WE-Prime Submodules and WE-Semi-Prime Submodules

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Abstract
In this article, we introduce the concept of a WE-Prime submodule, as a stronger form of a weakly prime submodule. And as a generalization of WE-Prime submodule, we introduce the concept of WE-Semi-Prime submodule, which is also a stronger form of a weakly semi-prime submodule. Various basic properties of these two concepts are discussed. Furthermore, the relationships between WE-Prime submodules and weakly prime submodules and studied. On the other hand, the relation between WE-Prime submodules and WE-Semi-Prime submodules are consider. Also the relation of "WE – Sime - Prime submodules and weakly semi-prime submodules" are explained. Behind that, some characterizations of these concepts are investigated.

Keywords: weakly prime submodules, weakly semi-prime submodules, WE-Prime submodules, WE-Semi-Prime submodules.

1. Introduction
Weakly prime submodule have been introduced and studied by Hadi M. A in [1], where a proper submodule $K$ of an $R$-module $X$ is called a weakly prime, if wherever $0 \neq rx \in K$, where $r \in R, x \in X$, implies that either $x \in K$ or $r \in [K : X]$, where $[K : X] = \{a \in R : aX \subseteq K\}$. Weakly semi-prime submodule have been introduced and studied by Farzalipour F in [2], where a proper submodule $K$ of an $R$-module $X$ is called a weakly semi-prime if wherever $0 \neq r^2x \in K$, where $r \in R, x \in X$, implies that $rx \in K$. Throughout this note all rings will be commutative with identity, and all $R$-modules are left unitary. A proper submodule $K$ of an $R$-module $X$ is said to be fully invariant if $f(K) \subseteq K$ for each $f \in \text{End}(X)$ [3]. An $R$-module $M$ is called X-Injective, if for every $R$-homomorphism $g : N \rightarrow M$, and every $R$-homomorphism $f : N \rightarrow X$, there exists an $R$-homomorphism $h : X \rightarrow M$, where $N$ is an $R$-module such that $hof = g$ [5]. An $R$-module $P$ is called X-Projective if for every $R$-homomorphism $f : P \rightarrow N$ and every $R$-epimorphism $g : M \rightarrow N$, there exists an $R$-homomorphism $h : P \rightarrow M$ such that $goh = f$ [5]. An $R$-module $X$ is called a scalar module if for each $f \in \text{End}(X)$, there exists $r \in R$ such that $f(m) = rm$ for each $m \in X$ [6].

2. WE-Prime Submodules
In this section, we introduce the concept WE-Prime submodule as a stronger form of a weakly prime submodule, and established some of its basic properties, examples and characterizations.
Definition (1)
A proper submodule $K$ of an $R$-module $X$ is said to be a weakly endo-prime (for a short WE-Prime), where $E = \text{End}(X)$, if wherever, $0 \neq \psi(x) \in K$, where $\psi \in \text{End}(X)$, $x \in X$, implies that either $x \in K$ or $\psi(x) \leq K$. And an ideal $I$ of a ring $R$ is said to be a weakly endo-prime ideal (WE-Prime ideal), if $I$ is a WE-Prime as an $R$-submodule of an $R$-module $R$.

The following proposition gives relation of WE-Prime submodules and weakly prime submodules.

Proposition (2)
Every WE-Prime submodule of an $R$-module $X$ is a weakly prime submodule of $X$.

Proof
Assume that $K$ is a WE-Prime submodule of $X$, and $0 \neq rx \in K$, where $r \in R, x \in X$, with $x \notin K$ . Now, let $\psi : X \to X$ be a mapping defined by $\psi(x) = rx$ for all $x \in X$. Clearly $\psi \in \text{End}(X)$. In fact we have $0 \neq rx = \psi(x) \in K$. But $K$ is a WE-Prime submodule of $X$, and $x \notin K$, implies that $\psi(x) \leq K$, hence $rx \leq K$, so $r \in [K : X]$. Therefore $K$ is a weakly prime submodule of $X$.

The converse of Proposition (2) is not true in general, as the following example shows.

Example (3)
Let $X = Z_3 \bigoplus Z$ and $R = Z, K = \langle 0 \rangle \bigoplus 3Z$. Clearly $K$ is a weakly prime submodule of $X$, but $K$ is not WE-Prime submodule of $X$. Since we define $\psi : X \to X$ by $\psi(\bar{a}, b) = (\bar{0}, b)$ for all $(\bar{a}, b) \in X$. Clearly $\psi \in \text{End}(X)$. Now $(0, 0) \neq \psi(\bar{1}, 3) = (\bar{0}, 3) \in K$, but $(\bar{1}, 3) \notin K$ and $\psi(X) = \langle 0 \rangle \bigoplus Z \leq K$.

The converse of Proposition (2) is true in the class of cyclic $R$-modules, as the following proposition shows.

Proposition (4)
Let $X$ be a cyclic $R$-module, and $K$ is a proper submodule of $X$ such that $K$ is a weakly prime submodule of $X$. Then $K$ is a WE-Prime submodule of $X$.

Proof
Assume that $K$ is a weakly prime submodule of cyclic $R$-module $X$, where $X = Rm, m \in X$. Suppose that $0 \neq \psi(x) \in K$, where $\psi \in \text{End}(X), x \in X$ and $x \notin K$. Now, let $y \in X$, then $y = rm$ and $x = r_1m$ for some $r, r_1 \in R$. Thus, $0 \neq \psi(x) = r_1\psi(m) \in K$, but $K$ is a weakly prime submodule of $X$, then either $r_1 \in [K : X]$ or $\varphi(m) \in K$. But $r_1 \notin [K : X]$ for $x = r_1m \notin K$. Hence $\psi(m) \in K$, hence $\psi(y) = r\psi(m) \in K$. Therefore $\psi(X) \leq K$.

Corollary (5)
Let $K$ be a proper submodule of a cyclic $R$-module $X$. Then $K$ is a WE-Prime if and only if $K$ is a weakly prime submodule of $X$.

Proposition (6)
Let $X$ be a faithful $R$-module, and $K$ is a WE-Prime submodule of $X$. Then $[K : X]$ is a WE-Prime ideal of $R$. 
Proof

Since K is a WE-Prime submodule of X, then by Proposition (2.2), K is a weakly prime submodule of X. Hence by [1, Prop.2.4], we get \([K : X]\) is a weakly prime ideal of R. But R is a cyclic R-module, then by Proposition (2.4), we get \([K : X]\) is a WE-Prime ideal of R.

We need to recall the following result before we introduce the next proposition.

Lemma (7) [3]
Let N and K be two submodules of an R-module X, then

1. If \(N \subseteq K\), then \([N : X] \leq [K : X]\).
2. If \(N \subseteq K\), then \([N : X] \leq [N : K]\).

The following proposition is a characterization of a WE-Prime submodules.

Proposition (8)
Let K be a proper fully invariant submodule of an R-module X. Then K is a WE-Prime submodule of X if and only if \([K : \psi(X)] = [K : \psi(H)]\) for all \(\psi \in \text{End}(X)\) and a non-zero submodule H of X with \(K < H\).

Proof

\((\Rightarrow)\) Assume that K is a WE-Prime submodule of X, and H is a non-zero submodule of X such that \(K < H\). Let \(\psi \in \text{End}(X)\), then by Lemma (2.7)(2) we have \([K : \psi(X)] \leq [K : \psi(H)]\), since \(K < H\), then there exists \(x \in H\) and \(x \not\in K\). Now, suppose that b is a non-zero element in \([K : \psi(H)]\), then \(0 \neq b\psi(H) \subseteq K\), implies that \(0 \neq b\psi(x) \in K\), where \(x \in H \leq X\). Define \(\psi : X \to X\) by \(\psi(y) = b\psi(y)\) for all \(y \in X\), clearly \(\psi \in \text{End}(X)\), also \(0 \neq b\psi(x) = \psi(x) \in K\). But K is a WE-Prime submodule of X, and \(x \not\in K\), then \(\psi(X) \subseteq K\), implies that \(b\psi(X) \subseteq K\) and hence \(b \in [K : \psi(X)]\). Thus \([K : \psi(H)] \leq [K : \psi(X)]\), and it follows that \([K : \psi(X)] = [K : \psi(H)]\).

\((\Leftarrow)\) Assume that \(0 \neq \psi(x) \in K\), where \(x \in X\) and \(\psi \in \text{End}(X)\), and suppose that \(x \not\in K\), we want to show that \(\psi(X) \subseteq K\). Since \(x \not\in K\), then \(K < K + Rx\), where \(K + Rx\) is a non-zero submodule of X. Thus by our hypothesis, we get \([K : \psi(X)] = [K : \psi(K + Rx)]\). Since K is a fully invariant, then \(\psi(K) \subseteq K\) and \(\psi(Rx) \subseteq K\), it follows that \(\psi(K + Rx) \subseteq K\). Hence \([K : \psi(K + Rx)] = R\), therefore \(1 \in [K : \psi(K + Rx)]\), implies that \(1 \in [K : \psi(X)]\), hence \(\psi(X) \subseteq K\). Thus K is a WE-Prime submodule of X.

Proposition (9)
Let X be an R-module, and L, H are submodules of X, with H is a fully invariant submodule of X and \(H \subseteq L\). If \(\frac{L}{H}\) is a WE-Prime submodule of \(\frac{X}{H}\), then L is a WE-Prime submodule of X.

Proof

Assume that \(0 \neq \psi(x) \in L\), where \(x \in X\) and \(\psi \in \text{End}(X)\). If \(x \not\in L\), then we must show that \(\psi(X) \subseteq L\). Define \(\psi_1 : \frac{X}{H} \to \frac{X}{H}\) by \(\psi_1(x + H) = \psi(x) + H\) for all \(x \in X\). To prove that \(\varphi_1\) is well define, suppose that \(x_1 + H = x_2 + H\) where \(x_1, x_2 \in X\), then \(x_1 - x_2 \in H\), hence \(\psi(x_1 - x_2) \in \psi(H) \subseteq H\) because H is a fully invariant. It follows that \(\psi(x_1) - \psi(x_2) \in H\). Hence \(\psi(x_1) + H = \psi(x_2) + H\), implies that \(\psi_1(x_1) + H = \psi_1(x_2) + H\). Since \(0 \neq \psi(x) \in \frac{L}{H}\), and \(\psi_1(x) \in \frac{L}{H}\), then \(\psi_1(x) \in L\). Therefore \(\psi_1\) is well defined.
Let $L$, implies that $0 \neq \psi(x) + H = \psi_1(x + H) \in \frac{L}{H}$. But $\frac{L}{H}$ is a WE-Prime submodule of $\frac{X}{H}$ and $x + H \notin \frac{L}{H}$, implies that $\psi_1(\frac{x}{H}) \leq \frac{L}{H}$, thus, we have $\frac{\psi(x) + H}{H} \leq \frac{L}{H}$, it follows that $\psi(X) + H \leq L$. Thus $\psi(X) \leq L$. Hence $L$ is a WE-Prime submodule of $X$.

**Proposition (10)**

Let $L$ and $K$ are submodules of an $R$-module $X$, with $L$ is an $X$-injective, and $K$ is a WE-Prime submodule of $X$. Then either $L \leq K$ or $K \cap L$ is a WE-Prime submodule of $L$.

**Proof**

Assume that $L \nleq K$, then $K \cap L$ is a proper submodule of $L$. Now, let $0 \neq \psi(x) \in K \cap L$, where $x \in L$ and $\psi \in \text{End}(L)$. Suppose that $x \notin K \cap L$, then $x \notin K$. Now, consider the following diagram, where $i$ is the inclusion map. Since $L$ is an $X$-injective then there exists $\phi: X \to L$ such that $\phi \circ i = \psi$. Clearly $\phi \in \text{End}(X)$, but $0 \neq \psi(x) = (\phi \circ i)(x) = \phi(x) \in K$, implies that $0 = \phi(x) \in K$. But $K$ is a WE-Prime submodule of $X$ and $x \notin K$, then $\phi(X) \leq K$. Also, we have $\psi(L) = (\phi \circ i)(L) = \phi(L) \leq L$ and $\psi(L) = \phi(L) \leq \phi(X) \leq K$. Hence $\psi(L) \leq K \cap L$, it follows that $K \cap L$ is a WE-Prime submodule of $L$.

**Proposition (11)**

Let $X$ be an $R$-module and $K$, $L$ are non-trivial submodules of $X$ such that $L$ is a WE-Prime submodule of $X$ and $IK$ is a non-zero submodule of $L$ for some ideal $I$ of $R$. If $I \leq [L:X]$ then $K \leq L$.

**Proof**

Suppose that $y \in K$, since $I \nleq [L:X]$, then there exists $i \in I$ and $i \notin [L:X]$. Now, let $\psi: X \to X$ define by $\psi(x) = ix$ for all submodule $x \in X$, clearly $\psi \in \text{End}(X)$. Since $IK$ is a non-zero submodule of $L$, then $iy$ is a non-zero element in $K$. That is $0 \neq \psi(y) = iy \in IK \leq L$, implies that $0 \neq iy \in L$, but $L$ is a WE-Prime submodule of $X$, and $iX = \psi(X) \nleq L$, implies that $y \in L$. Thus $K \leq L$.

**Proposition (12)**

Let $X$ be an $R$-module and $\psi: X \to X$ be an $R$-homomorphism, and $K$ be a proper fully invariant WE-Prime submodule of $X$ with $\psi(X) \nleq K$. Then $\psi^{-1}(K)$ is a WE-Prime submodule of $X$.

**Proof**

Clearly $\psi^{-1}(K)$ is a proper submodule of $X$. Now, assume that $0 \neq \phi(x) \in \psi^{-1}(K)$ where $x \in X$, $\phi \in \text{End}(X)$. If $x \notin \psi^{-1}(K)$, then $\psi(x) \notin K$, it follows that $x \notin K$ because $K$ is a fully invariant submodule of $X$. We must prove that $\phi(X) \leq \psi^{-1}(K)$. Since $0 \neq \psi \circ \phi(x) = \psi(\phi(x)) \in K$. That is $0 \neq \psi(\phi(x)) \in K$. But $K$ is a WE-Prime submodule of $X$, and $x \notin K$, it follows that $(\psi \circ \phi)(X) \leq K$, implies that $\phi(X) \leq \psi^{-1}(K)$. Hence $\psi^{-1}(K)$ is a WE-Prime submodule of $X$.

### 3. WE-Semi-Prime Submodules

In this section, we introduce the concept of WE-Semi-Prime submodule as a generalization of a WE-Prime submodule and stronger form of a weakly semi-prime submodule and give some basic properties, examples and characterizations of this concept.
Definition (13)
A proper submodule $K$ of an $R$-module $X$ is said to be a weakly endo semi-prime submodule of $X$ (for a short WE-Semi-Prime), where $E = \text{End}(X)$, if, wherever $0 \neq \psi^2(x) \in K$, where $x \in X$ and $\psi \in \text{End}(X)$, implies that $\psi(m) \in K$. And an ideal $I$ of a ring $R$ is said to be a weakly endo semi-prime ideal of $R$, if $I$ is a weakly endo semi-prime as an $R$-submodule of $R$.

Proposition (14)
Every WE-Prime submodule of an $R$-module $X$ is a WE-Semi-Prime submodule of $X$.

Proof
Let $K$ be a WE-Prime submodule of $X$, and $0 \neq \psi^2(x) \in K$, where $x \in X$, $\psi \in \text{End}(X)$. Since $K$ is a WE-Prime submodule, and $0 \neq \psi(\psi(x)) \in K$, then either $\psi(x) \in K$ or $\psi(x) \leq K$. Thus in any case $\psi(x) \in K$. Hence $K$ is a WE-Semi-Prime submodule of $X$.

Example (15)
Let $X = \mathbb{Z}$ and $R = \mathbb{Z}$, $K = 10\mathbb{Z}$ as a $\mathbb{Z}$-module of $X$. Then $K$ is a WE-Semi-Prime but not WE-Prime submodule of $X$, since if we defined $\psi: \mathbb{Z} \rightarrow \mathbb{Z}$ by $\psi(x) = x$, $\psi \in \text{End}(X)$ and $0 \neq 2\psi(5) = 10 \in K$, but $5 \notin K$ and $\psi(\mathbb{Z}) = \mathbb{Z} \nsubseteq K = 10\mathbb{Z}$, hence $K$ is not WE-Prime submodule of $X$. But $K$ is a WE-Semi-Prime, since $0 \neq \psi^2(10) = \psi(\psi(10)) = 10 \in K$, implies that $\psi(10) = 10 \in K$.

Proposition (16)
Every WE-Semi-Prime submodule of an $R$-module $X$ is a weakly semi-prime submodule of $X$.

Proof
Let $K$ be a WE-Semi-Prime submodule of $X$, and $0 \neq r^2x \in K$, where $r \in R, x \in K$. Now, let $\psi: X \rightarrow X$ defined by $\psi(x) = rx$ for all $x \in X$, clearly $\psi \in \text{End}(X)$. Now, $0 \neq r^2x = \psi^2(x) \in K$, but $K$ is a WE-Semi-Prime submodule of $X$, implies that $\psi(x) = rx \in K$. Thus $K$ is a weakly semi-prime submodule of $X$.

Example (17)
Let $X = \mathbb{Z} \oplus \mathbb{Z}$, $R = \mathbb{Z}$, $K = \mathbb{Z} \oplus 10\mathbb{Z}$, $K$ is a weakly semi-prime submodule of $X$ but not WE-Semi-Prime: Let $r = 2 \in \mathbb{Z}$ and $x = (3,5) \in X$, then $0 \neq 2^2(3,5) = (12,20) \in K$, implies that $2(3,5) = (6,10) \in K$. To show that $K$ is not WE-Semi-Prime: Let $\psi: X \rightarrow X$ defined by $\psi(x,y) = (y,x)$ for all $x,y \in \mathbb{Z}$. Clearly $\psi \in \text{End}(X)$. Now, take $\psi(0,5) = (5,0) \notin K$ but $\psi^2(0,5) = \psi(\psi(0,5)) = \psi(5,0) = (0,5) \in K$. Hence $K$ is not WE-Semi-Prime submodule of $X$.

Proposition (18)
Let $K$ be a submodule of an $R$-module $X$ with $K = \cap_{\alpha \in \Lambda} L_{\infty}$, where each $L_{\infty}$ is a WE-Prime submodule of $X$. Then $K$ is a WE-Semi-Prime submodule of $X$.

Proof
Suppose that $0 \neq \psi^2(x) \in K$, where $x \in X$, $\psi \in \text{End}(X)$, then $0 \neq \psi^2(x) \in L_{\infty}$ for each $\alpha \in \Lambda$. But $L_{\infty}$ is a WE-Prime submodule of $X$, hence by Proposition (3.2) $L_{\infty}$ is a WE-Semi-
Prime. Thus $\psi(x) \in L_\infty$ for each $x \in \Lambda$. Therefore $\psi(x) \in \cap_{\infty \in \Lambda} L_\infty$. Hence K is a WE-Semi-Prime submodule of X.

The following proposition shows that in the class of scalar modules, weakly semi-prime submodule and WE-Semi-Prime submodules are coinciding.

**Proposition (19)**

Let X be a scalar module, and L is a proper submodule of X. Then L is a WE-Semi-Prime submodule of X, if and only if L is a weakly semi-prime submodule of X.

**Proof**

($\Rightarrow$) Follows from Proposition (3.4).

($\Leftarrow$) Suppose that L is a weakly semi-prime submodule of X, and $0 \neq \phi^2(x) \in L$, where $x \in X$ and $\phi \in \text{End}(X)$. Since X is a scalar module, then there exists $r \in R$ such that $\phi(x) = rx$ for each $x \in X$. Now, $0 \neq \phi^2(x) = \phi(\phi(x)) = \phi(rx) = r^2x \in L$. But L is a weakly semi-prime submodule of X, implies that $rx \in L$. Hence $\phi(x) \in L$. Thus L is a WE-Semi-Prime submodule of X.

The following propositions are characterizations of WE-Semi-Prime submodules.

**Proposition (20)**

Let X be an R-module, and L is a proper submodule of X. Then L is a WE-Semi-Prime submodule if and only if $0 \neq \phi^2(K) \leq L$, where K is a submodule of X and $\phi \in \text{End}(X)$, implies that $\phi(K) \leq L$.

**Proof**

($\Rightarrow$) Assume that $0 \neq \phi^2(K) \leq L$, where K is a submodule of X, $\phi \in \text{End}(X)$, implies that $0 \neq \phi^2(x) \in L$ for all $x \in K \leq X$. Since L is a WE-Semi-Prime submodule of X, then $\phi(x) \in L$ for all $x \in X$. Thus $\phi(K) \leq L$.

($\Leftarrow$) Suppose that $0 \neq \phi^2(x) \in L$, where $x \in X$, and $\phi \in \text{End}(X)$, then by hypothesis, we have $K = (x)$ is a submodule of X, and $0 \neq \phi^2(K) \in L$, implies that $\phi(K) \leq L$, it follows that $\phi(x) \in L$. Hence L is a WE-Semi-Prime submodule of X.

**Proposition (21)**

Let X be an R-module, and L is a proper submodule of X. Then L is a WE-Semi-Prime submodule of X, if and only if, wherever $0 \neq \phi^n(x) \in L$, $x \in X$, $\phi \in \text{End}(X)$, and for $n \geq 2$, implies that $\phi(x) \in L$.

**Proof**

($\Rightarrow$) Follows by induction on $n \in Z^+$. 

($\Leftarrow$) Direct from definition of WE-Semi-Prime submodule.

In the class of scalar module, we get the following characterizations of WE-Semi-Prime submodules.

**Proposition (22)**

Let X be a scalar R-module, and L be a proper submodule of X. Then the following statements are equivalent:

1. L is a WE-Semi-Prime submodule of X.
2. $[L; r^2] = [(0); r^2] \cup [L; r]$ for non-zero r in R.
3. $[L; r^2] = [(0); r^2]$ or $[(0); r^2] = [L; r^2]$ for non-zero r in R.
Proof
(1) $\Rightarrow$ (2) Since $L$ is a WE-Semi-Prime submodule of $X$, then by Proposition (3.4) $L$ is a weakly semi-prime submodule of $X$. Now, let $x \in [L:r^2]$, implies that $r^2x \in L$, either $0 \neq r^2x \in L$ or $r^2x = 0$. If $0 \neq r^2x \in L$, implies that $rx \in L$, hence $x \in [L:r]$. If $r^2x = 0$, implies that $x \in [(0):r^2]$, hence, we get $[L:r^2] \subseteq [L:r] \cup [(0):r^2]$. Clearly we have by Lemma (2.7), $[L:r] \leq [L:r^2]$, and $[(0):r^2] \leq [L:r^2]$, hence $[L:r] \cup [(0):r^2] \leq [L:r^2]$. Thus the equality holds.

(2) $\Rightarrow$ (3) Direct.

(3) $\Rightarrow$ (1) To prove first $L$ is a weakly semi-prime submodule of $X$. Suppose that $0 \neq r^2x \in L$, where $x \in X$, $r \in R$, implies that $x \in [L:r^2]$ and $x \notin [(0):r^2]$. Thus by hypothesis, we get $x \in [L:r]$, implies that $rx \in L$, hence $L$ is a weakly semi-prime submodule of $X$.

Thus by Proposition (3.7), we have $L$ is a WE-Semi-Prime submodule of $X$.

Recall that an element $x$ in $R$-module $X$ is called torsion if $0 \neq a^n(x) = \{r \in R : rx = 0\}$. The set of all torsion elements denoted by $T(X)$, which is a submodule of $X$. If $T(X) = (0)$, then $X$ is called torsion free [3].

Proposition (23)
Let $X$ is a torsion free scalar $R$-module, and $L$ be a proper submodule of $X$, such that $L$ is a WE-Semi-Prime submodule of $X$. Then $[L:I]$ is a WE-Semi-Prime submodule of $X$ for any non-zero ideal $I$ of $R$.

Proof
Since $L$ is a WE-Semi-Prime submodule of $X$, then by Proposition (3.4) $L$ is a weakly semi-prime submodule of $X$. Thus by [2, Prop.27] we get $[L:I]$ is a weakly semi-prime submodule of $X$. But $X$ is a scalar module, hence by Proposition (3.7), we have $[L:I]$ is a WE-Semi-Prime submodule of $X$.

Proposition (24)
Let $\phi: X \rightarrow X'$ be an $R$-epimorphism, and $L$ is a WE-Semi-Prime submodule of $X$ with $Ker\phi \leq L$. Then $\phi(L)$ is a WE-Semi-Prime submodule of $X'$, where $X'$ is an X-projective $R$-module.

Proof
Clearly $\phi(L)$ is a proper submodule of $X'$. Assume that $0 \neq f^2(x') \in \phi(L)$ where $x' \in X'$, and $f \in End(X')$, we prove that $f(x') \in \phi(L)$, since $\phi$ is an epimorphism, and $x' \in X'$, then there exists $x \in X$ such that $\phi(x) = x'$. Consider the following diagram since $X'$ is X-projective, then there exists a homomorphism $h$ such that $\phi oh = f$. Now, $0 \neq f'(x') = f(f(x')) \in \phi(L)$, implies that $0 \neq \phi \circ h \circ \phi \circ h(x') \in \phi(L)$, and hence $0 \neq \phi(h \circ \phi)^2(x) \in \phi(L)$. But $Ker\phi \leq L$, then $0 \neq (h \circ \phi)^2(x) \in L$. Since $L$ is a WE-Semi-Prime submodule of $X$, then $(\phi \circ h)(x)$, implies that $\phi(h \circ \phi)(x) \in \phi(L)$ hence $(\phi \circ h)(\phi(x)) \in \phi(L)$ implies that $f(x') \in \phi(L)$. Therefore $\phi(L)$ is a WE-Semi-Prime submodule of $X'$.

As a direct consequence of Proposition (3.12) we get the following corollary.

Corollary (25)
Let $L$ and $K$ be a submodule of an $R$-module $X$ with $K \leq L$, and $L$ is a WE-Semi-Prime submodule of $X$. Then $L_K$ is a WE-Semi-Prime submodule of $\frac{X}{K}$, where $\frac{X}{K}$ is an X-projective $R$-module.
Recall that an R-module $X$ is multiplication if every submodule $K$ of $X$ is of the form $K=IX$ for some ideal $I$ of $R$ [7].

**Proposition (26)**

Let $X$ be a multiplication $R$-module and $L$ is a weakly semi-prime submodule of $X$, then $L$ is a WE-Semi-Prime submodule of $X$.

**Proof**

Suppose that $0 \neq f^2(x) \in L$, where $x \in X$, $f \in \text{End}(X)$. Since $X$ is a multiplication, then by [8, Coro.1.2] there exists $s \in R$ such that $f(x) = sx$ for all $x \in X$. Hence $0 \neq f(f(x)) = s^2x \in L$. But $L$ is a weakly semi-prime, implies that $sx \in L$. Thus $f(x) \in L$, so $L$ is a WE-Semi-Prime submodule of $X$.

It is well-known every cyclic $R$-module is a multiplication [7], we get the following result.

**Corollary (27)**

Let $X$ be a cyclic $R$-module, and $L$ is a proper submodule of $X$. Then $L$ is a WE-Semi-Prime submodule if and only if $L$ is a weakly semi-prime.

We end this section by the following result.

**Proposition (28)**

Let $X$ be a faithful multiplication $R$-module, and $L$ is a proper submodule of $X$. Then $L$ is a WE-Semi-Prime submodule of $X$ if and only if $[L:X]$ is a WE-Semi-Prime ideal of $R$.

**Proof**

$(\Rightarrow)$ Since $L$ is a WE-Semi-Prime submodule of $X$, then by Proposition (3.4) $L$ is a weakly semi-prime submodule of $X$. Hence by [2, Prop.29], we have $[L:X]$ is a weakly semi-prime ideal of $R$. Therefore $[L:X]$ is a weakly semi-prime as $R$-submodule of $R$-module $R$. But $R$ is cyclic $R$-module, implies that by Corollary (27) $[L:X]$ is a WE-Semi-Prime $R$-submodule of $R$-module $R$. Hence $[L:X]$ is a WE-Semi-Prime ideal of $R$.

$(\Leftarrow)$ Since $[L:X]$ is a WE-Semi-Prime ideal of $R$, implies that $[L:X]$ is a weakly semi-prime ideal of $R$. Hence by [2, Theo.30] we have $L$ is a weakly semi-prime submodule of $X$. But $X$ is a multiplication, then by Proposition (26) $L$ is a WE-Semi-Prime submodule of $X$.

As a direct consequence of Proposition (27), we get the following result.

**Corollary (3.17)**

Let $X$ be a faithful cyclic $R$-module, and $L$ is a proper submodule of $X$. Then $L$ is a WE-Semi-Prime submodule of $X$ if and only if $[L:X]$ is a WE-Semi-Prime ideal of $R$.

**References**