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On SAH – Ideal of BH – Algebra

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Abstract

The aim of this investigation is to present the idea of SAH – ideal, closed SAH – ideal and closed SAH – ideal with respect to an element, $\overline{SAH}$ – ideal and s- $\overline{SAH}$ – ideal of BH – algebra.

We detail and show theorems which regulate the relationship between these ideas and provide some examples in BH – algebra.

Keywords: BH – algebra, SAH – ideal of BH – algebra, closed SAH – ideal with respect to an element of BH – algebra, $\overline{SAH}$ – ideal .

1. Introduction

After founding of fuzzy subset by Zadeh L. A [1]. Several researchers presented the generalizations of the idea of fuzzy subsets. Imai and Iseki K. established two classes BCK algebra and BCI – algebra [2, 3]. Jun Y. B., Rogh E. H. And Kin H. S. produced a new concept, named a BH – algebra [4]. In this paper, we will recall some basic definitions. A BH – algebra is a nonempty set Ψ with a binary operation * satisfies the conditions: $ж * ж = 0$, for all $ж \in \Psi$, $ж * ц = 0$ and $ц * ж = 0 \rightarrow ж = ц$ for all $ж , ц \in \Psi$ and $ж * 0 = ж$, for all $ж \in \Psi$ [4]. we will use Ψ for representing a BH – algebra $(\Psi ; * ,0)$. Let $\mathfrak{S}$ a nonempty subset of Ψ . then $\mathfrak{S}$ is named an ideal of Ψ if it holds: $0 \in \mathfrak{S}$; $ж * ц \in \mathfrak{S}$ and $ц \in \mathfrak{S} \rightarrow ж \in \mathfrak{S}$ [4]. Let Ψ and Φ be BH – algebras.

A mapping $\delta : \Psi \rightarrow \Phi$ is named ahomomorphism if: $\delta (ж * ц) = \delta (ж) * \delta (ц)$, $\forall ж,ц \in \Psi$. A homomorphism δ is titled a monomerphism (resp, epimorphism) if it injective (resp., surjective). A bijective homomorphism is titled an isomorphism. Two BH – algebras Ψ and Φ are said to be isomorphic, written $\Psi \cong \Phi$, if there exists an isomorphism $\delta : \Psi \rightarrow \Phi$. For any homomorphism $: \Psi \rightarrow \Phi$, the set $\{ ж \in \Psi : \delta (ж) = 0' \}$ is titled the kernel of δ , symbolized by $\ker(\delta)$, and the set $\{\delta(ж): ж \in \Psi\}$ is named the image of δ, represented by $Im(\delta)$. Sign that $\delta (0) = 0'$, $\forall$ homomorphism δ [5]. An ideal $\mathfrak{S}$ of Ψ is known as closed ideal of Ψ if: for each $ж \in \mathfrak{S}$.

We requisite $0 * ж \in \mathfrak{S}$ [6]. Let $\mathfrak{S}$ be an ideal of Ψ . It is named a closed ideal with respect to an element $s \in \Psi$ (symbolized by $s-$ closed ideal) if $s * (0 * ж) \in \mathfrak{S}, \forall ж \in \mathfrak{S}$ [7]. An ideal $\mathfrak{S}$ of Ψ is known as completely closed ideal if $ж * ц \in \mathfrak{S}$, $\forall$, $ц \in \mathfrak{S}$ [7] . Let $\mathfrak{S}$ be an ideal of Ψ and $s \in \mathfrak{S}$. It is named a completely closed with respect to an element s (know by $s-$ completely closed ideal) if: $s * (ж * ц) \in \mathfrak{S}, \forall ж, ц \in \mathfrak{S}$ [7]. In the next parts of our research, we will symbolize to BH- algebra $(€ ; * ,0)$ *by* € .

2. Closed SAH – Ideal with Respect to an Element of BH – Algebra

Definition (1)

An ideal $\mathfrak{Y}$ of € is named a SAH – ideal of € if it fillfulls the requirement:

$\forall \varsigma , \zeta \in \mathfrak{Y}$, if $(\varsigma^* * \zeta) \in \mathfrak{Y} , \zeta^* \in \mathfrak{Y} \to (\zeta^* * \varsigma) \in \mathfrak{Y}$, where $\varsigma^* = e * \varsigma$, and e is unit number, i.e: $\varsigma * e = 0$

Example (2)

Assume $€ = \{0, \mathfrak{w}, \mathfrak{v}\}$ with the binary operation $*$ symbolized by the subsequent table:

Table 1.

*	0	$\mathfrak{w}$	$\mathfrak{v}$
0	0	0	0
$\mathfrak{w}$	$\mathfrak{w}$	0	0
$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$	0

Then the ideal $\mathfrak{Y} = \{0, \mathfrak{v}\}$ is a SAH – ideal of €.

Definition (3)

Assume $\mathfrak{Y}$ is SAH – ideal of € , then $\mathfrak{Y}$ is known as closed SAH – ideal if it fulfills the requirement:

$\forall \varsigma , \zeta \in \mathfrak{Y}$ if $0 * (\varsigma^* * \zeta) \in \mathfrak{Y} \wedge 0 * \zeta^* \in \mathfrak{Y} \to 0 * (\zeta^* * \varsigma) \in \mathfrak{Y}$

Example (4)

Assume $€ = \{0,1,2,3\}$ with the binary operation $*$ definition by the ensuing table:

Table 2

*	0	1	2	3
0	0	1	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then, the ideal $\mathfrak{Y} = \{0,3\}$ is a closed SAH – ideal of €.

Remark (5)

We know that every SAH – ideal in € is closed SAH – ideal. But the converse not correct.

Example (6)

Consider $€ = \{0,1,2,3\}$ with a binary operation $*$ connoted by the ensuing table:

Table 3.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	2	0

$\mathfrak{Y} = \{0,3\}$ is a closed SAH – ideal of € but $\mathfrak{Y}$ doesn't SAH – ideal, because:

when $\varsigma = 2, \zeta = 1 \rightarrow \varsigma^* = 2, \zeta^* = 2$

$(0 * 2 = 0) \in \mathfrak{Y}, 0 * 2 = 0 \in \mathfrak{Y} \rightarrow (0 * 0 = 0) \in \mathfrak{Y}$, while

$(2 * 1 = 2) \notin \mathfrak{Y}, 2 \notin \mathfrak{Y} \rightarrow (2 * 2 = 0) \in \mathfrak{Y}$

Theorem (7)

Assume $\{\mathfrak{Y}_\lambda, \lambda \in \Lambda\}$ is a collocation of closed SAH – ideal of €. Then $\left(\bigcap_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$ is a closed SAH – ideal of € .

Proof

$\forall \varsigma, \zeta \in \left(\bigcap_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$

$\therefore \varsigma, \zeta \in \mathfrak{Y}_\lambda, \forall \lambda \in \Lambda$

$\Longrightarrow 0 * (\varsigma^* * \zeta) \in \mathfrak{Y}_\lambda$ and $0 * \zeta^* \in \mathfrak{Y}$ then $0 * (\zeta^* * \varsigma) \in \mathfrak{Y}_\lambda, \ \forall \lambda \in \Lambda$

Since each $\mathfrak{Y}$ is closed SAH – ideal $\forall \lambda \in \Lambda$

$\Rightarrow 0 * (\varsigma^* * \zeta) \in \left(\bigcap_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$ and $0 * \zeta^* \in \left(\bigcap_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$ then $0 * (\zeta^* * \varsigma) \in \left(\bigcap_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$

$\therefore \left(\bigcap_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$ is closed SAH – ideal of BH – algebra € . ■

Theorem (8)

Assume $\{\mathfrak{Y}_\lambda, \lambda \in \Lambda\}$ is a collocation of closed SAH – ideals of €. Then $\left(\bigcup_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$ is a closed SAH – ideal of € .

Proof

To prove that $\left(\bigcup_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$ is closed SAH – ideal

$\forall\, \varsigma, \zeta \in \left(\bigcup_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$

$\Rightarrow \exists\, \mathfrak{Y}_j \in \{\mathfrak{Y}_\lambda\}_{\lambda\in\Lambda}$ is a c − SAH – ideal

Such that $\forall\, \varsigma, \zeta \in \mathfrak{Y}_j$

$\Longrightarrow 0 * (\varsigma^* * \zeta) \in \mathfrak{Y}_j$ and $0 * \zeta^* \in \mathfrak{Y}$ so $0 * (\zeta^* * \varsigma) \in \mathfrak{Y}_j$

$\Rightarrow 0 * (\zeta^* * \varsigma) \in \left(\bigcup_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$

$\Rightarrow \left(\bigcup_{\lambda\in\Lambda} \mathfrak{Y}_\lambda\right)$ is closed SAH – ideal of € . ■

Theorem (9)

Assume $\{€_\lambda\}_{\lambda\in\Lambda}$ is a collocation of € and $\mathfrak{Y}_\lambda$ be a closed SAH – ideal of €, $\forall\lambda \in \Lambda$. Then ($\prod_{\lambda\in\Lambda} \mathfrak{Y}_\lambda$) is a closed SAH – ideal of the direct product of €.

Proof

$\forall\, (\varsigma_\lambda), (\zeta_\lambda) \in \mathfrak{Y}_\lambda$

$(0)(\varsigma_\lambda^*)(\zeta_\lambda) \in \prod_{\lambda\in\Lambda} \mathfrak{Y}_\lambda \wedge (0)(\zeta_\lambda^*) \in \prod_{\lambda\in\Lambda} \mathfrak{Y}_\lambda$

$\Rightarrow (0 * \varsigma^* * \zeta) \in \prod_{\lambda\in\Lambda} \mathfrak{Y}_\lambda \;\wedge\; (0 * \zeta^*) \in \prod_{\lambda\in\Lambda} \mathfrak{Y}_\lambda$

$0 * \varsigma^* * \zeta \in \mathfrak{Y}_\lambda \;\wedge\; 0 * \zeta^* \in \mathfrak{Y}_\lambda$ and

Since $\mathfrak{Y}$ is closed SAH – ideal $\forall\lambda \in \Lambda$,then

$\therefore 0 * \zeta^* * \varsigma \in \mathfrak{Y}_\lambda, \forall\lambda \in \Lambda$

$\Longrightarrow (0 * \zeta^* * \varsigma) \in \prod_{\lambda\in\Lambda} \mathfrak{Y}_\lambda$

$\Longrightarrow \prod_{\lambda\in\Lambda} \mathfrak{Y}_\lambda$ is closed SAH – ideal of € . ■

Definition (10)

Assume $\mathfrak{Y}$ is a closed SAH – ideal of € . Then $\mathfrak{Y}$ is named closed SAH – ideal with respect to an element s ∈ € (represented by s −closed SAH – ideal) if:

$s * \left(0 * (\varsigma^* * \zeta)\right) \in \mathfrak{Y} \ \wedge \ s * (0 * \zeta^*) \in \mathfrak{Y}$. Then $s * \left(0 * (\zeta^* * \varsigma)\right) \in \mathfrak{Y}$

Example (11)

Consider € = {0,1,2,3} with binary operation $*$ defined by the ensuing table:

Table 4.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	3	0	3
3	3	0	0	0

$\mathfrak{Y} = \{0,2\}$, $s = 3$ and $\mathfrak{Y}$ is 3 – closed SAH – ideal of €.

3. Completely Closed SAH – Ideal with Respect to an Element of BH – Algebra

Definition (12)

A SAH – ideal $\mathfrak{Y}$ of € is known as completely closed SAH – ideal if

$\varsigma * \zeta \in \mathfrak{Y} \ , \forall \varsigma, \zeta \in \mathfrak{Y}$ (represented by $\overline{SAH}$ –ideal).

Example (13)

In example (11), we have $\mathfrak{Y}$ is $\overline{SAH}$ – ideal of € since:

$0 * 0 = 0 \in \mathfrak{Y} \ , \ 0 * 2 = 0 \in \mathfrak{Y}$

$2 * 0 = 2 \in \mathfrak{Y} \ , \ 2 * 2 = 0 \in \mathfrak{Y}$

Definition (14)

A SAH – ideal $\mathfrak{Y}$ of € and s ∈ € , then $\mathfrak{Y}$ is named a completely closed SAH – ideal with respect to an element s ∈ € (represented by s − $\overline{SAH}$ – ideal)

If $s * 0 * (\varsigma * \zeta) \in \mathfrak{Y} \ , \ \forall \varsigma, \zeta \in \mathfrak{Y}$

Example (15)

In example (11), we have:

$\mathfrak{Y} = \{0,2\}$ and $s = 2$, then $\mathfrak{Y}$ is 2 – $\overline{SAH}$ – ideal since:

$2 * 0 * (0 * 0) = 2 \in \mathfrak{Y} \ \ , 2 * 0 * (0 * 2) = 2 \in \mathfrak{Y}$

$2 * 0 * (2 * 0) = 2 \in \mathfrak{Y} \ \ , 2 * 2 * (2 * 2) = 2 \in \mathfrak{Y}$

Remark (16)

In € every s − $\overline{SAH}$ – ideal is a s − closed SAH – ideal.

Proposition (17)

Assume 𝔜 is a $\overline{SAH}$ – ideal of € . Then 𝔜 is a s − $\overline{SAH}$ – ideal, ∀s ∈ 𝔜 **.**

Proof

Assume ∀ ς , ζ ∈ 𝔜

Mean while 𝔜 is $\overline{SAH}$ – ideal and s ∈ 𝔜

Then $s * 0 * (ς * ζ) ∈ 𝔜$. ∎

Theorem (18)

Assume (€;∗ ,0) and (∉;⊛, 0′) are BH – algebras and 𝔥 : € → ∉ is a BH – epimorphism and 𝔜 is a SAH – ideal in €, then 𝔥 (𝔜) is a SAH – ideal in ∉.

Proof

Assume (ς* ⊛ ζ) ∈ 𝔥(𝔜) ∧ ζ* ∈ 𝔥(𝔜) to prove (ζ* ⊛ ς) ∈ 𝔥(𝔜) , ∀ς , ζ ∈ 𝔜

⟹ ∃ a, b ∈ 𝔜 such that

𝔥 (a) = ς , 𝔥 (b) = ζ ,

$\left((𝔥(a))^* ⊛ 𝔥(b)\right) ∈ 𝔥(𝔜) \;\; ∧ \;\; (𝔥(b))^* ∈ 𝔥(𝔜)$

$\left((𝔥(a)^* ⊛ 𝔥(b)\right) ∈ 𝔥(𝔜) \;\; ∧ \;\; 𝔥(b)^* ∈ 𝔥(𝔜)$

$𝔥(a^* * b) ∈ 𝔥(𝔜) \;\; ∧ \;\; 𝔥(b^*) ∈ 𝔥(𝔜)$

$⟹ a^* * b ∈ 𝔜 \;\; ∧ \;\; b^* ∈ 𝔜$

$→ b^* * a ∈ 𝔜$

$→ 𝔥(b^* * a) ∈ 𝔥(𝔜)$

∵ 𝔥 is epimorphism

$⟹ 𝔥(b^*) ⊛ 𝔥(a) ∈ 𝔥(𝔜)$

$⟹ (𝔥(b))^* ⊛ (𝔥(a)) ∈ 𝔥(𝔜)$

$(ζ^* ⊛ ς) ∈ 𝔥(𝔜)$

∴ 𝔥(𝔜) is SAH – ideal in ∉ . ∎

Theorem (19)

Assume (€;∗ ,0) and (∉;⊛, 0′) are BH – algebras and 𝔥 : € → ∉ an epimorphism and 𝔜 is a SAH – ideal in € . Then 𝔥 (𝔜) is a closed SAH – ideal in ∉.

Proof

Assume 𝔜 is a SAH – ideal in €

𝔥 (𝔜) is SAH – ideal (theorem (18))

And by using remark (5)

$\mathfrak{h}(\mathfrak{Y})$ is a closed SAH – ideal in $\not\in$. ■

Remark (20)

Now each SAH – ideal of € is a $s - \overline{SAH}$ – ideal of €, $\forall s \in \mathfrak{Y}$.

Theorem (21)

Assume (€;∗ ,0) and ($\not\in$;⊛, 0′) are BH – algebras and : € → $\not\in$ is a epimorphism , if $\mathfrak{Y}$ is a $s - \overline{SAH}$ – ideal in € , then $\mathfrak{h}$ ($\mathfrak{Y}$) is a $\mathfrak{h}$ (s) $\overline{SAH}$ – ideal in $\not\in$.

Proof

Assume $\mathfrak{Y}$ is a $s - \overline{SAH}$ – ideal in € , then $s * (a * c) \in \mathfrak{Y}$, $\forall$ a , c $\in \mathfrak{Y}$

Since $\mathfrak{Y}$ is SAH – ideal , then $\mathfrak{h}$ ($\mathfrak{Y}$) is a SAH – ideal (theorem 18)

Assume ς , $\zeta \in \mathfrak{h}(\mathfrak{Y})$

$\Longrightarrow \exists$ m, n $\in \mathfrak{Y}$ such that $\mathfrak{h}(m) = \varsigma$, $\mathfrak{h}(n) = \zeta$

$\mathfrak{h}(s) \circledast (\varsigma \circledast \zeta) = \mathfrak{h}(s) \circledast (\mathfrak{h}(m) \circledast \mathfrak{h}(n))$

$= \mathfrak{h}(s) \circledast \mathfrak{h}(m * n)$

$= \mathfrak{h}(s * (m * n)) \in \mathfrak{h}(\mathfrak{Y})$ [since $s * (m * n) \in \mathfrak{Y}$]

$\therefore$ $\mathfrak{h}(\mathfrak{Y})$ is a $\mathfrak{h}(s)$ $\overline{SAH}$ – ideal. ■

Proposition (22)

Assume $\mathfrak{Y}$ is a SAH – ideal of € such that $\mathfrak{Y} \subseteq €_+$. Then $\mathfrak{Y}$ is s − closed SAH – ideal $\forall s \in \mathfrak{Y}$. Where $€_+ = \{\varsigma \in €: 0 * \varsigma = 0\}$.

Proof

Assume $s \in \mathfrak{Y}$ and $\subseteq €_+$.

Then $s * (0 * \varsigma) = s * 0$ [since $\mathfrak{Y} \subseteq €_+$] = s$\in \mathfrak{Y}$

$\therefore$ $\mathfrak{Y}$ is s − closed SAH – ideal . ■

4. Conclusion

In this paper , we constructed the idea of SAH – ideal , closed SAH – ideal ,s- closed SAH – ideal , $\overline{SAH}$ – ideal and s- $\overline{SAH}$ – ideal of BH – algebra which are presented with some of their properties , examples and theorems . In our future work, we introduce the concept of fuzzy SAH – ideal of BH – algebra. It is our optimism that this effort grows into other fundamentals for further study of ideas of BH-algebra.

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