\lim_{n\to\infty} (\,\Upsilon\,(q_n,q_{n+1},q_{n+1}))^p\,F\left(\Upsilon\,(q_n,q_{n+1},q_{n+1})\right) = 0 \tag{8}$$

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From (6) we have

$$\left(\Upsilon\left(q_n,q_{n+1},q_{n+1}\right)\right)^p F\left(\Upsilon\left(q_n,q_{n+1},q_{n+1}\right)\right)-F\left(\Upsilon\left(q_0,q_1,q_1\right)\right)$$
$$\leq -\left(\Upsilon\left(q_n,q_{n+1},q_{n+1}\right)\right)^p n\tau \leq 0. \qquad (9)$$

By (7), (8) and letting $n\to\infty$, in (9) we get

$$\lim_{n\to\infty}\left(n\left(\Upsilon\left(q_n,q_{n+1},q_{n+1}\right)\right)^p = 0\,. \qquad (10)$$

we observe that from (10), then $\exists\, n_1 \in \mathbb{N} \ni n\left(\Upsilon(q_n,q_{n+1},q_{n+1})\right)^p \leq 1$ $,\forall\ n\geq n_1$ we have

$$\Upsilon\left(q_n,q_{n+1},q_{n+1}\right)\leq \frac{1}{n^{\frac{1}{p}}}\ ,\ \forall\, n\geq n_1 \qquad (11)$$

Now, $m,n\in\ \mathbb{N}\ni m>n\geq n_1$. Then, by properties of Υ and (11) we obtain

$$\Upsilon\left(q_n,q_m,q_m\right)\ \leq\ \Upsilon\left(q_n,q_{n+1},q_{n+1}\right)+\ \Upsilon\left(q_{n+1},q_{n+2},q_{n+2}\right)+\ \dots +\Upsilon\left(q_{m-1},q_m,q_m\right)$$
$$=\sum_{j=n}^{m-1}\Upsilon\left(q_j,q_{j+1},q_{j+1}\right)$$
$$\leq\ \sum_{j=n}^{\infty}\Upsilon\left(q_j,q_{j+1},q_{j+1}\right)$$
$$\leq\ \sum_{j=n}^{\infty}\frac{1}{j^{\frac{1}{p}}} \qquad (12)$$

The series $\sum_{j=n}^{\infty}\frac{1}{j^{\frac{1}{p}}}$ is $\Upsilon-$ convergent.

as $n\to\infty$, from (12) we get $\{q_n\}$ is a Υ-Cauchy sequence since

$$\lim_{n,m\to\infty}\Upsilon\left(q_n,q_m,q_m\right)=0\,.$$

By completeness of $\mathcal{M}$, $\exists r^*\in\overline{B(r_0,\epsilon)}\ \ni q_n\to r^*\ as\ n\to\infty$. Since f is $\Upsilon-$ continuous. Then

$,q_{n+1}=fq_n\to fr^*$ as $n\to\infty$, that is, $r^*=fr^*$.

Hence r^* is a fixed point of f. To prove uniqueness, let $q,u\in\overline{B_l(q_0,\epsilon)}$ and $q\neq u$ be any two fixed point of f. Then from (2) we have

$$\tau+F\left(\Upsilon\left(fq,fu,fu\right)\right)\leq F(h\,\Upsilon\left(q,u,u\right)),$$

we obtain,

$$\tau+F\left(\Upsilon\left(fq,fu,fu\right)\right)\leq F\left(\Upsilon\left(q,u,u\right)\right).$$

which is contradiction, so, $q=u$.

For more illustration we give the following example.

Example 6

Let $\mathcal{M}$=R^+ and $\Upsilon(q,u,v)=|q-u|\ +\ |u-v|\ +\ |q-v|$. Then $(\mathcal{M},\Upsilon)$ is a complete G-metric space. Define the mapping $f\colon\mathcal{M}\to\mathcal{M}$ by,

$$f(q)=\begin{cases}\dfrac{q}{4}, & r\in[0,1]\\ q-\dfrac{1}{2}, & r\in(1,\infty).\end{cases}$$

$r_0=1,\ \ \epsilon=3\quad ,\overline{B(r_0,\epsilon)}=\left[\frac{-1}{2},\frac{5}{2}\right]$.

If $F(\alpha)=\ln\alpha,\ \ \alpha>0$ and $\tau>0$, then $\Upsilon\left(1,f1,f1\right)\ =\frac{9}{4}\ <\epsilon$.

If $q,s,t\in\ \overline{B(r_0,\epsilon)}$ then

$$\frac{1}{4}\ \left(|q-u|\ +\ |u-v|\ +\ |q-v|\right)\ <\frac{1}{2}\left(|q-u|\ +\ |u-v|\ +\ |q-v|\right)$$

So, $\qquad \Upsilon\left(fq,fu,fv\right)<h\,\Upsilon\left(q,u,v\right)$

Hence

$$\tau + F\left(\Upsilon\left(f\,q, f\,u, f\,v\right)\right) = \tau + \ln\left(\Upsilon\left(f\,q, f\,u, f\,v\right)\right)$$
$$\leq \ln h\left(\Upsilon\left(q, u, v\right)\right)$$
$$= F(h\,\Upsilon\left(q,\ u, v\right))$$

If $q, u, v \in (1, \infty)$ then

$$\left|\left(q-\tfrac{1}{2}\right)-\left(u-\tfrac{1}{2}\right)\right|+\left|\left(u-\tfrac{1}{2}\right)-\left(v-\tfrac{1}{2}\right)\right|+\left|\left(v-\tfrac{1}{2}\right)-\left(q-\tfrac{1}{2}\right)\right| = |r-u|+|u-v|+|r-v| \qquad \tau + |f\,q - f\,u| + |f\,u - f\,v| + |f\,q - f\,v| \ > \ |q\ - u| + |u\ - v| + |q\ - v|$$

So, $\tau\ + F\left(\Upsilon\left(f\,q, f\,u, f\,v\right)\right) > F\left(\Upsilon\left(q\,, u, v\right)\right)$.

Then the contraction does not hold on $\mathcal{M}$.

Now, we present two properties of f, $f\colon \mathcal{M}\ \to\ \mathcal{M}$. We say that f satisfies the condition:

I- $\omega(fq, fu, fv) \geq 1,\ \forall\, q, u, v \in \mathcal{M}$ whenevere $\omega:\ \mathcal{M}^3 \to R^+,\ \ \omega(q, u, v) \geq 1$.

II- for given a sequence $\{q_n\} \subset \mathcal{M}$ with $q_n \to q \in \mathcal{M}$ as $n \to \infty$, if $\omega\ \ (q_n\,, q_{n+1}, q_{n+1}) \geq \varphi\ \ (q_n\,, q_{n+1}, q_{n+1}),\ \ \forall n \in N\ \Rightarrow\ f\,q_n \to fq$, where $\omega, \varphi\colon \mathcal{M}^3 \to R^+$ are two functions.

If $\varphi(q, u, v) = 1$ then (II) reduces (I).

Let $\Delta\psi = \{\,\psi\colon R^{+4}\ \to\ R^+ :\ \forall\, t_1, t_2, t_3, t_4 \in R^+,\ t_1 t_2 t_3 t_4\ = 0,\ \exists\tau > 0\ \ni\ \psi(t_1, t_2, t_3, t_4) = \tau\,\}$.

Definition 7

Let f be a self- mapping on a G-metric space $(\mathcal{M}, \Upsilon)$ and $r_0 \in\mathcal{M}$ with $\epsilon > 0$. Suppose that $\omega\colon \mathcal{M}^{\ 3} \to (0, +\infty)$, $\varphi\colon \mathcal{M}^{\ 3} \to R^+$ two functions. We say that f is called ω-φ-$\psi F-$ contraction on a closed ball if for all $q, u, v \in \overline{B(q_0, \epsilon)} \subseteq \mathcal{M}$, with

$$\omega[\,(q, fq, fq), (u, fu, fu), (v, fv, fv)] \leq\ \varphi\ (q, u, v)$$

and

$\Upsilon(fq, fu, fv) > 0$, we have

$$\psi[\,\Upsilon\ (q, fq, fq), \Upsilon\ (u, fu, fu), \Upsilon\ (v, fv, fv)] + F\ \left(\Upsilon\ (fq, fu, fv)\right)$$
$$\leq F\left(h\,\Upsilon(q, u, v)\right), \tag{13}$$

and

$$\Upsilon(q_0, fq_0, fq_0) \leq (1-k)\epsilon, \tag{14}$$

where $0 \leq k < 1, \psi \in \Delta\psi$ and $F \in D$.

Definition 8

Let $f\colon \mathcal{M}\ \to\ \mathcal{M}$ be a self –mapping and ω, φ: $\mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be two functions. f Is

called that is ω- admissible mapping with respect to φ if $q, u, v \in \mathcal{M}$, $\varphi\ (q, u, v) \leq \omega(q, u, v)$ implies that $\varphi\ (fq, fu, fv) \leq\ \ \omega(fq, fu, fv)$ and $\Upsilon(fq, fu, fv) >\ 0$, we have

$$\psi[\Upsilon\ (q, fq, fq), \Upsilon\ (u, fu, fu), \Upsilon\ (v, fv, fv), \Upsilon\ (q, fu, fv), \Upsilon\ (u, fq, fv), \Upsilon\ (v, fq, fu)] + F\left(\Upsilon\ (fq, fu, fv)\right) \leq F\ (M\ (q, u, v)) \tag{15}$$

where $M(q, u, v) = max\{\ \Upsilon\ (q, u, v), \Upsilon\ (q, fq, fq), \Upsilon\ (u, fu, fu),$

$$\Upsilon\ (v, fv, fv), \frac{\left(\Upsilon\ (q, fu, fv) +\ \Upsilon\ (u, fq, fv) +\ \Upsilon\ (v, fq, fu)\right)}{3}\ \}$$

and

$$\sum_{j=0}^{N} \Upsilon(r_0, fr_0, fr_0) \leq, \quad \forall j \in N \text{ and } \epsilon > 0. \tag{16}$$

$\psi \in \Delta\psi$ and$F \in D$.

Theorem 9

Let $f: \mathcal{M} \to \mathcal{M}$ be ω-φ- $\psi F -$ contraction mapping on a closed ball where
(i) f is an ω- admissible mapping with respect to φ,
(ii) there exists $r_0 \in \mathcal{M}$ such that $\omega(r_0, fr_0, fr_0) \geq$ φ(r_0, fr_0, fr_0) ,
(iii) f is an ω-φ- continuous.
Then there exists a point r_0 in $\overline{B(r_0,\epsilon)}$ such that $fr = r$

Proof: Let r_0 in $\mathcal{M}$ such that $\omega(r_0, fr_0, fr_0) \geq$ φ(r_0, fr_0, fr_0). For r_0 $\in\mathcal{M}$. Let us construct a sequence $\{r_n\}_{n=1}^{\infty}$ such that
$r_1 = f\, r_0$, $r_2 = f\, r_1 = f^2\, r_0$ continuing this way, $r_{n+1} = f\, r_n = f^{n+1}\, r_0$, $\forall\, n \in N$.
Since, f is an ω- admissible mapping with respect to φ, then

$$\omega\,(r_0\,, r_1, r_1) = \ \omega\,(r_0, fr_0, fr_0) \geq \ \varphi\,(r_0, fr_0, fr_0) = \varphi\,(r_0, r_1, r_1)\,.$$

Continuous we get

$$\varphi\,(r_{n-1}, f\,r_{n-1}, f\,r_{n-1}) = \varphi\,(r_{n-1}, r_n, r_n) \leq\ \omega\,(r_{n-1}, r_n, r_n)\quad, \forall\, n \in N \tag{17}$$

If $\exists\, n \in N \ni\ \Upsilon\,(r_n, f\,r_n, f\,r_n) = 0$, there is nothing to prove.
So, suppose that $r_n \neq r_{n+1}$ with

$$\Upsilon(fr_{n-1}, fr_n, fr_n) = \Upsilon(fr_{n=1}, fr_n, fr_n) > 0\,, \quad \forall\, n \in N.$$

First, we see that $r_n \in \overline{B(r_0,\epsilon)}$, $\forall\, n \in N$.
Since f be a ω-φ- $\psi F -$ contraction mapping on a closed ball, we get

$$\Upsilon(r_0, r_1, r_1) = \Upsilon(r_0, fr_0, fr_0) \leq (1-k)\epsilon < \epsilon \tag{18}$$

Thus
$r_1 \in \overline{B(r_0,\epsilon)}$. Suppose $r_2, \dots, r_j \in \overline{B(r_0,\epsilon)}$ for some $j \in N$, such that

$$\psi(\ \Upsilon\,(r_{j-1}, fr_{j-1}, fr_{j-1}), \Upsilon\,(r_j, fr_j, fr_j), \Upsilon\,(r_{j-1}, fr_j, fr_j), \Upsilon\,(r_j, fr_{j-1}, fr_{j-1}))$$
$$+F\ \ (\Upsilon\,(fr_{j-1}, fr_j, fr_j)) \leq F\,(h\,\Upsilon\,(r_{j-1}, r_j, r_j)).$$

This implies,

$$\psi(\ \Upsilon\,(r_{j-1}, r_j, r_j), \Upsilon\,(r_j, r_{j+1}, r_{j+1}), \Upsilon\,(r_{j-1}, r_{j+1}, r_{j+1}), 0)$$
$$+\,F(\,\Upsilon\,(fr_{j-1}, fr_j, fr_j)) \leq F\,(h\,\Upsilon(r_{j-1}, r_j, r_j)).$$

By definition of ψ,
$(\Upsilon\,(r_{j-1}, r_j, r_j), \Upsilon\,(r_j, r_{j+1}, r_{j+1}), \Upsilon\,(r_{j-1}, r_{j+1}, r_{j+1}), 0) = 0,$
So, $\exists\tau > 0$ such that,
$\psi\,(\Upsilon\,(r_{j-1}, r_j, r_j), \Upsilon\,(r_j, r_{j+1}, r_{j+1}), \Upsilon\,(r_{j-1}, r_{j+1}, r_{j+1}).\,0) = \ \tau.$
Therefore,

$$F\,(\Upsilon\,(r_j, r_{j+1}, r_{j+1})F\,\left(\Upsilon\,(r_j, r_{j+1}, r_{j+1})\right) = F\left(h\,\Upsilon\,(r_{j-1}, r_j, r_j)\right) - \tau \tag{19}$$

To complete, we follow the same steps of the theorem (9),since $\mathcal{M}$ is complete $G -$ metric space there exists $r \in \overline{B(r_0,\epsilon)}$ such that
$r_n \to r$ as $n \to \infty$. f is an ω-φ- continuous and

$$\varphi(r_{n-1}, r_n, r_n) \leq\ \omega(r_{n-1}, r_n, r_n), \quad \forall\, n \in N.$$

Then

$$r_{n+1} = f\,r_n \to f\,r \text{ as } n \to \infty.$$

That is, $r = f\,r$ hence r is a fixed point of f.

To illustrate theorem 9, we give the following example

Example 10

Let $\mathcal{M}=R^+$ and Υ be G −metric on $\mathcal{M}$ as in Example (6) Define $f:\mathcal{M} \to \mathcal{M}$, $\omega:\mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \{-\infty\} \cup (0,+\infty)$, $\varphi:\mathcal{M} \times \mathcal{M} \times \mathcal{M} \to R^+$, $\psi:(R^+)^4 \to R^+$, and $F:R \to R^+$ by

$$f(r) = \begin{cases} \sqrt{r}, & r \in [0,1], \\ 2r, & r \in (1,\infty) \end{cases} \qquad , \omega(r,u,v) = \begin{cases} e^{r+u+v}, & r \in [0,1] \\ \frac{1}{5} & , \text{ otherwise} \end{cases}$$

$\varphi\,(r,u,v) = \frac{1}{3}$ for all $r,u,v \in \mathcal{M}$, $\psi\,(t_1, t_2\,, t_3, t_4) = \tau > 0$

and $F(q) = \ln q$ with $q > 0$.

$$r_0 = \frac{1}{3}, \qquad \epsilon = 1\,, \qquad \overline{B(r_0,\epsilon)} = [\frac{-1}{6},\frac{5}{6}]$$

then

$$\Upsilon\,(\tfrac{1}{3}, f\tfrac{1}{3}, f\tfrac{1}{3}) = 0.732 < \epsilon.$$

If $r,u,v \in \overline{B(r_0,\epsilon)}$ then $\omega\,(r,u,v) = e^{r+u+v} > \frac{1}{3} = \varphi\,(r,u,v)$.

On the other hand, $f(r) \in \overline{B(r_0,\epsilon)}\,,\ \forall\ r \in \overline{B(r_0,\epsilon)}$.

Then, $\omega\ (fr,fu,fv) \geq \varphi\,(r,fr,fr)$ with $\Upsilon\,(f\,r,fu,fv) = |\sqrt{r}-\sqrt{u}| + |\sqrt{u}-\sqrt{v}| + |\sqrt{r}-\sqrt{v}| > 0$.

Clearly $\omega\,(0,f1,f1) \geq \varphi\,(0,f1,f1)$, then we have

$$\Upsilon(fr,fu,fv)$$
$$= \left|\frac{(\sqrt{r}-\sqrt{u})\,(\sqrt{r}+\sqrt{u})}{\sqrt{r}+\sqrt{u}}\right| + \left|\frac{(\sqrt{u}-\sqrt{v})\,(\sqrt{u}+\sqrt{v})}{\sqrt{u}+\sqrt{v}}\right| + \left|\frac{(\sqrt{r}-\sqrt{v})\,\,(\sqrt{r}+\sqrt{v})}{\sqrt{r}+\sqrt{v}}\right|$$
$$= \left|\frac{r-u}{\sqrt{r}+\sqrt{u}}\right| + \left|\frac{u-v}{\sqrt{u}+\sqrt{v}}\right| + \left|\frac{r-v}{\sqrt{r}+\sqrt{v}}\right| < h\,(|r-u| + |u-v| + |r-v|)$$

Consequently

$$\tau+F\big(\Upsilon\,(f\,r,fu,fv)\big) = \tau + \ln\Upsilon\,(f\,r,fu,fv) \leq \ln h\,\Upsilon(r\,,u,v) = F\big(h\,\Upsilon(r\,,u,v)\big).$$

If $r \notin \overline{B(r_0,\epsilon)}$ or $u \notin \overline{B(r_0,\epsilon)}$ or $v \notin \overline{B(r_0,\epsilon)}$ then

$$\omega(r\,,u,v) = \frac{1}{5} \ngeq \frac{1}{3} = \varphi\,(r\,,u,v)$$

$$2(|r-u| + |u-v| + |r-v|) > |r-u| + |u-v| + |r-v|)$$
$$|f\,r - f\,u| + |f\,u - f\,v| + |f\,r - f\,v|) > |r-u| + |u-v| + |r-v|$$
$$\tau + F\,\big(\Upsilon\,(f\,r, f\,u, f\,v)\big) \geq F\,\big(\Upsilon\,(r\,,u,v)\big)$$

The contraction does not hold.

3.Conclusion and Open Problem

This research focus on introducing new idea of F-contraction on a closed ball which is different from F-contraction given in [4]. Therefore a generalization of results is very useful so far as it requires the F-contraction mapping only on a closed ball rather the whole space. This new idea however guides the researcher towards futher investigations and applications.At the same time,it will be interesting to apply these concepts in a various spaces. In future,we suggest study the results in [8] to verify the extent achieved in the setting of $F-$ contraction mappings in modular spaces.

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