Abstract

The objective of this paper is, first, study a new collection of sets such as $\delta$–field and we discuss the properties of this collection. Second, introduce a new concepts related to the $\delta$–field such as measure on $\delta$–field, outer measure on $\delta$–field and we obtain some important results deals with these concepts. Third, introduce the concept of null-additive on $\delta$–field as a generalization of the concept of measure on $\delta$–field. Furthermore, we establish new concept related to $\delta$- field noted by weakly null-additive on $\delta$–field as a generalizations of the concepts of measure on and null-additive. Finally, we introduce the restriction of a set function $\Psi$ on $\delta$–field and many of its properties and characterizations are given.

Keywords: $\sigma$–field, measure on $\sigma$–field, monotone measure, null-additive.

1. Introduction

The theory of measure is an important subject in mathematics. In 1972, Robret [1], discusses many details about measure and proves some important results in measure theory. The notion of $\sigma$–field was studied by Robret and Dietmar, where $\mathbb{K}$ be a nonempty set. A collection $\mathcal{G}$ is said to $\sigma$–field iff $\mathbb{K}\in\mathcal{G}$ and $\mathcal{G}$ is closed under complementation and countable union [1, 2]. Zhenyuan and George in 2009 and Junhi, Radko and Endre in 2014 are used the concept of null-additive on $\sigma$–field, where $\mathcal{G}$ be a $\sigma$–field, then a set function $\Psi:\mathcal{G}\to[-\infty,\infty]$ is called null-additive on $\mathcal{G}$ if $A, B$ are disjoint sets in $\mathcal{G}$ and $\Psi(B) = 0$, then $\Psi(A\cup B) = \Psi(A)$ [3,4]. In 2016, Juha used the concept of $\sigma$–field to define measure, where $\mathcal{G}$ be a $\sigma$–field, then a measure on $\mathcal{G}$ is a set function $\Psi:\mathcal{G}\to[0,\infty]$ such that $\Psi(\emptyset) = 0$ and if $A_1, A_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{G}$, then $\Psi(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Psi(A_n)$ [5]. and also used power set to define outer measure, where $\mathbb{K}$ be a non-empty set, then a set function $\Psi:\mathcal{P}(\mathbb{K})\to[0,\infty]$ is called outer measure, if $\Psi(\emptyset) = 0$ and if $A, B \subseteq \mathbb{K}$ such that $A \subset B$, then $\Psi(A) \leq \Psi(B)$ and if $A_1, A_2, \ldots$ are subsets of $\mathbb{K}$, then $\Psi(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \Psi(A_n)$ [5]. The concept of monotone measure was studied by Peipe, Minhao and Jun in 2018, where $\mathcal{G}$ be a $\sigma$–field, then a set function $\Psi:\mathcal{G}\to[0,\infty]$ is called monotone measure, if $\Psi(\emptyset) = 0$ and if $A, B\in\mathcal{G}$ such that $A \subset B$, then $\Psi(A) \leq \Psi(B)$ [6].
The main aim of this paper is to introduce and study new concepts such as $\delta$–field, measure on $\delta$–field, outer measure on $\delta$–field and null-additive on $\delta$–field and we give basic properties, characterizations and examples of these concepts.

2. The Main Results

Let $\mathcal{X}$ be a nonempty set. Then a collection of all subsets of a set $\mathcal{X}$, denoted by $\mathcal{P}(\mathcal{X})$, and it’s called a power set of $\mathcal{X}$.

Definition 1

Let $\mathcal{X}$ be a nonempty set. A collection $\mathcal{F} \subseteq \mathcal{P}(\mathcal{X})$ is said to be $\delta$–field of a set $\mathcal{X}$ if the following conditions are satisfied:

1. $\Phi \in \mathcal{F}$.
2. If $A$ is a nonempty set in $\mathcal{F}$ and $A \subset B \subseteq \mathcal{X}$, then $B \in \mathcal{F}$.
3. If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Proposition 2

For any $\delta$–field $\mathcal{F}$ of a set $\mathcal{X}$, the following hold:

1. $\mathcal{X} \in \mathcal{F}$.
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
3. If $A_1, A_2, \ldots, A_n \in \mathcal{F}$, then $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$.
4. If $A_1, A_2, \ldots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$.
5. If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Proof

It is easy, so we omitted.

Example 3

Let $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\Phi, \{1,2\}, \{1,2,3\}, \{1,2,4\}, \mathcal{X}\}$. Then $\mathcal{F}$ is a $\delta$–field of a set $\mathcal{X}$.

Definition 4

Let $\mathcal{X}$ be a nonempty set and $\mathcal{F}$ is a $\delta$–field of a set $\mathcal{X}$. Then a pair $(\mathcal{X}, \mathcal{F})$ is called measurable space and any member of $\mathcal{F}$ is called a measurable set.

Proposition 5

Let $(\mathcal{F}_i)_{i \in I}$ be a sequence of $\delta$–field of a set $\mathcal{X}$. Then $\bigcap_{i \in I} \mathcal{F}_i$ is a $\delta$–field of a set $\mathcal{X}$.

Proof

Since $\mathcal{F}_i$ is $\delta$–field $\forall i \in I$, then $\Phi, \mathcal{X} \in \mathcal{F}_i \forall i \in I$, hence $\mathcal{F}_i \neq \Phi \forall i \in I$ and $\bigcap_{i \in I} \mathcal{F}_i \neq \Phi$, therefore $\Phi, \mathcal{X} \in \bigcap_{i \in I} \mathcal{F}_i$. Let $A \in \bigcap_{i \in I} \mathcal{F}_i$ such that $\Phi \neq A \subset B \subset \mathcal{X}$, then $A \in \mathcal{F}_i \forall i \in I$, but $A \subset B$. So, we get $B \in \mathcal{F}_i \forall i \in I$, hence $B \in \bigcap_{i \in I} \mathcal{F}_i$. Let $A_1, A_2, \ldots \in \bigcap_{i \in I} \mathcal{F}_i$. Then $A_1, A_2, \ldots \in \mathcal{F}_i$, $\forall i \in I$ and $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}_i$, $\forall i \in I$ which is implies that $\bigcap_{i=1}^{\infty} A_i \in \bigcap_{i \in I} \mathcal{F}_i$. Hence $\bigcap_{i \in I} \mathcal{F}_i$ is a $\delta$–field.

Definition 6

Let $\mathcal{F}$ be a $\delta$–field of a set $\mathcal{X}$ and let $K$ be a non-empty subset of $\mathcal{X}$. Then the restriction of $\mathcal{F}$ on $K$ is denoted by $\mathcal{F}|K$ and define as:

$\mathcal{F}|K = \{B: B = A \cap K, \text{ for some } A \in \mathcal{F}\}$.

Proposition 7

Let $\mathcal{F}$ be a $\delta$–field of a set $\mathcal{X}$ and $K$ be a non-empty subset of $\mathcal{X}$ such that $K \in \mathcal{F}$. Then $\mathcal{F}|K = \{A \subseteq K: A \in \mathcal{F}\}$. 

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Proof

Let \( B \in \mathcal{A} \mid K \). Then \( B = A \cap K \), for some \( A \in \mathcal{A} \), hence \( B \in \mathcal{A} \). Therefore \( B \in \{ A \subseteq K : A \in \mathcal{A} \} \) and \( \mathcal{A} \mid K \subseteq \{ A \subseteq K : A \in \mathcal{A} \} \). Let \( C \in \{ A \subseteq K : A \in \mathcal{A} \} \). Then \( C \subseteq K \) and \( C \in \mathcal{A} \), hence \( C = C \cap K \), but \( \mathcal{A} \mid K \), then \( C \in \mathcal{A} \) which implies that \( \{ A \subseteq K : A \in \mathcal{A} \} \in \mathcal{A} \mid K \), therefore \( \mathcal{A} \mid K = \{ A \subseteq K : A \in \mathcal{A} \} \).

Corollary 8

Let \( \mathcal{A} \) be a \( \delta \)–field of a set \( \mathcal{K} \) and \( K \) a non-empty subset of \( \mathcal{K} \) such that \( K \in \mathcal{A} \). Then \( \mathcal{A} \mid K \subseteq \mathcal{A} \).

Proof

From Proposition 7, we have \( \mathcal{A} \mid K = \{ A \subseteq K : A \in \mathcal{A} \} \). Now, for any \( B \in \mathcal{A} \mid K \), then \( B \in \{ A \subseteq K : A \in \mathcal{A} \} \). Hence \( B \subseteq K \) and \( B \in \mathcal{A} \), therefore \( \mathcal{A} \mid K \subseteq \mathcal{A} \).

Proposition 9

Let \( \mathcal{A} \) be a \( \delta \)–field of a set \( \mathcal{K} \) and let \( K \) be a non-empty subset of \( \mathcal{K} \) such that \( K \in \mathcal{A} \). Then \( \mathcal{A} \mid K \) is a \( \delta \)–field of a set \( K \).

Proof

Since \( \mathcal{A} \) is a \( \delta \)–field of \( \mathcal{K} \), then \( \Phi \subseteq \mathcal{K} \). Since \( K \subseteq \mathcal{K} \), then \( \mathcal{A} = \mathcal{K} \cap \Phi \) and \( \mathcal{A} \mid K \subseteq \mathcal{A} \). Since \( \mathcal{A} = \Phi \cap K \), then \( \Phi \subseteq \mathcal{A} \mid K \). Let \( B \in \mathcal{A} \mid K \) such that \( \Phi \not\subseteq B \subseteq D \subseteq K \) Then \( B \in \mathcal{A} \). But \( B \subseteq D \subseteq K \subseteq \mathcal{K} \) and \( \mathcal{A} \) is a \( \delta \)–field of a set \( \mathcal{K} \), then \( D \in \mathcal{A} \). Now, \( D \subseteq K \) and \( D \in \mathcal{A} \), then \( D \in \mathcal{A} \mid K \). Let \( B_1, B_2, \ldots, \epsilon \mathcal{A} \mid K \). Then there exist \( A_1, A_2, \ldots, \epsilon \mathcal{A} \) such that \( B_1 = A_1 \cap K \) where \( i = 1, 2, \ldots \), now \( \bigcap_{i=1}^{\infty} B_i = (\bigcap_{i=1}^{\infty} A_i) \cap K \). But, \( \mathcal{A} \) is a \( \delta \)–field, then \( \bigcap_{i=1}^{\infty} A_i \in \mathcal{A} \). Hence \( \bigcap_{i=1}^{\infty} B_i \epsilon \mathcal{A} \mid K \). Therefore \( \mathcal{A} \mid K \) is a \( \delta \)–field of a set \( K \).

If we take Example 3 and if we assume that \( K = \{1, 2, 4\} \), then \( \mathcal{A} \mid K = \{ \Phi, \{1, 2\}, K \} \) is a \( \delta \)–field of a set \( K \) and \( \mathcal{A} \mid K \subseteq \mathcal{A} \).

Definition 10

Let \( \mathcal{A} \) be a \( \delta \)–field of a set \( \mathcal{K} \). A measure on \( \mathcal{A} \) is a set function \( \Psi : \mathcal{A} \rightarrow [0, \infty] \) such that \( \Psi(\Phi) = 0 \) and if \( C_1, C_2, \ldots \) form a finite or countably infinite collection of disjoint sets in \( \mathcal{A} \), then \( \Psi(\bigcup_{n=1}^{\infty} C_n) = \sum_{n=1}^{\infty} \Psi(C_n) \).

Example 11

Let \( \mathcal{A} \) be a \( \delta \)–field of a set \( \mathcal{K} \) and define \( \Psi : \mathcal{A} \rightarrow [0, \infty] \) by \( \Psi(C) = 0 \), for all \( C \in \mathcal{A} \). Then \( \Psi \) is a measure on \( \mathcal{A} \).

A measure space is a triple \( (\mathcal{K}, \mathcal{A}, \Psi) \) where \( \mathcal{K} \) is a nonempty set and \( \mathcal{A} \) is a \( \delta \)–field of a set \( \mathcal{K} \) and \( \Psi \) is a measure on \( \mathcal{A} \).

Definition 12

Let \( \mathcal{A} \) be a \( \delta \)–field of a set \( \mathcal{K} \). A countably subadditive on \( \mathcal{A} \) is a set function \( \Psi : \mathcal{A} \rightarrow [0, \infty] \) such that \( \Psi(\Phi) = 0 \) and if \( C_1, C_2, \ldots \) are disjoint sets in \( \mathcal{A} \), then \( \Psi(\bigcup_{n=1}^{\infty} C_n) \leq \sum_{n=1}^{\infty} \Psi(C_n) \). If this requirement holds only for finite collection of disjoint sets in \( \mathcal{A} \), then \( \Psi \) is said to be finitely subadditive on a \( \delta \)–field \( \mathcal{A} \).

Definition 13

Let \( \mathcal{A} \) be a \( \delta \)–field of a set \( \mathcal{K} \). Then a set function \( \Psi : \mathcal{A} \rightarrow [0, \infty] \) is said to be monotone measure, if it satisfies the following requirements:

1. \( \Psi(\Phi) = 0 \).
2. If \( B \in \mathcal{A} \) and \( B \subseteq D \subseteq \mathcal{K} \), then \( \Psi(B) \leq \Psi(D) \).
Definition 14
Let $\mathcal{F}$ be a $\delta$–field of a set $\mathfrak{X}$. Then a set function $\Psi: \mathcal{F} \to [0, \infty]$ is called outer measure, if it satisfies the following requirements:

1- $\Psi(\Phi) = 0$.

2- If $B \in \mathcal{F}$ and $B \subseteq D \subseteq \mathfrak{X}$, then $\Psi(B) \leq \Psi(D)$.

3- If $C_1, C_2, \ldots \in \mathcal{F}$, then $\Psi(\bigcup_{n=1}^{\infty} C_n) \leq \sum_{n=1}^{\infty} \Psi(C_n)$.

Lemma 15
Let $\Psi$ be an outer measure on $\delta$–field $\mathcal{F}$ of a set $\mathfrak{X}$ and $t \in [0, \infty)$. If $t\Psi: \mathcal{F} \to [0, \infty]$ is defined by

\[ (t\Psi)(A) = t \cdot \Psi(A) \quad \forall A \in \mathcal{F}, \]

then $(t\Psi)$ is an outer measure on $\mathcal{F}$.

Proof
Since $\Psi$ is an outer measure on $\mathcal{F}$ and $\Phi \in \mathcal{F}$, then $\Psi(\Phi) = 0$ and $(t\Psi)(\Phi) = 0$.

Let $B \in \mathcal{F}$ and $B \subseteq D \subseteq \mathfrak{X}$, then $D \in \mathcal{F}$ and $\Psi(D) \leq \Psi(D)$. Since

\[ (t\Psi)(B) = t \cdot \Psi(B) \leq t \cdot \Psi(D) = (t\Psi)(D) \]

Let $C_1, C_2, \ldots \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} C_n \in \mathcal{F}$.

So, we have

\[ t \cdot \sum_{n=1}^{\infty} \Psi(C_n) = t \cdot \Psi(\bigcup_{n=1}^{\infty} C_n) \leq t \cdot \sum_{n=1}^{\infty} \Psi(C_n) \]

But, $t \cdot \sum_{n=1}^{\infty} \Psi(C_n) = \sum_{n=1}^{\infty} t \cdot \Psi(C_n) = \sum_{n=1}^{\infty} (t\Psi)(C_n)$. Therefore $t\Psi$ is an outer measure on $\mathcal{F}$.

Lemma 16
Let $\Psi_1$ and $\Psi_2$ be two outer measures on a $\delta$–field $\mathcal{F}$ of a set $\mathfrak{X}$. If $\Psi_1 + \Psi_2: \mathcal{F} \to [0, \infty]$ is defined by

\[ (\Psi_1 + \Psi_2)(C) = \Psi_1(C) + \Psi_2(C), \forall C \in \mathcal{F}, \]

then $\Psi_1 + \Psi_2$ is an outer measure on $\mathcal{F}$.

Proof
Since $\Psi_1$ and $\Psi_2$ are outer measure on $\delta$–field $\mathcal{F}$ and $\Phi \in \mathcal{F}$, then $\Psi_1(\Phi) = \Psi_2(\Phi) = 0$ and

\[ (\Psi_1 + \Psi_2)(\Phi) = 0. \]

Let $B \in \mathcal{F}$ and $B \subseteq D \subseteq \mathfrak{X}$, then $D \in \mathcal{F}$ and $\Psi_1(D) \leq \Psi_1(D)$ and $\Psi_2(D) \leq \Psi_2(D)$. So we have,

\[ (\Psi_1 + \Psi_2)(B) = \Psi_1(B) + \Psi_2(B) \leq \Psi_1(D) + \Psi_2(D) = \Psi_1(D) + \Psi_2(D) = \Psi_1(D) + \Psi_2(D) \]

Let $C_1, C_2, \ldots \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} C_n \in \mathcal{F}$. So, we have

\[ \sum_{n=1}^{\infty} (\Psi_1 + \Psi_2)(C_n) = \sum_{n=1}^{\infty} \Psi_1(C_n) + \sum_{n=1}^{\infty} \Psi_2(C_n) \]

\[ = \sum_{n=1}^{\infty} \Psi_1(C_n) + \sum_{n=1}^{\infty} \Psi_2(C_n) = \sum_{n=1}^{\infty} [\Psi_1(C_n) + \Psi_2(C_n)] \]

Therefore $\Psi_1 + \Psi_2$ is an outer measure on $\mathcal{F}$.

The proof of the following proposition consequence from Lemma (15 and 16) with mathematical induction.

Proposition 17
Let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be outer measure on a $\delta$–field $\mathcal{F}$ of a set $\mathfrak{X}$ and $t_i \in [0, \infty)$ for all $i = 1, 2, \ldots, n$. If a set function $\sum_{i=1}^{n} t_i \Psi_i: \mathcal{F} \to [0, \infty]$ is defined by

\[ (\sum_{i=1}^{n} t_i \Psi_i)(C) = \sum_{i=1}^{n} t_i \Psi_i(C), \forall C \in \mathcal{F}, \]

then $\sum_{i=1}^{n} t_i \Psi_i$ is an outer measure on $\delta$–field $\mathcal{F}$.

Proof
Since $t_i \in [0, \infty)$ and $\Psi_i$ is an outer measure on a $\delta$–field $\mathcal{F}$ for all $i = 1, 2, \ldots, n$. Then by Lemma15 we get $t_i \Psi_i$ is an outer measure on a $\delta$–field $\mathcal{F}$.
Let $\psi_i = t_i \psi_i \ \forall i = 1,2,\ldots,n$. Then we prove that $(\sum_{i=1}^{n} \psi_i)$ is an outer measure on $\mathcal{F}$ by mathematical induction. If $n = 2$, then $\psi_1 + \psi_2$ is an outer measure on $\mathcal{F}$ by Lemma 16. Suppose that $(\sum_{i=1}^{k} \psi_i)$ is an outer measure on $\mathcal{F}$, then we must prove that $(\sum_{i=1}^{k+1} \psi_i)$ is an outer measure on $\mathcal{F}$, whenever $\psi_i$ is an outer measure on $\mathcal{F}$ for all $i = 1,2,\ldots,k,k+1$. $(\sum_{i=1}^{k+1} \psi_i)(\Phi) = (\sum_{i=1}^{k} \psi_i + \psi_{k+1})(\Phi) = (\sum_{i=1}^{k} \psi_i)(\Phi) + \psi_{k+1}(\Phi)$

Let $B, D \in \mathcal{F}$ and $B \subset D$. Then $(\sum_{i=1}^{k} \psi_i)(B) \leq (\sum_{i=1}^{k} \psi_i)(D)$ and $\psi_{k+1}(B) \leq \psi_{k+1}(D)$. 

$(\sum_{i=1}^{k+1} \psi_i)(B) = (\sum_{i=1}^{k} \psi_i)(B) + \psi_{k+1}(B) \leq (\sum_{i=1}^{k} \psi_i)(D) + \psi_{k+1}(D)$ since $(\sum_{i=1}^{k} \psi_i)$ and $\psi_{k+1}$ are outer measure $= (\sum_{i=1}^{k} \psi_i + \psi_{k+1})(D) = (\sum_{i=1}^{k} \psi_i)(D) + \psi_{k+1}(D)$.

Therefore, $\sum_{i=1}^{k+1} t_i \psi_i$ is an outer measure on $\mathcal{F}$.

**Definition 18**

Let $\mathcal{F}$ be a $\delta$–field of a set $\mathcal{K}$. Then a set function $\Psi: \mathcal{F} \to [0,\infty]$ is called null-additive on $\mathcal{F}$ iff $C, D$ are disjoint sets in $\mathcal{F}$ and $\Psi(D) = 0$, then $\Psi(C \cup D) = \Psi(C)$.

**Example 19**

Let $\mathcal{K} = \{1,2\}$ and $\mathcal{F} = \{ \Phi, \{1\}, \{2\}, \mathcal{K} \}$ and define $\Psi: \mathcal{F} \to [0,\infty]$ by:

$\Psi(C) = \begin{cases} 0 & \text{if } C = \Phi \\ 1 & \text{if } C \neq \Phi \end{cases}$. Then $\Psi$ is a null-additive.

**Proposition 20**

Let $\mathcal{F}$ be a $\delta$–field of a set $\mathcal{K}$. Then every measure is null-additive.

**Proof**

Let $\Psi$ be a measure on $\delta$–field $\mathcal{F}$ and let $C, D$ are disjoint sets in $\mathcal{F}$ and $\Psi(D) = 0$. Then $\Psi(C \cup D) = \Psi(C) + \Psi(D) = \Psi(C)$. Hence $\Psi$ is a null-additive.

While the converse is not true and Example 19 indicate that $\Psi$ is null-additive but not measure, because $\{1\}, \{2\}$ are disjoint sets in $\mathcal{F}$ but $\Psi(\{1\} \cup \{2\}) \neq \Psi(\{1\}) + \Psi(\{2\})$.

**Lemma 21**

Let $\Psi$ be a null-additive on a $\delta$–field $\mathcal{F}$ of a set $\mathcal{K}$ and $t \in (0,\infty)$. If $t \Psi: \mathcal{F} \to [0,\infty]$ is defined by:

$(t \Psi)(C) = t \cdot \Psi(C) \ \forall C \in \mathcal{F}$, then $(t \Psi)$ is a null-additive on $\mathcal{F}$.

**Proof**

Let $C, D$ be disjoint sets in $\mathcal{F}$ such that $(t \Psi)(D) = 0$. Then $t \cdot \Psi(D) = 0$ and hence $\Psi(D) = 0$ since $t > 0$. Now, $(t \Psi)(C) = t \cdot \Psi(C)$

Therefore, $t \Psi$ is a null-additive on $\mathcal{F}$.
Lemma 22
Let $\Psi_1$ and $\Psi_2$ be two null-additives on a $\delta$-field $\mathcal{G}$ of a set $\mathcal{X}$. If $\Psi_1 + \Psi_2: \mathcal{G} \rightarrow [0, \infty]$ is defined by:
$$(\Psi_1 + \Psi_2)(C) = \Psi_1(C) + \Psi_2(C) \quad \forall C \in \mathcal{G},$$
then $\Psi_1 + \Psi_2$ is a null-additive on $\mathcal{G}$.

**Proof**
Let $C, D$ be disjoint sets in $\mathcal{G}$ such that $(\Psi_1 + \Psi_2)(D) = 0$. Then $\Psi_1(D) + \Psi_2(D) = 0$, hence $\Psi_1(D) = \Psi_2(D) = 0$ since $\Psi_1$ and $\Psi_2$ are null-additive on $\mathcal{G}$.

Now, $(\Psi_1 + \Psi_2)(C \cup D) = \Psi_1(C \cup D) + \Psi_2(C \cup D)$
$$= \Psi_1(C) + \Psi_2(C)$$
$$= (\Psi_1 + \Psi_2)(C).$$
Therefore, $\Psi_1 + \Psi_2$ is a null-additive on $\mathcal{G}$.

Proposition 23
Let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be a null-additive on a $\delta$-field $\mathcal{G}$ of a set $\mathcal{X}$ and $t_i \in (0, \infty)$ for all $k = 1, 2, \ldots, n$. If a set function $\sum_{k=1}^n t_k \Psi_k: \mathcal{G} \rightarrow [0, \infty]$ is defined by:
$$(\sum_{k=1}^n t_k \Psi_k)(C) = \sum_{k=1}^n t_k \Psi_k(C) \quad \forall C \in \mathcal{G},$$
then $\sum_{k=1}^n t_k \Psi_k$ is a null-additive on $\mathcal{G}$.

**Proof**
Since $t_k \in (0, \infty)$ and $\Psi_k$ is null-additive on $\mathcal{G}$ for all $k = 1, 2, \ldots, n$, then by Lemma 21, we get $t_k \Psi_k$ is null-additive on $\mathcal{G}$ for all $k = 1, 2, \ldots, n$. Let $\Psi_k = t_k \Psi_k$

If $n = 2$, then $\Psi_1 + \Psi_2$ is a null-additive on $\mathcal{G}$ by Lemma 22. Let $C, D$ are disjoint sets in $\mathcal{G}$ such that $(\sum_{k=1}^n \Psi_k)(D) = 0$. Then $\Psi_k(D) = 0$ for all $k = 1, 2, \ldots, n$.

$$(\sum_{k=1}^n \Psi_k)(C \cup D) = \sum_{k=1}^n \Psi_k(C \cup D)$$
$$= \Psi_1(C) + \cdots + \Psi_n(C)$$ since $\Psi_k$ is a null-additive and $\Psi_k(D) = 0$, $\forall k$
$$= (\sum_{k=1}^n \Psi_k)(C).$$ Hence $\sum_{k=1}^n t_k \Psi_k$ is a null-additive on $\mathcal{G}$.

Definition 24
Let $\mathcal{G}$ be a $\delta$-field of a set $\mathcal{X}$ and let $\Psi: \mathcal{G} \rightarrow [0, \infty]$ be a set function and $B \in \mathcal{G}$. If $\Psi_B: \mathcal{G} \rightarrow [0, \infty]$ is define by $\Psi_B(C) = \Psi(C \cap B)$ for all $C \in \mathcal{G}$, then $\Psi_B$ is called $B$-restriction of $\Psi$.

Proposition 25
Let $\mathcal{G}$ be a $\delta$-field of a set $\mathcal{X}$ and $B \in \mathcal{G}$. If $\Psi$ is a measure on $\mathcal{G}$, then:

1. $\Psi_B$ is a measure on $\mathcal{G}$.
2. $\Psi_B(C) = \Psi(C)$, whenever $C \subseteq B$.
3. $\Psi_B(C) = 0$, whenever $C, B$ are disjoint sets in $\mathcal{G}$.

**Proof**

1. Since $\mathcal{G}$ is a $\delta$-field, then $\Phi \in \mathcal{G}$ and $\Psi(\Phi) = 0$. From definition of $\Psi_B$ we get, $\Psi_B(\Phi) = \Psi(\Phi \cap B) = \Psi(\Phi) = 0$. Let $C_1, C_2, \ldots$ are disjoint sets in $\mathcal{G}$, then $\cup_{n=1}^\infty C_n \in \mathcal{G}$. Since $B, C_n \in \mathcal{G} \quad \forall \quad n = 1, 2, \ldots$, then $C_n \cap B \in \mathcal{G}$ and hence $\cup_{n=1}^\infty (C_n \cap B) \in \mathcal{G}$. So, we have $\Psi_B(\bigcup_{n=1}^\infty C_n) = \sum_{n=1}^\infty \Psi((C_n \cap B))$
$$= \sum_{n=1}^\infty \Psi(C_n \cap B)$$
$$= \sum_{n=1}^\infty \Psi_B(C_n).$$ Therefore, $\Psi_B$ is a measure on $\mathcal{G}$.

2. Since $C \subseteq B$, then $C \cap B = C$. So, we have $\Psi_B(C) = \Psi(C \cap B) = \Psi(C)$.

3. Since $C, B$ are disjoint sets in $\mathcal{G}$, then $C \cap B = \Phi$ and $\Psi_B(C) = \Psi(C \cap B) = \Psi(\Phi) = 0$. 


Proposition 26
Let $\mathcal{G}$ be a $\delta$–field of a set $\mathcal{K}$ and $B \in \mathcal{G}$. If $\Psi$ is an outer measure on $\mathcal{G}$, then $\Psi_B$ is an outer measure on $\mathcal{G}$.

**Proof**
Since $\mathcal{G}$ is a $\delta$–field, then $\Phi \in \mathcal{G}$ and $\Psi(\Phi) = 0$. From definition of $\Psi_B$ we get, $\Psi_B(\Phi) = \Psi(\Phi \cap B) = \Psi(\Phi) = 0$. Let $A \in \mathcal{G}$ and $A \subseteq C \subseteq \mathcal{K}$, then $A \cap B \subseteq C \cap B$ and each of $C$, $A \cap B$, $C \cap B \in \mathcal{G}$. Since $\Psi$ is an outer measure on $\mathcal{G}$, then $\Psi(A \cap B) \leq \Psi(C \cap B)$ . So, we have $\Psi_B(A) \leq \Psi_B(C)$. Let $C_1, C_2, \ldots \in \mathcal{G}$. Then $\bigcup_{n=1}^{\infty} C_n \in \mathcal{G}$ and $C_n \cap B \in \mathcal{G}$ for all $n=1, 2, \ldots$, hence $\bigcup_{n=1}^{\infty} (C_n \cap B) \in \mathcal{G}$. So, we have,

$$\Psi_B\left( \bigcup_{n=1}^{\infty} C_n \cap B \right) = \Psi\left( \bigcup_{n=1}^{\infty} (C_n \cap B) \right) \leq \sum_{n=1}^{\infty} \Psi(C_n \cap B) = \sum_{n=1}^{\infty} \Psi_B(C_n).$$

Therefore, $\Psi_B$ is an outer measure on $\mathcal{G}$.

From Proposition 26, we conclude that if $\Psi$ is a monotone measure on $\mathcal{G}$, then $\Psi_B$ is a monotone measure on $\mathcal{G}$, where $\mathcal{G}$ is a $\delta$–field of a set $\mathcal{K}$ and $B \in \mathcal{G}$.

Proposition 27
Let $\mathcal{G}$ be a $\delta$–field of $\mathcal{K}$ and $B \in \mathcal{G}$. If $\Psi$ is a null-additive on $\mathcal{G}$, then $\Psi_B$ is a null-additive on $\mathcal{G}$.

**Proof**
Let $A, C$ be disjoint sets in $\mathcal{G}$ and $\Psi_B(C) = 0$. Then $\Psi(C \cap B) = 0$.

Now, $\Psi_B(A \cup C) = \Psi\left( (A \cup C) \cap B \right) = \Psi\left( (A \cap B) \cup (C \cap B) \right) = \Psi(A \cap B)$ since $\Psi$ is a null-additive on $\mathcal{G}$

Hence, $\Psi_B$ is a null-additive on $\mathcal{G}$.

Proposition 28
Let $\mathcal{G}$ be a $\delta$–field of $\mathcal{K}$ and $B \in \mathcal{G}$. If $\Psi$ is a measure on $\mathcal{G}$, then $\Psi_B$ is a null-additive on $\mathcal{G}$.

**Proof**
It is easy, so we omitted.

Definition 29
Let $\mathcal{G}$ be a $\delta$–field of a set $\mathcal{K}$ and $\Psi: \mathcal{G} \rightarrow [0, \infty]$ be a set function and $\mathcal{K}$ be a non-empty subsets of $\mathcal{G}$ such that $K \in \mathcal{G}$. If $\Psi|K: \mathcal{G}|K \rightarrow [0, \infty]$ is define by:

$$\Psi|K(A) = \Psi(A)$$

for all $A \in \mathcal{G}|K$, then $\Psi|K$ is called the restriction of $\Psi$ on $\mathcal{G}|K$

Proposition 30
Let $\Psi$ be a measure on $\delta$–field $\mathcal{G}$ of a set $\mathcal{K}$ and $\Phi \neq K \subseteq \mathcal{K}$ such that $K \in \mathcal{G}$. Then $\Psi|K$ is a measure on a $\delta$–field $\mathcal{G}|K$ of a set $K$.

**Proof**
Since $\mathcal{G}$ is a $\delta$–field of a set $\mathcal{K}$, then $\Phi \in \mathcal{G}$ and $\Psi(\Phi) = 0$. Since $\Phi \in \mathcal{G}|K$, then by definition of $\Psi|K$, we get $\Psi|K(\Phi) = \Psi(\Phi) = 0$. Let $C_1, C_2, \ldots$ be disjoint sets in $\mathcal{G}|K$. Then $C_n \subseteq K$ and $C_n \in \mathcal{G}$ for all $n=1, 2, \ldots$, hence $\bigcup_{n=1}^{\infty} C_n \in \mathcal{G}|K$. So, we have

$$\Psi|K\left( \bigcup_{n=1}^{\infty} C_n \right) = \Psi\left( \bigcup_{n=1}^{\infty} C_n \right) = \sum_{n=1}^{\infty} \Psi(C_n)$$

since $\Psi$ is a measure on $\mathcal{G}$

$$=\sum_{n=1}^{\infty} \Psi|K(C_n)$$

Therefore, $\Psi|K$ is a measure on a $\delta$–field $\mathcal{G}|K$ of a set $K$. 68
If $\Psi$ is an outer measure on $\delta$–field $\mathcal{Q}$ of a set $\mathcal{K}$, then we need the following two facts to prove that $\Psi|\mathcal{K}$ is an outer measure on a $\delta$–field $\mathcal{Q}|\mathcal{K}$ of a set $\mathcal{K}$.

**Lemma 31**

Let $\Psi$ be a monotone measure on $\delta$–field $\mathcal{Q}$ of a set $\mathcal{K}$ and $\Phi \neq K \subseteq K$ such that $K \in \mathcal{Q}$. Then $\Psi|\mathcal{K}$ is a monotone measure on a $\delta$–field $\mathcal{Q}|\mathcal{K}$ of a set $\mathcal{K}$.

**Proof**

Let $\Psi$ be a monotone measure on $\mathcal{Q}$, then $\Psi(\Phi) = 0$. Since $\mathcal{Q}|\mathcal{K}$ is a $\delta$–field, then $\Phi \in \mathcal{Q}|\mathcal{K}$. From definition of $\Psi|\mathcal{K}$, we get $\Psi|\mathcal{K}(\Phi) = \Psi(\Phi) = 0$.

Let $B \subseteq \mathcal{Q}|\mathcal{K}$ such that $B \subseteq C \subseteq K$, then $B \subseteq \mathcal{Q}$ and $B \subseteq C \subseteq K$. Since $\Psi$ is a monotone measure on $\mathcal{Q}$, then $\Psi(B) \leq \Psi(C)$. But $B, C \in \mathcal{Q}|\mathcal{K}$, then $\Psi|\mathcal{K}(B) = \Psi(B)$ and $\Psi|\mathcal{K}(C) = \Psi(C)$, hence $\Psi|\mathcal{K}(B) \leq \Psi|\mathcal{K}(C)$ and $\Psi|\mathcal{K}$ is monotone measure on $\mathcal{Q}|\mathcal{K}$ of $\mathcal{K}$.

**Lemma 32**

Let $\Psi$ be a countably subadditive on $\delta$–field $\mathcal{Q}$ of a set $\mathcal{K}$ and $\Phi \neq K \subseteq K$ such that $K \in \mathcal{Q}$, then $\Psi|\mathcal{K}$ is a countably subadditive on a $\delta$–field $\mathcal{Q}|\mathcal{K}$ of a set $\mathcal{K}$.

**Proof**

Let $C_1, C_2, \ldots \in \mathcal{Q}|\mathcal{K}$ and $C = \bigcup_{n=1}^{\infty} C_n$, then $C_1, C_2, \ldots \in \mathcal{Q}$ and $C \in \mathcal{Q}$. Since $\Psi$ is a countably subadditive on $\mathcal{Q}$, then $\Psi(C) \leq \sum_{n=1}^{\infty} \Psi(C_n)$, but $C, C_1, C_2, \ldots \in \mathcal{Q}|\mathcal{K}$. So, we have $\Psi(C) = \Psi|\mathcal{K}(C)$ and $\Psi(C_n) = \Psi|\mathcal{K}(C_n)$ for all $n=1,2,\ldots$, hence $\Psi|\mathcal{K}(C) \leq \sum_{n=1}^{\infty} \Psi|\mathcal{K}(C_n)$ and $\Psi|\mathcal{K}$ is a countably subadditive on $\mathcal{Q}|\mathcal{K}$ of $\mathcal{K}$.

**Proposition 33**

Let $\Psi$ be an outer measure on $\delta$–field $\mathcal{Q}$ of a set $\mathcal{K}$ and $\Phi \neq K \subseteq K$ such that $K \in \mathcal{Q}$. Then $\Psi|\mathcal{K}$ is an outer measure on a $\delta$–field $\mathcal{Q}|\mathcal{K}$ of a set $\mathcal{K}$.

**Proof**

Since $\Psi$ is an outer measure on $\mathcal{Q}$, then $\Psi$ is a monotone measure and countably subadditive. By Lemma 31 and Lemma 32 we have $\Psi|\mathcal{K}$ is a monotone measure and countably subadditive on $\mathcal{Q}|\mathcal{K}$ of $\mathcal{K}$. Therefore $\Psi|\mathcal{K}$ is an outer measure on $\mathcal{Q}|\mathcal{K}$ of $\mathcal{K}$.

**Proposition 34**

Let $\Psi$ be a null-additive on $\delta$–field $\mathcal{Q}$ of a set $\mathcal{K}$ and $\Phi \neq K \subseteq K$ such that $K \in \mathcal{Q}$. Then $\Psi|\mathcal{K}$ is a null-additive on $\delta$–field $\mathcal{Q}|\mathcal{K}$.

**Proof:**

Let $C, D$ be disjoint sets in $\mathcal{Q}|\mathcal{K}$ and $\Psi|\mathcal{K}(D) = 0$. Then $\Psi(D) = 0$.

Now, $\Psi|\mathcal{K}(C \cup D) = \Psi((C \cup D))$

$= \Psi(C)$ since $\Psi$ is a null-additive on $\mathcal{Q}$

$= \Psi|\mathcal{K}(C)$ by definition of $\Psi|\mathcal{K}$.

Hence, $\Psi|\mathcal{K}$ is a null-additive on $\mathcal{Q}$.

3. Conclusions

The main results of this paper are the following:

1. Let $\mathcal{K}$ be a nonempty set. A collection $\mathcal{Q} \subseteq P(\mathcal{K})$ is said to be $\delta$–field of a set $\mathcal{K}$ if the following conditions are satisfied:
   1. $\Phi \in \mathcal{Q}$.
   2. If $A$ is a nonempty set in $\mathcal{Q}$ and $A \subseteq B \subseteq K$, then $B \in \mathcal{Q}$.
   3. If $A_1, A_2, \ldots \in \mathcal{Q}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{Q}$.

2. Let $\{\mathcal{Q}_i\}_{i \in I}$ be a sequence of $\delta$–field of a set $\mathcal{K}$. Then $\bigcap_{i \in I} \mathcal{Q}_i$ is a $\delta$–field of a set $\mathcal{K}$.
(3) Let $\mathcal{G}$ be a $\delta$–field of a set $\mathcal{X}$ and let $K$ be a non-empty subset of $\mathcal{X}$. Then the restriction of $\mathcal{G}$ on $K$ is denoted by $\mathcal{G}|K$ and $\mathcal{G}|K = \{B: B= A \cap K, \text{ for some } A \in \mathcal{G}\}$.

(4) Let $\mathcal{G}$ be a $\delta$–field of a set $\mathcal{X}$. Then every measure is null-additive.

(5) Let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be null-additive on a $\delta$–field $\mathcal{G}$ of a set $\mathcal{X}$ and $t_i \in (0, \infty)$ for all $k = 1,2, \ldots, n$. If a set function $\sum_{k=1}^{n} t_k \Psi_k: \mathcal{G} \to [0, \infty]$ is defined by:

$$\left(\sum_{k=1}^{n} t_k \Psi_k\right)(C) = \sum_{k=1}^{n} t_k \Psi_k(C) \quad \forall C \in \mathcal{G},$$

then $\sum_{k=1}^{n} t_k \Psi_k$ is a null-additive on $\mathcal{G}$.

(6) Let $\mathcal{G}$ be a $\delta$–field of a set $\mathcal{X}$ and $B \in \mathcal{G}$. If $\Psi$ is a measure on $\mathcal{G}$, then:

1. $\Psi_B$ is a measure on $\mathcal{G}$.
2. $\Psi_B(C) = \Psi(C)$, whenever $C \subseteq B$.
3. $\Psi_B(C) = 0$, whenever $C, B$ are disjoint sets in $\mathcal{G}$.

(7) Let $\mathcal{G}$ be a $\delta$–field of a set $\mathcal{X}$ and $B \in \mathcal{G}$. If $\Psi$ is an outer measure on $\mathcal{G}$, then $\Psi_B$ is an outer measure on $\mathcal{G}$.

(8) Let $\mathcal{G}$ be a $\delta$–field of $\mathcal{X}$ and $B \in \mathcal{G}$. If $\Psi$ is a null-additive on $\mathcal{G}$, then $\Psi_B$ is a null-additive on $\mathcal{G}$.

(9) Let $\Psi$ be a measure on $\delta$–field $\mathcal{G}$ of a set $\mathcal{X}$ and $\Phi \neq K \subseteq \mathcal{X}$ such that $K \in \mathcal{G}$. Then $\Psi|K$ is a measure on a $\delta$–field $\mathcal{G}|K$ of a set $K$.

(10) Let $\Psi$ be a monotone measure on $\delta$–field $\mathcal{G}$ of a set $\mathcal{X}$ and $\Phi \neq K \subseteq \mathcal{X}$ such that $K \in \mathcal{G}$. Then $\Psi|K$ is a monotone measure on a $\delta$–field $\mathcal{G}|K$ of a set $K$.

References