2-Regular Modules

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Abstract

In this paper we introduced the concept of 2-pure submodules as a generalization of pure submodules, we study some of its basic properties and by using this concept we define the class of 2-regular modules, where an R-module M is called 2-regular module if every submodule is 2-pure submodule. Many results about this concept are given.

Key Words: 2-pure submodules, 2-regular modules, pure submodules, regular modules.
Introduction

Throughout this paper, R denotes a commutative ring with identity and every R-module is a unitary. It is well-known that the pure submodules were given by several authors. For example [1] and [2].

Definition (0.1): [1]
Let M be an R-module. A submodule N of M is called pure if the sequence $0 \rightarrow E \otimes N \rightarrow E \otimes M$ is exact for every R-module E.

Proposition (0.2): [1]
Let N be a submodule of M. The following statements are equivalent:
(1) N is a pure submodule of M.
(2) For each $\sum_{i=1}^{n} r_{ji} m_{i} \in N$, $r_{ji} \in R$, $m_{i} \in M$, $j = 1,2,...,k$, there exists $x_{i} \in N$, $i = 1,2,...,n$ such that $\sum_{i=1}^{n} r_{ji} m_{i} = \sum_{i=1}^{n} r_{ji} x_{i}$ for each j.

(3)

Proposition (0.3): [2]
Let N be an R-submodule of M. Consider the following statements:
(1) N is a pure submodule of M.
(2) $N \cap IM = IN$ for each ideal I of R.
(3) $N \cap IM = IN$ for each finitely generated ideal I of R.
(4) $N \cap (r)M = (r)N$ for each principal ideal (r) of R.
(5) $N \cap rM = rN$ for each $r \in R$.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). And if M is flat then (1) $\Leftrightarrow$ (2).

Notice that: Anderson was called the submodule N of M pure if it satisfies (2), see [3].

Recall that an R-module M is called regular module if every submodule of M is pure [2]. M is called a Von Neumann regular module if every cyclic submodule of M is a direct summand of M, [4].

This paper is structured in three sections. In section one we give a comprehensive study of 2-pure submodules. Some results are analogous to the properties of pure submodules. In section two, we study the concept of 2-regular modules. It is clear that every regular module is 2-regular, but the converse is not true (see Remarks and Examples (2.2)(1)). Section three is concerned with the direct sum of 2-regular modules. It is shown under certain condition, the direct sum of 2-regular modules is 2-regular (see corollary 3.3). Also we show that the 2-regular property of a module is inherited by its submodules (see Corollary 3.7). Other results are given in this section.

0- 2-Pure Submodules

In this section we introduce the concept of 2-pure submodules. We investigate the basic properties of this type of submodules, some of these properties are analogous to the properties of pure submodules.

Definition (1.1):
Let M be an R-module. A submodule N of M is called a 2-pure submodule of M if for each ideal I of R, $I^{2}M \cap N = I^{2}N$.

Remarks and Examples (1.2):
(1) It is clear that every pure submodule is a 2-pure, but not the converse. For example: the submodule $\{0,2\}$ of the module $Z_{4}$ as Z-module is 2-pure submodule since if I= 2Z is an
ideal of $\mathbb{Z}$, then $I^2\mathbb{Z}_4 \cap \langle \overline{0}, 2 \rangle = \langle \overline{0} \rangle$. On the other hand $I^2 = 4 \langle \overline{0}, 2 \rangle = \langle \overline{0} \rangle$.

By the similar simple calculation one can easily to show that $I^2\mathbb{Z}_4 \cap \langle \overline{0}, 2 \rangle = \langle \overline{0} \rangle$ for every ideal $I = n\mathbb{Z}$ of $\mathbb{Z}$ where $n$ is any positive integer. Thus $\langle \overline{0}, 2 \rangle$ is a 2-pure submodule of $\mathbb{Z}_4$ but is not pure since if $I = 2\mathbb{Z}$, then $IZ_4 \cap \langle \overline{0}, 2 \rangle = 2\mathbb{Z}_4 \cap \langle \overline{0}, 2 \rangle = \langle 0 \rangle$.

(2) In any $R$-module $M$, the submodules $M$ and $\{0\}$ are always 2-pure submodules in $M$.

(3) In the module $\mathbb{Z}$ as $\mathbb{Z}$-module, the only 2-pure submodules are $\{0\}$ and $\mathbb{Z}$. To see this, for every submodule $n\mathbb{Z}$ of $\mathbb{Z}$, $n^2 = n^2 1 < n >^2 \mathbb{Z} \cap n\mathbb{Z}$, but $n^2 \not\in n^2(n\mathbb{Z}) = n^3\mathbb{Z}$.

(4) Every nonzero cyclic submodule of the module $\mathbb{Q}$ as $\mathbb{Z}$-module is a non 2-pure submodule.

**Proof:**

Let $N$ be a cyclic submodule of $\mathbb{Q}$ as $\mathbb{Z}$-module, generated by an element $\frac{a}{b}$ where $a$ and $b$ are two nonzero elements in $\mathbb{Z}$. If we take an ideal $<n>$ of $\mathbb{Z}$ where $n$ is greater than one, then $<n^2> \cap \langle \frac{a}{b} \rangle = \langle \frac{n^2 a}{b} \rangle$.

Also, $Q = <n^2> \cdot Q$, because for any element $\frac{c}{d} \in Q$ we have $\frac{c}{d} = \frac{c}{n^2 d} n^2 < n^2 > \cdot Q$, thus $Q = <n^2> \cdot Q$. Therefore $<n^2> \cdot Q \cap \langle \frac{a}{b} \rangle = <\frac{a}{b} >$, implies that $<n^2> \cdot Q \cap <\frac{a}{b} > \neq <n^2 > \cdot <\frac{a}{b} >$.

(5) It is clear every direct summand is 2-pure since every direct summand is pure submodule, hence it is a 2-pure submodule, but the converse is not true, for example: the submodule of the module $\mathbb{Z}_9$ as $\mathbb{Z}$-module. It is easily to check that $I^2\mathbb{Z}_9 \cap \langle \overline{0}, 3, 6 \rangle = I^2 \langle \overline{0}, 3, 6 \rangle$ for each $I$ of $\mathbb{Z}$. So, $\langle \overline{0}, 3, 6 \rangle$ is 2-pure in $\mathbb{Z}_9$ but not pure and hence not direct summand. Since if we take $I = 3\mathbb{Z}$, then $IZ_9 \cap \langle \overline{0}, 3, 6 \rangle = \langle 0, 3, 6 \rangle$ and $I = \langle \overline{0}, 3, 6 \rangle = \langle 0 \rangle$.

(6) Let $N$ be a 2-pure submodule of $M$ such that $N \cong K$ for some submodule $K$ of $M$, then $K$ may not be a 2-pure. For example: consider the module $\mathbb{Z}$ as $\mathbb{Z}$-module. Let $N = \mathbb{Z}$ and $K = 2\mathbb{Z}$. It is clear $\mathbb{Z} \cong 2\mathbb{Z}$ but $2\mathbb{Z}$ is not 2-pure in $\mathbb{Z}$.

The following propositions give some properties of 2-pure submodules.

**Proposition (1.3):**

Let $M$ be an $R$-module and $N$ be a 2-pure submodule of $M$. If $A$ is a 2-pure submodule in $N$, then $A$ is a 2-pure submodule in $M$.

**Proof:**

Let $I$ be an ideal of $R$. Since $N$ is a 2-pure submodule in $M$ and $A$ is a 2-pure submodule in $N$, then $I^2M \cap N = I^2N$ and $I^2N \cap A = I^2A$. But $A \subseteq N$, implies $I^2A = I^2N \cap A = (I^2M \cap N) \cap A = I^2M \cap (N \cap A) = I^2M \cap A$. 

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Proposition (1.4):
Let $M$ be an $R$-module and $N$ is a 2-pure submodule of $M$. If $A$ is a submodule of $M$ containing $N$, then $N$ is a 2-pure submodule in $A$.

Proof:
Let $I$ be an ideal of $R$. Since $N$ is a 2-pure submodule in $M$, hence $I^2M \cap N = I^2N$ and since $N \subseteq A \subseteq M$ implies $I^2A \cap N = (I^2A \cap I^2M) \cap N = I^2A \cap (I^2M \cap N) = I^2A \cap I^2N = I^2N$.

Proposition (1.5):
Let $M$ be an $R$-module and $N$ is a 2-pure submodule of $M$. If $H$ is a submodule of $N$, then $N/H$ is a 2-pure submodule in $M/H$.

Proof:
Let $I$ be an ideal of $R$. Since $I^2M + H \cap N = I^2N$, hence $I^2M + H \cap N/H = (I^2M \cap N) + (H \cap N) = I^2N + H$ by Modular law.

Recall that a ring $R$ is called an arithmetical ring if every finitely generated ideal of $R$ is a multiplication ideal, where an ideal $I$ of $R$ is called a multiplication ideal if every ideal $J \subseteq I$ there exists an ideal $K$ of $R$ such that $J = IK$, see [5].

The following proposition gives a characterization of 2-pure submodules of modules over some classes of rings. First let us state the following theorem, which can be found in [6].

Theorem (1.6):
Let $I = (a_1, a_2, \ldots, a_n)$ be a multiplication ideal in the ring $R$. Then for each positive integer $k$, $(a_1, a_2, \ldots, a_n)^k = (a_1^k, a_2^k, \ldots, a_n^k)$.

Proof: see [6].

Proposition (1.7):
Let $M$ be a module over arithmetical ring $R$. The following statements are equivalent:

(1) $N$ is a 2-pure submodule of $M$.
(2) For each $\sum_{i=1}^{m} x_i \epsilon N$, $x_i \epsilon M$, $j = 1, 2, \ldots, m$, there exists $x'_i \epsilon N$, $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} r_{ij} x'_i = \sum_{i=1}^{n} r_{ij} x_i$ for each $j$.

Proof:

(1) $\Rightarrow$ (2) Assume that $N$ is a 2-pure submodule of $M$, let $y_i = \sum_{i=1}^{m} r_{ij} x_i \epsilon N$ for any finite sets, $\{x_i\}_{i=1}^{m}$ in $M$, $\{y_j\}_{j=1}^{n}$ in $N$ and $\{r_{ij}\}$ in $R$ where $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. Let $I$ be an ideal of $R$
generated by the finite set \( \{r_{ij}, r_{ij}, \ldots, r_{ij}\} \), then \( r_{ij} \in I \) and \( r_{ij}^2 \in I^2 \) imply \( r_{ij}^2 x_i \in I^2 M \). Thus \( y_j = \sum_{i=1}^{n} r_{ij}^2 x_i \in I^2 M \), therefore \( y_j \in I^2 M \cap N \). But \( I^2 M \cap N = I^2 N \), implies \( y_i \in I^2 N \). Since \( R \) is arithmetical ring, hence by theorem (1.6), \( I^2 = (r_{ij}, r_{ij}, \ldots, r_{ij}) \). Therefore \( y_j = \sum_{i=1}^{n} r_{ij}^2 x_i' \) for some \( x_i' \in N \).

(2) \( \Rightarrow \) (1) Let \( N \) be any submodule of \( M \). Let \( y_j \in I^2 M \cap N \), \( y_j = \sum_{i=1}^{n} r_{ij}^2 x_i \) where \( \{x_i\}_{i=1}^{n} \subseteq M, \{r_{ij}\}_{j=1}^{m} \subseteq N, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \). Therefore by hypothesis, there exists \( x_i' \in N \) such that \( y_j = \sum_{i=1}^{n} r_{ij}^2 x_i = \sum_{i=1}^{n} r_{ij}^2 x_i' \in I^2 N \) implies \( y_j \in I^2 N \). Then \( I^2 M \cap N \subseteq I^2 N \). The reverse inclusion is clear. Thus \( I^2 M \cap N = I^2 N \), and hence \( N \) is a 2-pure submodule of \( M \).

1- 2-Regular Modules

In this section, we introduce and study the class of 2-regular modules.

**Definition (2.1):**

An \( R \)-module \( M \) is called **2-regular module** if every submodule of \( M \) is 2-pure.

**Remarks and Examples (2.2):**

(1) It is clear that the following implications hold:

- Von Neumman regular \( \Rightarrow \) regular \( \Rightarrow \) 2-regular

But non of these implications is reversible. For example: the module \( \mathbb{Z}_4 \) as \( \mathbb{Z} \)-module is 2-regular since every submodule of \( \mathbb{Z}_4 \) is 2-pure submodule in \( \mathbb{Z}_4 \), but \( \mathbb{Z}_4 \) is not regular since the submodule \( \{0, 2\} \) of \( \mathbb{Z}_4 \) is not pure, see remark and example (1.2)(1).

(2) The modules \( \mathbb{Z} \) and \( \mathbb{Q} \) as \( \mathbb{Z} \)-modules are not 2-regular modules, see remarks and examples (1.2)(3), and (4).

The following theorem shows that the cyclic 2-pure submodules is enough to make the module be 2-regular.

**Theorem (2.3):**

Let \( M \) be an \( R \)-module. The following statements are equivalent:

1. \( M \) is 2-regular module.
2. Every cyclic submodule of \( M \) is 2-pure submodule of \( M \).
3. Every finitely generated submodule of \( M \) is 2-pure submodule.
4. Every submodule of \( M \) is a 2-pure submodule of \( M \).

**Proof:**

(1) \( \Rightarrow \) (2) it follows by definition (2.1).

(2) \( \Rightarrow \) (1) Assume that every cyclic submodule of \( M \) is 2-pure. Let \( N \) be a submodule of \( M \) and \( I \) is an ideal of \( R \). Let \( x \in I^2 M \cap N \) implies \( x \in I^2 M \) and \( x \in N \). Therefore \( x \in I^2 M \cap \langle x \rangle = I^2 \langle x \rangle \subseteq I^2 N \).

(1) \( \Rightarrow \) (3) It follows by definition (2.1), and the proof of (2) \( \Rightarrow \) (1).

(3) \( \Rightarrow \) (2) It is clear.

(1) \( \Rightarrow \) (4) It follows by definition (2.1).
The Direct Sum of 2-Regular Modules—Basic Results

In this section, we study the direct sum and the epimorphic image of 2-regular module; various properties of 2-regular modules are discussed and illustrated. We start with the following proposition.

The following proposition shows that the factor module of a 2-regular module is also 2-regular module.

Proposition (3.1):
Let $M$ be an $R$-module. Then $M$ is a 2-regular if and only if $\frac{M}{N}$ is 2-regular for every submodule $N$ of $M$.

Proof:
($\Rightarrow$) Let $N$ be a submodule of $M$ and $K$ is any submodule of $M$ containing $N$. Since $M$ is 2-regular then $K$ is 2-pure in $M$. Thus $\frac{K}{N}$ is 2-pure in $\frac{M}{N}$ by proposition (1.5), therefore $\frac{M}{N}$ is 2-regular.

($\Leftarrow$) It is easily by taking $N = 0$.

Now, we have several consequences of the proposition (3.1), the first result shows that the epimorphic image of 2-regular module is 2-regular.

Corollary (3.2):
Let $M$ and $M'$ be $R$-modules and $f: M \longrightarrow M'$ be an $R$-epimorphism. If $M$ is 2-regular module then $M'$ is 2-regular.

Proof:
Since $f: M \longrightarrow M'$ is an $R$-epimorphism and $M$ is 2-regular. Then $\frac{M}{\ker f}$ is 2-regular module by proposition (3.1). But $\frac{M}{\ker f} \cong M'$ by the first isomorphism theorem. Therefore $M'$ is 2-regular.

Corollary (3.3):
Let $M_1$ and $M_2$ be $R$-modules. If $M = M_1 \oplus M_2$ is 2-regular $R$-module, then $M_1$ and $M_2$ are 2-regular $R$-modules. The converse is true provided $\text{ann}(M_1) + \text{ann}(M_2) = R$.

The following statements are equivalent:

Proof:
For the first assertion, assume that $M = M_1 \oplus M_2$ is 2-regular $R$-module. Let $\rho_i: M \longrightarrow M_i$ be the natural projective map of $M$ onto $M_i$ for each $i = 1, 2$. Since $\rho_i$ is an $R$-epimorphism then the epimorphic image of $M$ is 2-regular, implies that $M_i$ is 2-regular. Conversely, assume $M_1$ and $M_2$ are 2-regular $R$-modules and $M = M_1 \oplus M_2$. Let be a submodule of $M = M_1 \oplus M_2$. Since $\text{ann}(M_1) + \text{ann}(M_2) = R$ then by the same way of the proof of [7, prop. (4.2), CH.1], $N = N_1 \oplus N_2$ where $N_1$ is a submodule in $M_1$ and $N_2$ is a submodule in $M_2$. Let $I$ be an ideal of $R$. To show $I^2M \cap N = I^2N$. Since $I^2M \cap N_1 = I^2N_1$ and $I^2M \cap N_2 = I^2N_2$ implies that $(I^2M_1 \cap N_1) \oplus (I^2M_2 \cap N_2) = I^2N_1 \oplus I^2N_2$. Then $(I^2M_1 \oplus I^2M_2) \cap (N_1 \oplus N_2) = I^2(N_1 \oplus N_2)$, therefore $M$ is 2-regular module.

The proof of the following result is similar to that of corollary (3.3).
Corollary (3.4):
Let $M_1$ and $M_2$ be $R$-modules. If $N_1$ is a 2-pure submodule in $M_1$ and $N_2$ is a 2-pure submodule in $M_2$, then $N_1 \oplus N_2$ is a 2-pure submodule in $M_1 \oplus M_2$.

Corollary (3.5):
Let $M_1$ and $M_2$ be $R$-modules and $M_1 \oplus M_2$ is 2-regular $R$-module, then $M_1 + M_2$ is 2-regular.

Proof:
Define $f : M_1 \oplus M_2 \to M_1 + M_2$ by $f(m_1, m_2) = m_1 + m_2$. It is easily to check that $f$ is an epimorphism. Since $M_1 \oplus M_2$ is 2-regular, thus the epimorphic image of $M_1 \oplus M_2$ is 2-regular by corollary (3.2). Therefore $M_1 + M_2$ is 2-regular.

Corollary (3.6):
Let $M_1$ and $M_2$ be 2-regular $R$-modules such that $\text{ann}(M_1) + \text{ann}(M_2) = R$, then $M_1 + M_2$ is a 2-regular $R$-module.

Proof:
Since $M_1$ and $M_2$ are 2-regular $R$-modules then $M_1 \oplus M_2$ is 2-regular by corollary (3.3) implies that $M_1 + M_2$ is a 2-regular by corollary (3.5).

The following result shows that every submodule of a 2-regular module inherits the 2-regular property.

Corollary (3.7):
Every submodule of a 2-regular module is a 2-regular module.

Proof:
Let $N$ be a submodule of a 2-regular $R$-module $M$. To show that $N$ is 2-regular $R$-module. Let $K$ be a submodule in $N$ and $I$ is an ideal of $R$. Thus we have:

\[ I^2 N \cap K = (I^2 M \cap N) \cap K \]
\[ = I^2 M \cap (N \cap K) \]
\[ = I^2 M \cap K \]
\[ = I^2 K \]

Since $N$ is 2-pure in $M$ and $K$ is 2-pure in $M$.

Therefore $K$ is 2-pure in $N$ implies $N$ is 2-regular.

We end this paper by the following remark.

Remark (3.8):
If all proper submodules of an $R$-module $M$ are 2-regular then $M$ may not be 2-regular, for example: the module $Z_8$ as $Z$-module is not 2-regular. Since $\langle 4 \rangle$ is not 2-pure submodule of $Z_8$ because $2^2 \cdot Z_8 \cap \langle 4 \rangle = \langle 4 \rangle$ but $2^2 \cdot \langle 4 \rangle = \langle 0 \rangle$, while every proper submodule of $Z_8$ is 2-regular, since $\langle 2 \rangle \cong Z_4$ and $\langle 4 \rangle \cong Z_2$ are 2-regular modules.
References
المقاسات المنتظمة من النمط -٢

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الخلاصة

في بحثنا هذا نقدم مفهوم المقاسات الجزئية النظيفة من النمط -٢ وتعميم لمفهوم المقاسات الجزئية النظيفة وتعاملاً بهذا المفهوم تعرف المقاسات المنتظمة من النمط -٢ إذا قال ان المقاس M على الحلقة R بأنه منتظم من النمط -٢ إذا كان كل مقاس جزئي فيه يكون نظيفاً من النمط -٢. أعطينا العديد من النتائج حول هذا المفهوم.

الكلمات المفتاحية: المقاسات الجزئية النظيفة من النمط -٢، المقاسات المنتظمة من النمط -٢، المقاسات الجزئية النظيفة، المقاسات المنتظمة.