Convergence Comparison of two Schemes for Common Fixed Points with an Application

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Abstract

Some cases of common fixed point theory for classes of generalized nonexpansive maps are studied. Also, we show that the Picard-Mann scheme can be employed to approximate the unique solution of a mixed-type Volterra-Fredholm functional nonlinear integral equation.

Keywords: Banach space, common fixed point, strong convergence, condition \((C\_1)\).

1. Introduction

Let \(B\) be a non-empty subset of a Banach space \(M\). A map \(T\) on \(B\) is called quasi-nonexpansive \([1]\) if \(F(T) \neq \emptyset\) and \(\|T a - b\| \leq \|a - b\|\) for all \(a \in B\) and all \(b \in F(T)\), where \(F(T)\) denoted the set of all fixed points of \(T\).

In 2008, Suzuki \([2]\), introduced a condition on \(T\) which is stronger than quasi-nonexpansive and weaker than nonexpansive, called condition \((C)\) and presented some results about a fixed point for such maps.

In 2009, Dhompongsa et al \([3]\), extended Suzuki’s theorems to the general class of maps in Banach spaces. García-Falset et al \([4]\), defined two generalization of condition \((C)\), called condition \((E\_1)\) and condition \((C\_2)\). And studied their asymptotic behavior as well as the existence of fixed points. On the other hand, Bruck \([5]\), introduced a map called firmly nonexpansive map in Banach space. Of course, every firmly nonexpansive is nonexpansive.

To discuss about convergence theorem for two nonexpansive maps \(S\) and \(T\) on \(B\) to itself, Khan and Kim \([6]\), constricted the following iterative scheme to find a common fixed point of \(S\) and \(T\):

\[
x \in B
\]
\[
x_{n+1} = (1 - \alpha_n) Ty_n + \alpha_n S y_n
\]
\[
y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \in N
\]

Where \((\alpha_n)\) and \((\beta_n)\) \(\in (0,1)\).

This scheme is independent of both Ishikawa scheme and Yao-Chen scheme \([6]\).
In this paper, we prove some convergence theorems for approximating common fixed points of firmly nonexpansive and maps satisfied condition ($C_\lambda$).

2. Preliminaries

We will assume throughout this paper that $(M, \| \cdot \|)$ is a uniformly convex Banach space and $B$ is a non-empty closed convex subset of $M$. For maps $S, T : B \to B$ the set of all fixed points of $S$ and $T$ will be denoted by $F(T, S)$.

A sequence $(a_n)$ in $B$ is called:

- Picard-Mann hybrid [7].

\begin{align*}
a_{n+1} &= Sb_n \\
b_n &= (1 - \alpha_n)a_n + \alpha_n Ta_n, \quad \forall \, n \in N
\end{align*}

Where $(\alpha_n) \in (0, 1)$.

- Noor iterative scheme [8]. if

\begin{align*}
z_{n+1} &= (1 - \alpha_n)z_n + \alpha_n Su_n \\
u_n &= (1 - \beta_n)z_n + \beta_n Tu_n \\
v_n &= (1 - \gamma_n)z_n + \gamma_n Tz_n, \quad \forall \, n \in N
\end{align*}

Where $(\alpha_n), (\beta_n)$ and $(\gamma_n) \in [0, 1]$.

**Definition (1) [9].** A map $T : B \to B$ said to be Lipschitz continuous or $\text{liLipschitz}$ if $\exists \, K > 0$ such that $\|Ta - Tb\| \leq K\|a - b\|, \forall \, a, b \in B$.

If $K = 1$, then $T$ is nonexpansive.

**Definition (2) [10].** A map $T : B \to B$ is said to satisfying:

- 1-Condition ($C^1$) if $\frac{1}{2} \|a - Ta\| \leq \|a - b\| \longrightarrow \|Ta - Tb\| \leq \|a - b\|, \forall \, a, b \in B$.

- 2-Condition ($C^2$) if $\lambda \|a - Ta\| \leq \|a - b\| \longrightarrow \|Ta - Tb\| \leq \|a - b\|, \forall \, a, b \in B$ and $\lambda \in (0, 1)$.

**Definition (3)[5].** A map $T : B \to M$ is said to be firmly nonexpansive map if $\|Ta - Tb\| \leq \|(1 - t)(Ta - Tb) + t(a - b)\|, \forall \, a, b \in B$ and $t \geq 0$.

**Definition (4)[11].** Two maps are called:

- 1-Condition (A) if there is a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0, g(i) > 0, \forall \, i \in (0, \infty)$ such that:

\[
\text{Either } \|a - Ta\| \geq g(D(a, F)) \text{ or } \|a - Sa\| \geq g(D(a, F)), \forall \, a \in B,
\]

where $D(a, F) = \inf \{\|a - a^*\|; \ a^* \in F\}$ and $F = F(T) \cap F(S)$.

- 2-Condition (I) if $\|a - Tb\| \leq \|Sa - Tb\|, \forall \, a, b \in B$.

**Definition (5)[12].** A map $T : B \to B$ is called

- 1-Demiclosed at 0 if $\forall$ sequence $(a_n)$ in $B$ such that $(a_n)$ converges weakly to $(a)$ and $(Ta_n)$ converges strongly to 0, then $Ta = 0$.

- 2-Affine if $B$ is convex and

\[
T((1 - K)b) = KT(a) + (1 - K)Tb, \forall \, a, b \in B \text{ and } K \in [0, 1].
\]
Definition (6)[7]. Let \((f_n)\) and \((g_n)\) be two sequences of real numbers that converging to \(f\) and \(g\)
\[
\lim_{n \to \infty} \frac{\|f_n - f\|}{\|g_n - g\|} = 0.
\]
Then \((f_n)\) converges faster than \((g_n)\).

Lemma (7)[13]. Let \((\mu_n)_{n=0}^{\infty}\) and \((\omega_n)_{n=0}^{\infty}\) be nonnegative real sequences satisfying the inequality: \(\mu_{n+1} \leq (1 - \delta_n)\mu_n + \omega_n\)
Where \(\delta_n \in (0,1), \forall \ n \geq n_0, \sum_{n=1}^{\infty} \delta_n = \infty\) and \(\frac{\omega_n}{\delta_n} \to 0\) as \(n \to \infty\), then \(\lim_{n \to \infty} \mu_n = 0\).

Lemma (8)[10]. Let \(M\) be a uniformly convex Banach space and \(0 < l \leq t_n \leq k < 1, \forall \ n \in \mathbb{N}\).
Suppose that \((a_n)\) and \((b_n)\) are two sequences of \(M\) such that \(\lim_{n \to \infty} \|a_n\| \leq m, \lim_{n \to \infty} \|b_n\| \leq m\) and \(\lim_{n \to \infty} \|t_n a_n + (1-t_n) b_n\| = m\) hold for some \(m \geq 0\). Then \(\lim_{n \to \infty} \|a_n - b_n\| = 0\).

3. Two Lemmas
Lemma (9): Let \(B\) be a non-empty closed convex subset of a normed space \(M\), \(T: M \to B\) be a firmly nonexpansive and satisfying Lipschitz \(S: B \to B\) be satisfying condition \((C_\lambda)\) Let 1-(\(a_n\)) as in (1) where \((\alpha_n) \in (0,1), n \in \mathbb{N}\).
2-(\(z_n\)) as in (2) where \((\alpha_n), (\beta_n)\) and \((\gamma_n) \in [0,1]\).
If \(F(S,T) \neq \emptyset\), then \(\lim_{n \to \infty} \|a_n - a^*\|\) and \(\lim_{n \to \infty} \|z_n - a^*\|\) exist \(\forall \ a^* \in F(S,T)\).

Proof: Let \(a^* \in F(T,S)\).
By using condition \((C_\lambda)\), we have
\[
\lambda \|a^* - Sa^*\| = 0 \leq \|b_n - a^*\| \overset{\text{yields}}{\longrightarrow} \|Sb_n - a^*\| \leq \|b_n - a^*\|.
\]
Then
1-\(\|a_{n+1} - a^*\| = \|Sb_n - a^*\| \leq \|b_n - a^*\| \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\| \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n\|T(1-t)\|T \|a_n - a^*\| + \alpha_n t\|a_n - a^*\| \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n(1-t)\|T \|a_n - a^*\| + \alpha_n t\|a_n - a^*\| \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n(1-t)\|a_n - a^*\| \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n\|a_n - a^*\| \leq \|a_n - a^*\|
\]
Then \(\lim_{n \to \infty} \|a_n - a^*\|\) exists \(\forall a^* \in F(T,S)\).
2-\(\|v_n - a^*\| \leq (1 - \gamma_n)\|z_n + \gamma_n Tz_n - a^*\| \leq (1 - \gamma_n)\|z_n - a^*\| + \gamma_n\|Tz_n - a^*\| \leq (1 - \gamma_n)\|z_n - a^*\| + \gamma_n(1-t)\|Tz_n - a^*\| + \gamma_n t\|z_n - a^*\| \leq (1 - \gamma_n)\|z_n - a^*\| + \gamma_n(1-t)\|Tz_n - a^*\| + \gamma_n t\|z_n - a^*\| \leq (1 - \gamma_n)\|z_n - a^*\| + \gamma_n(1-t)\|z_n - a^*\| \leq \|z_n - a^*\| \leq \|u_n - a^*\| \leq (1 - \beta_n)\|z_n - a^*\| + \beta_n\|Tv_n - a^*\| \leq (1 - \beta_n)\|z_n - a^*\| + \beta_n(1-t)\|T(v_n - a^*\| \leq \|z_n - a^*\|
\]
Now
\(\|z_{n+1} - a^*\| \leq (1 - \alpha_n)\|z_n - a^*\| + \alpha_n\|Su_n - a^*\| \leq (1 - \alpha_n)\|z_n - a^*\| + \alpha_n\|u_n - a^*\|
\]
Lemma (10): Let $M$ be a uniformly convex Banach space and $B$ be a nonempty closed convex subset of $M$. Let:

1. $T : B \rightarrow B$ be firmly nonexpansive map and satisfying Lipschitz, $S : B \rightarrow B$ be affine and satisfying condition $(C_j)$ and $(a_n)$ be as in (1).

2. $T : B \rightarrow B$ be firmly nonexpansive map and satisfying Lipschitz, $S : B \rightarrow B$ be satisfying condition $(C_j)$ and $(z_n)$ be as in (2). Suppose that condition (I) holds. If $F(S, T) \neq \emptyset$, then $\lim_{n \to \infty} \| Tz_n - a^* \| = 0 = \lim_{n \to \infty} \| Sa_n - a^* \| \& \lim_{n \to \infty} \| Tz_n - a^* \| = 0 = \lim_{n \to \infty} \| Sz_n - a^* \|.

Proof: Let $a^* \in F(T, S)$.

1. As proved by lemma (9), $\lim_{n \to \infty} \| a_n - a^* \|$ exists. Suppose that $\lim_{n \to \infty} \| a_n - a^* \| = c, \forall c \geq 0$.

If $c = 0$, there is nothing to prove.

Now, suppose $c > 0$.

Since, $\| a_{n+1} - a^* \| = \| Sb_n - a^* \|$

$\| b_n - a^* \| \leq (1 - \alpha_n) \| a_n - a^* \| + \alpha_n \| Ta_n - a^* \| \leq \| a_n - a^* \|$

Then $\lim_{n \to \infty} \| b_n - a^* \| = c$.

Next consider

$c = \| b_n - a^* \| \leq (1 - \alpha_n) \| a_n - a^* \| + \alpha_n \| Ta_n - a^* \|$

By applying lemma (9), we obtain

$\lim_{n \to \infty} \| Ta_n - a_n \| = 0$.

Now

$c = \lim_{n \to \infty} \| a_{n+1} - a^* \| = \lim_{n \to \infty} \| Sb_n - a^* \|$

$\| Sb_n - a^* \| = \| S(1 - \alpha_n)a_n + \alpha_n Ta_n - a^* \|$

$\leq (1 - \alpha_n) \| Sa_n - a^* \| + \alpha_n \| STa_n - a^* \|$

By applying Lemma (8), we have

$\lim_{n \to \infty} \| Sa_n - STa_n \| = 0.$

Next, by using condition (I), we obtain

$\| Sa_n - a^* \| \leq \| Sa_n - STa_n \| + \| STa_n - a^* \| \leq 2 \| Sa_n - STa_n \| \rightarrow 0$ as $n \to \infty$

Thus $\lim_{n \to \infty} \| Sa_n - a_n \| = 0$

2. As proved by lemma (9), $\lim_{n \to \infty} \| z_n - a^* \|$ exists. Suppose that $\lim_{n \to \infty} \| z_n - a^* \| = c, \forall c \geq 0$.

If $c = 0$, there is nothing to prove.

Now, suppose $c > 0$.

Since $\| Tz_n - a^* \| \leq \| z_n - a^* \|$ and as proved by lemma (3.1)

$\| Su_n - a^* \| \leq \| u_n - a^* \|$ and $\| Tv_n - a^* \| \leq \| v_n - a^* \|.$

Then,

$\lim_{n \to \infty} \| Tz_n - a^* \| \leq c, \lim_{n \to \infty} \| Su_n - a^* \| \leq c and \lim_{n \to \infty} \| Tv_n - a^* \| \leq c.$

Moreover

$\lim_{n \to \infty} \| z_{n+1} - a^* \| = c$

$c = \| z_{n+1} - a^* \| \leq (1 - \alpha_n) \| z_n - a^* \| + \alpha_n \| Su_n - a^* \|$
By applying lemma (9), we get
\[ \lim_{n \to \infty} \| z_n - S u_n \| = 0 \]

Now
\[ \| u_n - z_n \| \leq (1 - \beta_n) \| z_n - z_n \| + \beta_n \| T v_n - z_n \| = 0 \]
Then, \( \lim_{n \to \infty} \| u_n - z_n \| = 0 \).

Since, \( \lim_{n \to \infty} \| u_n - a^* \| \leq c \) and \( \| z_n - a^* \| \leq \| z_n - S u_n \| + \| S u_n - a^* \| \),
which implies to
\[ c \leq \lim inf_{n \to \infty} \| u_n - a^* \| \]

That gives \( \lim_{n \to \infty} \| u_n - a^* \| = c \), so
\[ c = \| u_n - a^* \| \leq (1 - \beta_n) \| z_n - a^* \| + \beta_n \| T v_n - a^* \| \]
\[ \leq (1 - \alpha_n \beta_n) \| z_n - a^* \| + \alpha_n \beta_n \| T z_n - a^* \| \]

By lemma (9), we obtain:
\[ \lim_{n \to \infty} \| z_n - T z_n \| = 0. \]

Next
\[ \| z_n - S z_n \| \leq \| z_n - S u_n \| + \| S u_n - z_n \| + \| z_n - S z_n \| \]
Letting \( n \to \infty \), we have:
\[ \| z_n - S z_n \| \leq \| z_n - S z_n \| \]
That means \( \lim_{n \to \infty} \| z_n - S z_n \| = 0 \).

4. Convergence and Equivalence Results

**Theorem (11):** Let \( M \) be a uniformly convex Banach space. Let \( B, S, T, (a_n) \) and \( (z_n) \) be as in lemma (10) and \( T, S \) satisfying condition (A). If \( F(T, S) \neq \emptyset \), then \( (a_n) \) and \( (z_n) \) converge strongly to a common fixed point of \( T \) and \( S \).

**Proof:** Now, we will show that \( (a_n) \) is strong convergence. By lemma (10), \( \lim_{n \to \infty} \| a_n - a^* \| \) exists. Suppose that \( \lim_{n \to \infty} \| a_n - a^* \| = c \), \( c \geq 0 \).

From lemma (9), we have \( \| a_{n+1} - a^* \| \leq \| a_n - a^* \| \)
That gives
\[ \inf_{a^* \in F} \| a_{n+1} - a^* \| \leq \inf_{a^* \in F} \| a_n - a^* \| \]
Which means, \( d(a_{n+1}, F) \leq d(a_n, F) \) yields
\[ \lim_{n \to \infty} d(a_n, F) \]

By using condition (A), we have
\[ \lim_{n \to \infty} g(d(a_n, F)) = \lim_{n \to \infty} \| a_n - T a_n \| = 0. \]
Or
\[ \lim_{n \to \infty} g(d(a_n, F)) = \lim_{n \to \infty} \| a_n - S a_n \| = 0. \]
In both situations, we obtain
\[ \lim_{n \to \infty} g(d(a_n, F)) = 0 \]
Since \( g \) is a non-decreasing function and \( g(0) = 0 \). It follows that \( \lim_{n \to \infty} d(a_n, F) = 0 \).

Now to show that \( (a_n) \) is a Cauchy sequence in \( B \). Let \( \epsilon > 0 \), \( \lim_{n \to \infty} d(a_n, F) = 0 \), \( \exists \) a positive integer \( n_0 \), such that:
\[ d(a_n, F) < \frac{\epsilon}{4}, \quad \forall n \geq n_0 \]
In particular,
\[ \inf \{ \| a_n - a^* \|, a^* \in F \} < \frac{\varepsilon}{2} \]

Thus, it must exist \( a^{**} \in F(T,S) \) such that \( \| a_n - a^{**} \| < \frac{\varepsilon}{2} \).

Now, \( \forall n, w \geq n_0 \), we obtain:
\[ \| a_{n+w} - a_n \| \leq \| a_{n+w} - a^{**} \| + \| a_n - a^{**} \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

Hence, \( (a_n) \) is Cauchy sequence in the \( B \) of \( M \). Then \( (a_n) \) converges to a point \( p \in B \).

\[ \lim_{n \to \infty} d(a_n, F) = 0 \quad \text{yields} \quad d(p, F) = 0. \]

Since \( F \) is closed, hence \( p \in F(T,S) \).

By utilizing the same procedure, we can prove \( (z_n) \) convergence strongly.

**Theorem (12):** Let \( T: B \to B \) be a firmly nonexpansive and satisfying lipschitz, \( S: B \to B \) satisfying condition \( (C_{\lambda}) \), with \( F(S,T) \neq \emptyset \) and,

1- \( (a_n) \) be as in (1) and \( (a_n) \in (0,1) \) satisfying \( \sum_{k=0}^{\infty} \alpha_k = \infty \).

2- \( (z_n) \) be as in (2) and \( (a_n), (\beta_n) \) and \( (\gamma_n) \in [0,1] \) satisfying \( \sum_{k=0}^{\infty} \alpha_k = \infty \).

Then \( (a_n) \) & \( (z_n) \) converge to a unique common fixed point \( a^* \in F(S,T) \).

**Proof:**

1- \[ \| b_n - a^* \| \leq (1 - \alpha_n) \| a_n - a^* \| + \alpha_n \| Ta_n - a^* \| \]

Suppose \( \xi = (1 - t)K + t \)

\[ \| a_{n+1} - a^* \| = \| Sb_n - a^* \| \]

\[ \leq \| b_n - a^* \| \]

\[ \leq (1 - (1 - \xi)\alpha_n) \| a_n - a^* \| \]

By induction

\[ \| a_{n+1} - a^* \| \leq \prod_{i=0}^{n} (1 - (1 - \xi)\alpha_i) \| a_0 - a^* \| \]

\[ \leq \| a_0 - a^* \| e^{-(1-\xi)\sum_{i=0}^{\infty} \alpha_i} \]

Since \( \sum_{i=0}^{\infty} \alpha_i = \infty \), \( e^{-(1-\xi)\sum_{i=0}^{\infty} \alpha_i} \to 0 \) as \( n \to \infty \).

Thus \( \lim_{n \to \infty} \| a_n - a^* \| = 0. \)

2- \[ \| v_n - a^* \| \leq (1 - \gamma_n) \| z_n - a^* \| + \gamma_n \| Tz_n - a^* \| \]

Setting \( \xi = (1 - t)K + t \)

\[ \| u_n - a^* \| \leq (1 - \beta_n) \| z_n - a^* \| + \beta_n \| Tv_n - a^* \| \]

\[ \leq (1 - \beta_n) \| z_n - a^* \| + \beta_n \| v_n - a^* \| \]

\[ \leq (1 - \beta_n) \| z_n - a^* \| + \beta_n \xi \| v_n - a^* \| \]

\[ \leq (1 - \beta_n) \| z_n - a^* \| + \beta_n \xi (1 - \gamma_n + \gamma_n \xi) \| z_n - a^* \| \]

\[ \leq (1 - \beta_n) \| z_n - a^* \| + \beta_n (1 - \gamma_n + \gamma_n \xi) \| z_n - a^* \| \]

Now

\[ \| z_{n+1} - a^* \| \leq (1 - \alpha_n) \| z_n - a^* \| + \alpha_n \| Su_n - a^* \| \]

\[ \leq (1 - \alpha_n) \| z_n - a^* \| + \alpha_n \| u_n - a^* \| \]

\[ \leq (1 - \alpha_n) \| z_n - a^* \| + \alpha_n (1 - \beta_n) \| z_n - a^* \| \]
By induction
\[ \|z_{n+1} - a^*\| \leq \prod_{i=0}^{n} (1 - \alpha_i \beta_i \gamma_i) \|z_0 - a^*\| \]
\[ \leq \|z_0 - a^*\| e^{-\sum_{i=0}^{n} \alpha_i \beta_i \gamma_i} \]

Since \( \sum_{i=0}^{\infty} \alpha_i \beta_i \gamma_i = \infty, e^{-\sum_{i=0}^{n} \alpha_i \beta_i \gamma_i} \to 0 \) as \( n \to \infty \).

Thus, \( \lim_{n \to \infty} \|z_n - a^*\| = 0. \)

**Theorem (13):** Let \( T: B \to B \) be a firmly nonexpansive mapping and satisfying lipachitz, \( S: B \to B \) satisfying condition \((C_3)\) and \( a^* \in B \) be a common fixed point of \( S \) and \( T \). Let \( (\alpha_n) \) and \( (\gamma_n) \) be the Picard-Mann and Noor iterations defined in (1) and (2).

Suppose \( (\alpha_n), (\beta_n) \) and \( (\gamma_n) \) satisfied the following conditions:
1. \( \alpha_n \) and \( \beta_n \) \( \in (0,1), \forall \ n \geq 0. \)
2. \( \sum \alpha_n = \infty. \)
3. \( \sum \alpha_n \beta_n < \infty. \)

If \( z_0 = a_0 \) and \( R(T), R(S) \) are bounded, then the Picard-Mann iteration sequence \( (a_n) \) converges strongly to \( a^* \) \( (a_n \to a^*) \) and the Noor iteration sequence \( (\gamma_n) \) converges strongly to \( a^* \) \( (\gamma_n \to a^*) \).

**Proof:** Since the range of \( T \) and \( S \) is bounded, let:
\[ M = \sup_{a \in B} \{|Ta|\} + \|a_0\| < \infty \]
and
\[ M = \sup_{a \in B} \{|Tz|\} + \|z\| < \infty \]
Then
\[ \|a_n\| \leq M, \|b_n\| \leq M, \|z_n\| \leq M, \|u_n\| \leq M, \|v_n\| \leq M \]

Therefore
\[ \|Ta_n\| \leq M, \|Tz_n\| \leq M \]
\[ \|a_{n+1} - z_{n+1}\| = \|b_n - (1 - \alpha_n)z_n - \alpha_n Su_n\| \]
\[ \leq \|b_n - z_n\| + \alpha_n \|Su_n - z_n\| \]
\[ \leq \|b_n - a^*\| + \alpha_n \|u_n - a^*\| + (1 + \alpha_n) \|z_n - a^*\| \]
\[ \|v_n - a^*\| \leq (1 - \gamma_n) \|a_n - a^*\| + \alpha_n (M + \|a^*\|) \]
\[ \|u_n - a^*\| \leq (1 - \beta_n) \|z_n - a^*\| + \beta_n \|Tv_n - a^*\| \]
\[ \leq (1 - \beta_n) \|z_n - a^*\| + \beta_n \|v_n - a^*\| \]
\[ \leq \|z_n - a^*\| \]
\[ \leq M + \|a^*\| \]

Then
\[ \|a_{n+1} - z_{n+1}\| \leq \|b_n - a^*\| + \alpha_n \|u_n - a^*\| + (1 + \alpha_n) \|z_n - a^*\| \]
Let
\[ \mu_n = \|a_n - z_n\|, \quad \omega_n = (2+2\alpha_n)(M + \|a^*\|) \]
and \( \frac{\omega_n}{\delta_n} \to 0 \) as \( n \to \infty \). By applying lemma (7), we get:
\[ \lim_{n \to \infty} \|a_n - w_n\| = 0 \]
If \( a_n \to a^* \in F(T, S) \), then
\[ \|z_n - a^*\| \leq \|z_n - a_n\| + \|a_n - a^*\| \to 0 \text{ as } n \to \infty. \]
If \( z_n \to a^* \in F(T, S) \), then
\[ \|a_n - a^*\| \leq \|a_n - z_n\| + \|z_n - a^*\| \to 0 \text{ as } n \to \infty. \]

**Theorem (14):** Let \( T: B \to B \) be a firmly nonexpansive mapping and satisfying Lipschitz with \( Kt < 1 \) and \( S: B \to B \) satisfying condition \( (C_\delta) \). Suppose that the Picard-Mann and Noor iteration converge to the same common fixed point \( a^* \). Then picard-Mann iteration converges faster than Noor iteration.

**Proof:** Let \( a^* \in F(T, S) \). Then, for Picard-Mann iteration.
\[ \|b_n - a^*\| \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\| \]
Setting \( \xi = (1 - t)K + t \), then we have
\[ \leq (1 - (1 - \xi)\alpha_n)\|a_n - a^*\| \]
Next
\[ \|a_{n+1} - a^*\| = \|Sb_n - a^*\| \]
\[ \leq \|b_n - a^*\| \]
\[ \leq (1 - (1 - \xi)\alpha_n)\|a_n - a^*\| \]
\[ \leq \]
\[ \leq (1 - (1 - \xi)\alpha)^n\|a_1 - a^*\| \]
Let \( f_n = (1 - (1 - \xi)\alpha)^n\|a_1 - a^*\| \)

Now, Noor iteration.
\[ \|v_n - a^*\| \leq (1 - \gamma_n)\|z_n - a^*\| + \gamma_n\|Tz_n - a^*\| \]
\[ \leq (1 - \gamma_n)\|z_n - a^*\| + \gamma_n\xi\|z_n - a^*\| \]
\[ = \|z_n - a^*\| \]
\[ \|u_n - a^*\| \leq (1 - \beta_n)\|z_n - a^*\| + \beta_n\|Tv_n - a^*\| \]
\[ \leq \]
\[ \leq (1 - (1 - \xi)\beta_n)\|z_n - a^*\| \]
Then
\[ \|z_{n+1} - a^*\| \leq (1 - \alpha_n)\|z_n - a^*\| + \alpha_n\|Su_n - a^*\| \]
\[ \leq (1 - \alpha_n + \alpha_n(1 - (1 - \xi)\beta_n))\|z_n - a^*\| \]
Assume that
\[ \alpha_n \leq (1 - \alpha_n + \alpha_n(1 - (1 - \xi)\beta_n)) \]
\[ \leq \alpha_n\|z_n - a^*\| \]
\[ \leq \]
\[ \leq \alpha^n\|z_1 - a^*\| \]
Let $g_n = a^n \|z_1 - a^*\|$

Now,

$$\frac{f_n}{g_n} = \frac{(1 - (1 - \xi)n) \|a_1 - a^*\|}{a^n \|z_1 - a^*\|} \leq \left(1 - (1 - \xi)\right)^n \frac{\|a_1 - a^*\|}{\|z_1 - a^*\|} \to 0 \text{ as } n \to \infty.$$  

Then, $(a_n)$ converges faster than $(z_n)$ to $a^*$.

**Example (15):** Let $B = [0, \infty)$ and $T, S: B \to B$ be an mappings defined by $Ta = \frac{3 - a}{2}$ and $Sa = \frac{4 + 4a}{5}$ $\forall a \in B$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$, $\forall n$ with initial value $a_1 = 20$. The Picard-Mann iteration converges faster than Noor iteration, as shown in Table 1.

**Table 1.** Numerical results corresponding to $a_1 = 20$ for 30 steps.

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<tr>
<th>n</th>
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<th>Noor</th>
<th>n</th>
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</table>

**Figure 1:** Convergence behavior corresponding to $a_1 = 20$ for 30 steps.
5. Application
The following mixed type of Volterra-Fredholm functional nonlinear integral equation that is appeared in [14]. We use theorem (14) to solve it:

\[ a(t) = G(t, a(t)), \int_{x_1}^{t_1} \ldots \int_{x_n}^{t_n} K(t, r, a(r))dr, \int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} H(t, r, a(r))dr \]  \tag{3}

Where:
\[ [x_1, y_1] \times \ldots \times [x_n, y_n] \] be an interval in \( R^n \), \( K, H: [x_1, y_1] \times \ldots \times [x_n, y_n] \times [x_1, y_1] \times \ldots \times [x_n, y_n] \times R \rightarrow R \) continuous functions and \( G: [x_1, y_1] \times \ldots \times [x_n, y_n] \times R^3 \rightarrow R \).

Assume that the following conditions are accomplished:

i- \( K, H \in C([x_1, y_1] \times \ldots \times [x_n, y_n] \times [x_1, y_1] \times \ldots \times [x_n, y_n] \times R) \).

ii- \( G \in C([x_1, y_1] \times \ldots \times [x_n, y_n] \times R^3) \).

iii- \exists \text{positive constants } c, q, \varepsilon \text{ such that } |G(t, a_1, b_1, c_1) - G(t, a_2, b_2, c_2)| \leq c|a_1 - a_2| + q|b_1 - b_2| + \varepsilon|c_1 - c_2| \quad \forall t \in [x_1, y_1] \times \ldots \times [x_n, y_n], a_1, a_2, b_1, b_2, c_1, c_2 \in R.

iv- \exists \text{positive constants } S_K \text{ and } S_H \text{ such that } |K(t, r, a) - K(t, r, b)| \leq E_K|a - b|, |H(t, r, a) - H(t, r, b)| \leq E_H|a - b| \quad \forall t \in [x_1, y_1] \times \ldots \times [x_n, y_n] \text{ and } a, b \in R.

v- \exists \text{such that } |a(t) - a^*(t)| \leq 1, \quad a^* \in C([x_1, y_1] \times \ldots \times [x_n, y_n]).

Theorem (16) [14]. Suppose that conditions (i-v) are satisfied. Then, the equation (3) has a unique solution \( a^* \in C([x_1, y_1] \times \ldots \times [x_n, y_n]). \)

Theorem (17): We deem Banach space \( M = C([x_1, y_1] \times \ldots \times [x_n, y_n], ||.||) \), such that satisfying \( \sum_{k=0}^{\infty} a_k = \infty \). Let \( (a_n) \) be as shown in step (1) and a map \( T: M \rightarrow M \) is defined by

\[ T(a_n) = G(t, a_n(t)), \int_{x_1}^{t_1} \ldots \int_{x_n}^{t_n} K(t, r, a(r))dr, \int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} H(t, r, a(r))dr \]

Suppose that the conditions (i-v) are accomplished. Then, the equation (3) has a unique solution \( a^* \in C([x_1, y_1] \times \ldots \times [x_n, y_n]) \) and the Picard-Mann iteration converges to \( a^* \).

**Proof:** To prove \( a_n \rightarrow a^* \) as \( n \rightarrow \infty \). Let

\[ ||a_{n+1} - a^*|| = ||Sb_n - a^*|| \]

\[ = ||Sb_n(t) - S(a^*)(t)|| \]

\[ = G(t, b_n(t)), \int_{x_1}^{t_1} \ldots \int_{x_n}^{t_n} K(t, r, b_n(r))dr, \int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} H(t, r, b_n(r))dr - G(t, a^*(t)), \int_{x_1}^{t_1} \ldots \int_{x_n}^{t_n} K(t, r, a^*(r))dr, \int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} H(t, r, a^*(r))dr \]

\[ \leq c|b_n(t) - a^*(t)| + q \int_{x_1}^{t_1} \ldots \int_{x_n}^{t_n} K(t, r, b_n(r))dr, \int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} K(t, r, a^*(r))dr \]

\[ + \varepsilon \int_{x_1}^{t_1} \ldots \int_{x_n}^{t_n} H(t, r, b_n(r))dr, \int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} H(t, r, a^*(r))dr \]

\[ \leq [c + (qE_K + \varepsilon E_H)(y_1 - x_1) \ldots (y_n - x_n)] ||b_n - a^*|| \]

Since,

\[ ||b_n - a^*|| \leq (1 - a_n)||a_n - a^*|| + a_n||Ta_n(t) - Ta^*(t)|| \]

\[ 90 \]
\begin{align*}
&\leq (1 - \alpha_n)\|a_n - a^*\| \\
&\quad + \alpha_n \left| G(t, b_n(t)) \int_{x_1}^{x_1} \cdots \int_{x_n}^{x_n} \mathcal{K}(t, r, b_n(r)) dr \int_{y_1}^{y_1} \cdots \int_{y_n}^{y_n} \mathcal{H}(t, r, b_n(r)) - G(t, a^*(t)) \int_{x_1}^{x_1} \cdots \int_{x_n}^{x_n} \mathcal{K}(t, r, a^*(r)) dr \int_{y_1}^{y_1} \cdots \int_{y_n}^{y_n} \mathcal{H}(t, r, a^*(r)) dr \right| \\
&\leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n \left[ \zeta + \left( qE_K + E_{H} \right)(y_1 - x_1) \cdots (y_n - x_n) \right] \\
&\quad \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n \left[ \zeta + \left( qE_K + eE_H \right)(y_1 - x_1) \cdots (y_n - x_n) \right] \\
&\quad \leq \|a_0 - a^*\| \prod_{k=0}^{n} \left( 1 - (1 - \alpha_n)\left[ \zeta + \left( qE_K + eE_H \right)(y_1 - x_1) \cdots (y_n - x_n) \right] \right) \\
&\quad < 1
\end{align*}

By condition (v), \( 1 - \alpha_n\left[ \zeta + \left( qE_K + eE_H \right)(y_1 - x_1) \cdots (y_n - x_n) \right] < 1 \).

Now, under using theorem (12), we obtain that equation (3) has a unique solution \( a^* \in C([x_1, y_1] \times \cdots \times [x_n, y_n]) \) and Picard-Mann iteration converges to \( a^* \).

In the same scope you can see the results in [15] and [16], where Hasan and Abed established weak convergence theorems by using appropriate conditions for approximating common fixed points and equivalence between the convergence of the Picard-Mann iteration scheme and Liu et al. iteration scheme in Banach spaces.

6. Conclusion

In the setting of 2-normed spaces [16], we define firmly nonexpansive and generalized nonexpansive maps. Then, we study the convergence of Picard-Mann iteration and Noor iteration.

References


