WN-2-Absorbing Submodules and WNS-2-Absorbing Submodules

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Abstract
In this article, we study the concept of WN-2-Absorbing submodules and WNS-2-Absorbing submodules as generalization of weakly 2-absorbing and weakly semi 2-absorbing submodules respectively. We investigate some of basic properties, examples and characterizations of them. Also, prove, the class of WN-2-Absorbing submodules is contained in the class of WNS-2-Absorbing submodules. Moreover, many interesting results about these concepts, were proven.

Keywords: WN-2-Absorbing submodules, WNS-2-Absorbing submodules, Weakly 2-Absorbing submodules, Weakly Semi-2-Absorbing submodules.

1. Introduction
Weakly 2-absorbing submodules was introduced by Darani and Soheilinia, in 2011, where a proper submodule $B$ of an $R$-module $Y$ is called weakly 2-absorbing submodule, if whenever $0 \neq aby \in B$, with $a, b \in R, y \in Y$, implies that either $ay \in B$ or $by \in B$ or $ab \in [B:Y]$ [1]. And the concept of a weakly semi 2-absorbing submodule was introduce by Haibt and Khalaf in 2018, where a proper submodule $B$ of an $R$-module $Y$ is called a weakly semi 2-absorbing submodule, if whenever $0 \neq a^2y \in B$, with $a \in R, y \in Y$, implies that either $ay \in B$ or $a^2 \in [B:Y]$ [2].

These two concepts are generalized in this article, to WN-2-Absorbing submodules and WNS-2-Absorbing submodules, we prove that the class of WN-2-Absorbing submodules is contained in the class of WNS-2-Absorbing submodules while the converse is not true see example (3.14). Recall that a submodule $A$ of an $R$-module $Y$ is called small if for any submodule $B$ of $Y$, $Y = A + B$, implies that $A = Y$ [3]. Recall that an $R$-epimorphism $f : Y \to Y$ is called small if Ker$f$ is a small submodule of $Y$, and $f(j(M)) = j(M') = j(f(M))$ and $j(M) = f^{-1}(j(M))$ [3]. A ring $R$ is a good ring if $j(R)Y = j(Y)$, where $Y$ is an $R$-module equivalently $R$ is a good ring if $j(Y) \cap A = (A)$ for every submodule $A$ of $Y$ [3]. If $Y$ is an $R$-module and $A, B, C$ are submodules of $Y$ with $B \subseteq C$. Then $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$ [3].

Recall that an $R$-module $Y$ is regular if $R/ann(x)$ is regular ring [4]. Recall that a subset $S$ of a ring $R$ is called multiplicatively closed subset of $R$ if $1 \in S$ and $ab \in S$ for all $a, b \in S$ [5]. This note consists of two parts in the first part, we introduced the concept of WN-2-Absorbing submodule, and in the second part we introduced the concept of WNS-2-Absorbing submodule.
2. WN-2-Absorbing Submodules and Related Concept

In this part of the research, we introduce and studied the concept of WN-2-Absorbing submodules as a generalization of weakly 2-absorbing submodules.

**Definition 1**
A proper submodule $B$ of an $R$-module $Y$ is said to be WN-2-Absorbing submodules if whenever $0 \neq aby \in B$, where $a, b \in R, y \in Y$, implies that either $ay \in B + J(Y)$ or $by \in B + J(Y)$ or $ab \in [B + J(Y):Y]$, where $J(Y)$ is the Jacobsen radical of $Y$. An ideal $I$ of a ring $R$ is said to be WN-2-Absorbing ideal of $R$, if $I$ is a WN-2-Absorbing submodules of an $R$-module $R$.

**Remark 2**
Every weakly 2-absorbing submodule of an $R$-module $Y$ is WN-2-Absorbing submodules, while the converse is not true.

**Proof**
Clear. For the converse consider the following example : let $Y = \mathbb{Z}_{16}, R = \mathbb{Z}$ and $B = \langle 8 \rangle$ it is clear that $B$ is a WN-2-Absorbing submodules of $Y$ since $B + J(Y) = \langle 8 \rangle + \langle 2 \rangle = \langle 2 \rangle$. But $B$ is not weakly 2-absorbing submodule of $Y$ since, $0 \neq 2.2 \in B$, but $2.2 \notin B$ and $2.2 \notin [B:Y] = 8\mathbb{Z}$.

**Proposition 3**
Let $Y$ be an $R$-module, and $B$ a proper submodule of $Y$ with $J(Y) \subseteq B$ then $B$ is a weakly 2-absorbing submodule of $Y$ if and only if $B$ is a WN-2-Absorbing submodule of $Y$.

**Proof**
$(\Rightarrow)$ By remark (2.2).

$(\Leftarrow)$ since $J(Y) \subseteq B$ then $B + J(Y) = B$, hence proof is direct.

**Proposition 4**
Let $Y$ be an $R$-module, and $B$ a proper submodule of $Y$ with $A \subseteq B$. If $A$ is a WN-2-Absorbing submodule of $Y$ and $J(Y) \subseteq j(B)$, then $A$ is a WN-2-Absorbing submodule of $B$.

**Proof**
Let $0 \neq aby \in A$, where $a, b \in R, y \in B$, since $A$ is a WN-2-Absorbing submodule of $Y$ then either $ay \in A + J(Y)$ or $by \in A + J(Y)$ or $ab \in [A + J(Y):Y]$, but $J(Y) \subseteq j(B)$, so either $ay \in A + j(B)$ or $by \in A + j(B)$ or $ab \in [A + J(Y):Y] \subseteq [A + J(B):Y] \subseteq [A + j(B):B]$ since $B$ is a submodule of $Y$. Hence $A$ a WN-2-Absorbing submodule of $B$.

**Proposition 5**
Let $Y$ be an $R$-module, and $B$ a proper submodule of $Y$, if $B + J(Y)$ is a WN-2-Absorbing submodule of $Y$, then $B$ is a WN-2-Absorbing submodule of $Y$.

**Proof**
Since $B \subseteq B + J(Y)$, hence proof is clearly.
Remark 6

The intersection of two is a WN-2-Absorbing submodules of an R-module Y need not to be is a WN-2-Absorbing submodule. The following example explain that:
Let \( Y = \mathbb{Z}, R = \mathbb{Z}, A = 6\mathbb{Z}, B = 7\mathbb{Z}. \) Clearly A, B is a WN-2-Absorbing submodules since they are weakly 2-absorbing submodules of \( Y \) but \( A \cap B = 42\mathbb{Z} \) is not WN-2-Absorbing submodule of \( Y \) since, if \( 0 \neq 2.3.7 \in A \cap B \), but \( 2.7 \notin A \cap j(Y) \) and \( 3.7 \notin A \cap j(Y) \) and \( 2.3 \notin [A \cap j(Y) : Y] = 42\mathbb{Z}. \)

Proposition 7

Let \( Y \) be an R-module, and A, B are WN-2-Absorbing submodules of \( Y \) with \( A \subseteq j(Y) \) and \( B \subseteq j(Y) \), then \( A \cap B \) is WN-2-Absorbing submodules of \( Y \).

Proof

Let \( 0 \neq aby \in A \cap B \), with \( a, b \in R, y \in Y \), implies that \( 0 \neq aby \in A \) and \( 0 \neq aby \in B \). it follows that either \( ay \in A \) or \( by \in B \) or \( ab \in [A + j(Y) : Y] \), and either \( ay \in B \) or \( by \in A \) or \( ab \in [B + j(Y) : Y] \). But \( A \subseteq j(Y) \) and \( B \subseteq j(Y) \), then \( A + j(Y) = j(Y) \) and \( B + j(Y) = j(Y) \). Hence \( ay \in j(Y) \) or \( by \in j(Y) \) or \( ab \in \{ j(Y) : Y \} \). Thus \( A \cap B \subseteq \{ j(Y) : Y \} \), implies that \( A \cap B + j(Y) = j(Y) \) thus, we have \( ay \in A \cap j(Y) \) or \( by \in A \cap j(Y) \) or \( ab \in [A \cap B + j(Y) : Y] \). So, \( A \cap B \) is a WN-2-Absorbing submodule of \( Y \).

Proposition 8

Let \( Y \) be an R-module, over a good ring and \( A, B \) are submodules of \( Y \), \( A \nsubseteq B \) and \( j(Y) \nsubseteq A \), if \( B \) is WN-2-Absorbing submodules of \( Y \), then \( A \cap B \) is WN-2-Absorbing submodules of \( A \).

Proof

Since \( A \nsubseteq B \), then \( A \cap B \) is a proper submodule of \( A \), let \( 0 \neq aby \in A \cap B \), with \( a, b \in R, y \in Y \), then \( 0 \neq aby \in A \) and \( 0 \neq aby \in B \). Since \( B \) WN-2-Absorbing submodules of \( Y \), then either \( ay \in B + j(Y) \) or \( by \in B + j(Y) \) or \( ab \in [B + j(Y) : Y] \). That is either either \( ay \in (B + j(Y)) \cap A \) or \( by \in (B + j(Y)) \cap A \) or \( ab \in (B + j(Y)) \cap A \), hence by moduler law we have either \( ay \in A \cap B + j(A) \) or \( by \in A \cap B + j(A) \) or \( ab \in [A \cap B + j(A) : Y] \nsubseteq [A \cap B + j(A) : A] \), thus \( A \cap B \) is WN-2-Absorbing submodules of \( A \).

As a direct consequence of proposition 2.8, we get the following corollary

Corollary 9

Let \( Y \) be an R-module, over a good ring and \( A, B \) are submodules of \( Y \), \( A \nsubseteq B \) and \( A \) is a maximal submodule of \( Y \), if \( B \) is WN-2-Absorbing submodules of \( Y \), then \( A \cap B \) is WN-2-Absorbing submodules of \( A \).

Proposition 10

Let \( Y \) be an R-module, and \( A \) proper submodule of \( Y \). Then \( A \) is WN-2-Absorbing submodules of \( Y \) if and only if for each submodule \( B \) of \( Y \) with \( [A : Y] \subseteq [A : B] \) and for each \( a, b \in R \) with \( 0 \neq abB \subseteq A \), implies that either \( aB \subseteq A + j(Y) \) or \( bB \subseteq A + j(Y) \) or \( ab \in [A + j(Y) : Y] \).
Proof
Suppose that $0 \neq abB \subseteq A$ for each submodule $B$ of $Y$ and $a, b \in R$. then $0 \neq aby \in A$ for each submodule $B$ of $Y$. It follows that either $ab \in A + j(Y)$ or $b \in A + j(Y)$ or $ab \in [A + j(Y), Y]$. That is either $ab \in A + j(Y)$ or $b \in A + j(Y)$ or $ab \in [A + j(Y), Y]$. Thus A is WN-2-Absorbing submodules of $Y$.

Conversely:

Proposition 11
Let $Y$ be an $R$-module and $A$ is a proper submodule of $Y$. If $A$ is WN-2-Absorbing submodules of $Y$, then $S^{-1}A$ is WN-2-Absorbing submodules of an $S^{-1}R$-module $S^{-1}Y$, where $S$ is a multiplicatively closed subset of $R$.

Proof
Let $0 \neq \frac{r_1}{s_1} \frac{r_2}{s_2} \frac{Y}{s_3} \in S^{-1}_A$, where $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}_R$ and $\frac{Y}{s_3} \in S^{-1}Y$ with $r_1, r_2 \in R$, $s_1, s_2, s_3 \in S$, $y \in Y$. Then $0 \neq \frac{r_1 r_2 y}{t} \in S^{-1}_A$, where $t = s_1 s_2 s_3 \in S$, then there exists $t_1 \in S$ such that $0 \neq t_1 r_1 r_2 y \in A$. But A is WN-2-Absorbing submodules of $Y$, then either $t_1 r_1 y \in A + j(Y)$ or $t_1 r_2 y \in A + j(Y)$ or $t_1 r_1 r_2 \in [A + j(Y), Y]$. implies that $\frac{t_1 r_1 r_2 y}{t_1 s_1 s_2 s_3} \in S^{-1}_A + j(S^{-1}Y)$ or $\frac{t_1 r_2 y}{t_1 s_2 s_3} \in S^{-1} - A + j(S^{-1}Y)$. Thus either $\frac{r_1 y}{s_1} \in S^{-1}_A + j(S^{-1}Y)$ or $\frac{r_2 y}{s_2 s_3} \in S^{-1}_A + j(S^{-1}Y)$ or $\frac{r_1 r_2 y}{s_1 s_2 s_3} \in S^{-1}_A + j(S^{-1}Y) : Y$. Hence $S^{-1}_A$ is WN-2-Absorbing submodules of an $S^{-1}_R$-module $S^{-1}Y$.

Proposition 12
Let $h : Y \to Y'$ be a small R-epimorphism and $A$ is WN-2-Absorbing submodules of $Y$. If $A$ is WN-2-Absorbing submodules of $Y$, then $h^{-1}(A)$ is WN-2-Absorbing submodules of $Y$.

Proof
It is clear that $h(A)$ is a proper submodule of $Y'$, let $aby \in h(A)$, where $a, b \in R, y \in Y'$, then $h(y) = y$. for some $y \in Y$. thus $0 \neq abh(y) \in h(A)$, then $h(aby) = h(n)$ for some non-zero $n \in A$, since Ker $A$ it follows that $0 \neq aby \in A$, but $A$ is WN-2-Absorbing submodules of $Y$, then either $ay \in A + j(Y)$ or $by \in A + j(Y)$ or $ab \in [A + j(Y), Y]$. Thus either $ah(y) \in h(A) + h(j(Y))$ or $bh(y) \in h(A) + h(j(Y))$ or $abh(y) \subseteq h(A) + h(j(Y))$. hence $h(A)$ is WN-2-Absorbing submodules of of $Y'$.

Proposition 13
Let $h : Y \to Y'$ be a small R-epimorphism and $A$ is WN-2-Absorbing submodules of $Y$, then $h^{-1}(A)$ is WN-2-Absorbing submodules of $Y$.

Proof
Let $0 \neq aby \in h^{-1}(A)$, where $a, b \in R, y \in Y$, with $ay \notin h^{-1}(A) + j(Y)$ and by $\notin h^{-1}(A) + j(Y)$. It follows that $ah(y) \notin h(h^{-1}(A) + j(Y)) = A + j(Y')$ and $bh(y) \notin h(h^{-1}(A) + j(Y)) = A + j(Y')$. Hence $h(A)$ is WN-2-Absorbing submodules of of $Y'$. Mathematics
+ j(Y) because h is a small epimorphism. We have \( 0 \neq aby \in h^{-1}(A) \), implies that \( 0 \neq abh(y) \in A \), but A is WN-2-Absorbing submodules of \( Y \), then \( ab \in [A + j(Y) : Y] \) that is \( abY \subseteq A + j(Y) \), implies that \( abh(y) \subseteq A + j(Y) \), hence \( abY \subseteq h^{-1}(A + j(Y)) \subseteq h^{-1}(A) + j(Y) \). Thus \( ab \in [h^{-1}(A) + j(Y) : Y] \).

3. WNS-2-Absorbing Submodules and Related Concept

This section devoted to introduce and study the concept of WNS -2-Absorbing submodules as a generalization of a weakly semi 2-absorbing submodule.

Definition 14

A proper submodule \( B \) of an \( R \)-module \( Y \) is said to be a WNS-2-Absorbing submodule of \( Y \), if whenever \( 0 \neq a^2y \in B \), where \( a \in R \), \( y \in Y \), implies that either \( ay \in B + j(Y) \) or \( a^2 \in [B + j(Y) : Y] \). An ideal \( I \) of a ring \( R \) is called a WNS-2-Absorbing ideal if \( I \) is a WNS-2-Absorbing \( R \)-submodule of an \( R \)-module \( R \).

Remarks and Examples 15

1. It is clear that every weakly semi 2-absorbing submodule of an \( R \)-module \( Y \) is a WNS-2-Absorbing submodule of \( Y \) while the converse is not true.
2. In the \( Z \)-module \( Z_{16} \), the submodule \( B = \langle \bar{8} \rangle \) is a WNS-2-Absorbing submodule of \( Y \), but not weakly semi 2-absorbing since \( 0 \neq 2^2 \bar{2} \in B \), but \( 2, \bar{2} \not\in B \) and \( 2 \not\in [B : Y] \).
3. If \( Y \) be an \( R \)-module, with \( j(Y) = 0 \), then a WNS-2-Absorbing submodule of \( Y \), equivalent with a weakly semi 2-absorbing submodule of \( Y \).
4. If \( Y \) is semi simple (regular) \( R \)-module, then a WNS-2-Absorbing submodule of \( Y \) and weakly semi 2-absorbing submodule of \( Y \) are equivalent.
5. If \( Y \) is a \( R \)-module, and \( B \) a proper submodule of \( Y \), with \( j(Y) \subseteq B \). Then \( B \) is a WNS-2-Absorbing submodule of \( Y \) if and only if \( B \) is a weakly semi 2-absorbing submodule of \( Y \).
6. If \( B \) is a proper submodule of \( Y \), with \( B + j(Y) \) is a WNS-2-Absorbing submodule of \( Y \), then \( B \) is a WNS-2-Absorbing submodule of \( Y \).

Proposition 16

Let \( Y \) be an \( R \)-module and \( B \) be a proper submodule of \( Y \). Then \( B + j(Y) \) is a WNS-2-Absorbing submodule of \( Y \) if and only if for each non-zero \( a \in R [B + j(Y) : a^2y] = [B + j(Y) : ay] \) or \( a^2 \in [B + j(Y) : Y] \).

Proof

\( \Rightarrow \) Suppose that \( a^2 \notin [B + j(Y) : Y] \), and let \( c \in [B + j(Y) : a^2y] \), implies that \( 0 = a^2cy \in B + j(Y) \), but \( B + j(Y) \) is a WNS-2-Absorbing submodule of \( Y \) and \( a^2 \notin [B + j(Y) : Y] \), then \( acy \in B + j(Y) \), implies that \( c \in [B + j(Y) : ay] \). Thus \( [B + j(Y) : a^2y] \subseteq [B + j(Y) : ay] \). Clearly \( [B + j(Y) : ay] \subseteq [B + j(Y) : a^2y] \). Hence \( [B + j(Y) : a^2y] = [B + j(Y) : ay] \).

\( \Leftarrow \) let \( 0 \neq a^2y \in B + j(Y) \), where \( a \in R \), \( y \in Y \). By hypothesis, if \( [B + j(Y) : a^2y] = [B + j(Y) : ay] \) and \( 0 \neq a^2y \in B + j(Y) \), implies that \( [B + j(Y) : a^2y] = R \) implies that \( [B + j(Y) : ay] = R \), hence \( ay \in B + j(Y) \).
**Proposition 17**
Let $Y$ be an $R$-module and $A$, $B$ are submodules of $Y$, with $A$ is a subset of $B$. If $A$ is a WNS-2-Absorbing submodule of $Y$ and $j(Y) \subseteq (B)$, then $A$ is a WNS-2-Absorbing submodule of $B$.

**Proof**
Similarly as in proposition 2.4

**Proposition 18**
Let $Y$ be an $R$-module over a good ring $R$ and $A$, $B$ are proper submodules of $Y$. If $A$ is a WNS-2-Absorbing submodule of $Y$ then $A$ is a WNS-2-Absorbing submodule of $B$.

**Proof**
Let $0 \neq b^2y \in B$, for $b \in R$, $y \in B \subseteq Y$, it follows that either $by \in A + j(Y)$ or $b^2y \in (A + j(Y)) \cap B$, implies that by $by \in (A + j(Y)) \cap B$ or $b^2y \in (A + j(Y)) \cap B$ for each $y \in B$. Thus by modular law, by $by \in (A \cap B) + (j(Y) \cap B)$. But $R$ is a good ring, then $j(Y) \cap B = j(B)$ and $A \cap B$ is a proper subset of $A$, hence either by $by \in A + j(B)$ or $b^2y \in A + (B)$ for each $y \in B$. Thus either by $by \in A + (B)$ or $b^2y \in [A + (B):B]$. Hence $A$ is a WNS-2-Absorbing submodule of $B$.

**Remark 19**
The intersection of two WNS-2-Absorbing submodules of an $R$-module $Y$ is not necessary WNS-2-Absorbing submodules of $Y$.
The following example explain that:

**Proposition 20**
Let $Y$ be an $R$-module, and $A$, $B$ are proper submodules of $Y$ with $j(Y) \subseteq A$, or $j(Y) \subseteq B$, if $A$ and $B$ are WNS-2-Absorbing submodules of $Y$, then $A \cap B$ is a WNS-2-Absorbing submodule of $Y$.

**Proof**
Let $0 \neq r^2y \in A \cap B$, where $r \in R$, $y \in Y$, then $0 \neq r^2y \in A$ and $0 \neq r^2y \in B$, but both $A$ and $B$ are WNS-2-Absorbing submodules of $Y$ then either $ry \in A + j(Y)$ or $r^2y \in (A + j(Y)) \cap B$, implies that $ry \in A + j(Y) \cap B$ or $r^2y \in (A + j(Y)) \cap B$, it follows that either $ry \in A \cap B + j(Y)$ or $r^2y \in (A \cap B + j(Y)) \cap B$, that is $ry \in A \cap B + j(Y)$ or $r^2y \in (A \cap B + j(Y)) : Y$. Hence $A \cap B$ is a WNS-2-Absorbing submodule of $Y$. Similarly if $j(Y) \subseteq A$, we get $A \cap B$ is a WNS-2-Absorbing submodule of $Y$.

**Proposition 21**
Let $Y$ be an $R$-module, and $A$ is a proper submodule of $Y$ with $j(Y) \subseteq A$, if $A$ is a WNS-2-Absorbing submodules of $Y$, then $[A : Y]$ is a WNS-2-Absorbing deal of $R$. 
Proof
Since \( A \) is a WNS-2-Absorbing submodules of \( Y \), and \( j(Y) \subseteq A \), then by remarks and examples 15 (5), \( A \) a weakly semi 2-absorbing submodule of \( Y \), then by [2,prop. 4], [ \( A:Y \) ] is a weakly semi 2- absorbing ideal of \( R \), so by (1) [ \( A:Y \) ] is a WNS-2- Absorbing ideal of \( R \).

Proposition 22
Let \( Y \) be a cyclic \( R \)- module ,and \( A \) is a proper submodule of \( Y \) if [ \( A:Y \) ] is a WNS -2- Absorbing ideal of \( R \) with \( j(R) \subseteq [ A:Y ] \), then \( A \) is a WNS-2-Absorbing submodules of \( Y \).

Proof
Follows by remarks and examples 15(5)(1) and corollary [2, coro. 2.5].

Proposition 23
Let \( g:Y \rightarrow Y' \) be small \( R \)-epimorphism and \( A \) proper submodules of \( Y \), with \( \text{Ker}g \subseteq A \). If \( A \) is a WNS-2- Absorbing submodules of \( Y \), then \( g(A) \) is a WNS- 2- Absorbing submodules of \( Y' \).

Proof
Similarly, as in proposition 2.12.

Proposition 24
Let \( g:Y \rightarrow Y' \) be small \( R \)-epimorphism and \( A \) proper submodules of \( Y' \). If \( A' \) is a WNS-2 - Absorbing submodules of \( Y' \), then \( g^{-1}(A') \) is a WNS- 2- Absorbing submodules of \( Y \).

Proof
Similarly, as in proposition 2.13.

Proposition 25
Let \( Y \) be an \( R \)- module , and \( A \) is a proper submodule of \( Y \) if \( A \) is a WNS-2- Absorbing submodules of \( Y \), then \( S^{-1}A \) is is a WNS-2-Absorbing submodules of \( S^{-1}R \)- module \( S^{-1}Y \).

Proof
Similarly as in proposition 2.11

Proposition 26
Let \( Y \) be an \( R \)- module and \( A \) is a WN-2- Absorbing submodules of \( Y \),then \( A \) is a WNS- 2- Absorbing submodules of \( Y \).

Proof
Let \( 0 \neq a^2y \in A \), where \( a \in R \), \( y \in Y \) that is \( 0 \neq a,a y \in A \). Since \( A \) is a WN-2-Absorbing. Then either \( ay \in A + j(Y) \) or \( a^2 \in [A + j(Y):Y] \). Thus \( A \) is WNS-2-Absorbing submodules of \( Y \).

The converse of proposition 3.13 is not true. In general as the following examples shows that:
**Example 27**  
Let $Y = Z \oplus Z$, $R = R$, $A = 15Z \oplus (0)$, $A$ is WNS-2-Absorbing submodules of $Y$, but not is WN-2-Absorbing. Since $0 \neq 3.5(1,0) \in A$, but $3(1,0) \notin A + j(Y)$ and $5(1,0) \notin A + j(Y)$ and $3.5 \notin [A + j(Y) : Y] = (0)$.

**References**