Efficient Semi-Analytic Technique for Solving Nonlinear Singular Initial Value Problems

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Abstract
The aim of this paper is to present a semi-analytic technique for solving singular initial value problems of ordinary differential equations with a singularity of different kinds to construct polynomial solution using two point osculatory interpolation.

The efficiency and accuracy of suggested method is assessed by comparisons with exact and other approximate solutions for a wide classes of non–homogeneous, non–linear singular initial value problems.

A new, efficient estimate of the global error is used for adaptive mesh selection. Also, analyze some of the numerical aspects relevant for the implementation, describe measures to increase the efficiency of the code and compare its performance with the performance of established standard codes for singular initial value problems.

Many examples are presented to demonstrate the applicability and efficiency of the suggested method on one hand and to confirm the convergence order on the other hand.

Keyword: ODE, Singular initial value problems.
1. Introduction
The model of nonlinear phenomena in physics, engineering and other sciences, leads to singular initial value problems (SIVP) associated with nonlinear second order ordinary differential equations (ODE) [1].

The general form of the SIVP is:
\[ x^m y'' + f(x, y, y') = 0 \quad , \quad 0 < x < 1 \]  \hspace{1cm} (1)
with some initial conditions.

The object is to find a function \( y \) that satisfies the differential equation together with IC. We view \( f \) as a generally nonlinear function of \( y \) and \( y' \), but here it wins, we as look \( f = f(x) \) only. For such a problem to have a solution, it is generally necessary either that \( f(x) \neq 0 \) hold. When \( f(x) \equiv 0 \), and/or \( A = 0 , B = 0 \) the IVP is said to be homogeneous and will in general have only the trivial solution, \( y(x) \equiv 0 \) [2].

2. Solution of Second Order SIVP
In this section, we suggest a semi analytic technique which is based on osculatory interpolating polynomials \( P_{2n+1} \) and Taylor series expansion to solve second order SIVP, we consider the problem (1), where \( f \) are in general nonlinear functions of their arguments.

Now, to solve the problem by suggested method we do the following steps:

Step one :-
Evaluate Taylor series of \( y(x) \) abut \( x = 0 \):
\[ y = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \]  \hspace{1cm} (2)
where \( y(0) = a_0 \), \( y'(0) = a_1 \), \( y''(0) / 2! = a_2 \),…, \( y^{(i)}(0) / i! = a_i \), \( i = 3, 4,\ldots \)
And evaluate Taylor series of \( y(x) \) about \( x = 1 \):
\[ y = b_0 + b_1 (x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i \]  \hspace{1cm} (3)
where \( y(1) = b_0 \), \( y'(1) = b_1 \), \( y''(1) / 2! = b_2 \),… , \( y^{(i)}(1) / i! = b_i \), \( i = 3, 4,\ldots \)

Step two :-
Insert the series form (3) into equation (1) and put \( x = 1 \), then equate the coefficients of powers of \( (x-1) \) to obtain \( b_2 \).

Step three :-
Derive equation (1) with respect to \( x \), to get new form of equation say (4)
\[ x^m y''' + m x^{m-1} y'' + df(x, y, y') / dx = 0 \]  \hspace{1cm} (4)
then, insert the series form (2) into equation (4) and put \( x = 0 \) and equate the coefficients of powers of \( x \) to obtain \( a_2 \), again insert the series form (3) into equation (4) and put \( x = 1 \), then equate the coefficients of powers of \( (x-1) \) to obtain \( b_3 \).

Step four :-
Iterate the above process many times to obtain \( a_3 , b_4 \), then \( a_4 , b_5 \) and so on, that is, to get \( a_i \) and \( b_i \) for all \( i \geq 2 \), the resulting equations can be solved using MATLAB version 7.10, to obtain \( a_i \) and \( b_i \) for all \( i \geq 2 \).

Step five :-
The notation implies that the coefficients depend only on the indicated unknowns \( a_0 , a_1 , b_0 , b_1 \), and use the IC to get \( a_0 \) and \( a_1 \), therefore we have only two unknown coefficients \( b_0 \) and \( b_1 \). Now, we can construct two point osculatory interpolating polynomials \( P_{2n+1}(x) \) from these coefficients \( (a_i s \text{ and } b_i s) \) by the following:
\[ P_{2n+1} = \sum_{i=0}^{n} \{ a_i Q_i(x) + (-1)^i b_i Q_i(1-x) \} \quad \text{(5)} \]

where \( Q_j(x) / j! = (x^j / j!) (1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s \)

**Step six :-**

To find the unknown coefficients \( b_0 \) and \( b_1 \), integrate equation (1) on \([0, x]\) to obtain:

\[ x^m y'(x) - mx^{m-1}y(x) + m(m-1) \int_0^x x^{m-2} y(x) \, dx + \int_0^x f(x, y, y') \, dx = 0 \quad \text{(6a)} \]

and again integrate equation (6a) on \([0, x]\) to obtain:

\[ x^m y(x) - 2m \int_0^x x^{m-1} y(x) \, dx + m(m-1) \int_0^x x^{m-2} y(x) \, dx + \int_0^x (1-x) f(x, y, y') \, dx = 0 \quad \text{(6b)} \]

**Step seven :-**

Putting \( x = 1 \) in equations (6) to get:

\[ b_1 - mb_0 + m(m-1) \int_0^1 x^{m-2} y(x) \, dx + \int_0^1 f(x, y, y') \, dx = 0 \quad \text{(7a)} \]

and

\[ b_0 - 2m \int_0^1 x^{m-1} y(x) \, dx + m(m-1) \int_0^1 x^{m-2} y(x) \, dx + \int_0^1 (1-x) f(x, y, y') \, dx = 0 \quad \text{(7b)} \]

**Step eight :-**

Use \( P_{2n+1}(x) \) which defined in equation (5) as a replacement of \( y(x) \), we see that equations (7) have only two unknown coefficients \( b_0 \) and \( b_1 \). So, we can find \( b_0 \) and \( b_1 \) by solving the system of algebraic equations (7) using MATLAB, so insert the value of \( b_0 \) and \( b_1 \) into equation (5), thus equation (5) represents the solution of the problem.

**Note**

Extensive computations have shown that this generally provides a more accurate polynomial representation for a given \( n \).

3. **Examples**

In this section, many examples will be given to illustrate the efficiency, accuracy, implementation and utility of the suggested method. The bvp4c solver of MATLAB has been modified accordingly so that it can solve some class of SIVP as effectively as it previously solved nonsingular IVP.
Example 1

Consider the following 2nd order nonlinear SIVP:
\[ y'' + \frac{4}{x} y' + y^2 = x^6 + 4x^3 + 18x + 4 \]
with IC: \( y(0) = 2, y'(0) = 0 \)

It is clear that \( x = 0 \) is singular point and it is singularity of first kind and the exact solution is \( y(x) = x^3 + 2 \). Now, this example is solved by using semi-analytic technique, from equation (5) we have: \( P_3 = x^3 + 2 \)

Example 2

Consider the following 2nd order nonlinear SIVP:
\[ y'' + \frac{6}{x} y' + \frac{6}{x^2} y + y^2 = 20 + x^4 \]
with IC: \( y(0) = 0, y'(0) = 0 \)

It is clear that \( x = 0 \) is singular point and the exact solution is \( y(x) = x^2 \). Now, this example is solved by using semi-analytic technique, from equation (5) we have: \( P_3 = x^2 \)

Now, the suggested method is used to investigate the solution of Emden–Fowler equations as a class of nonlinear SIVP, which plays very important role because of the singularity behaviour at the origin. We study the Emden–Fowler equations of the form:
\[ y'' + \left( \frac{a}{x} \right) y' + b x^{m-1} y^n = g(x) ; \quad 0 \leq x \leq 1, a \geq 0, b \in \mathbb{R} \]  \( (8) \)
under the following initial conditions: \( y(0) = A, y'(0) = 0 \)

where \( A, a, b, m \) are constants, \( n \neq 0, n \neq 1 \). When \( m = 1, a = 2 \) and \( b = 1 \) equation (8) reduces to the Lane – Emden equation [3].

Example 3

Consider the following Emden – Fowler equation as a SIVP:
\[ y'' + \left( \frac{8}{x} \right) y' + x y^2 = x^5 + x^4 \]  \( ; \quad 0 \leq x \leq 1 \)
with IC: \( y(0) = 1, y'(0) = 0 \)

It is clear that \( x=0 \), is nonlinear singular point of the first kind, we solve this example by using semi-analytic technique, from equation (5), we have:
\[ P_9 = -0.00078826882x^9 + 0.0012952791x^8 + 0.00879760172x^7 + 0.014414828109x^6 - 0.000150192x^5 - 0.033333333333758x^3 + 1.0 \]
higher accuracy can be obtained by evaluating higher \( n \), now, take \( n=5 \), i.e.,
\[ P_{11} = -0.0021822695x^{11} + 0.0086427359x^{10} - 0.00189361335x^9 + 0.0014538812x^8 + 0.009581210401x^7 + 0.013781683x^6 - 0.033333333333758x^3 + 1.0 \]

Now, increase \( n \), to get higher accuracy, let \( n=6 \), i.e.,
Demir[3] solved this example by DTM and gave the following solution:

\[ y_{20}(x) = 1 - \frac{1}{30} x^3 + \frac{8}{585} x^4 + \frac{1}{98} x^5 - \frac{37}{187200} x^6 - \frac{1}{8330} x^9 + \frac{367}{64022400} x^{17} + \frac{23}{6497400} x^{13} \]

\[ - \frac{55049}{85830030000} x^{15} - \frac{2867}{3586564800} x^{19} + \frac{1618343}{102996036000000} x^{18} + \frac{164117}{885815056000} x^{10} + \frac{29}{5289883200} x^{20} \]

Also, this example is solved by numerical method by using the Maple11 as given in [3] and the results are compared with the suggested method and its results are given in table (1a) and (1b). Figure (1) give a comparison between the numerical solution in[3] and \( P_{13} \); table (2) gave the results obtained with DTM for different values of \( n \) which are given [3].

4. Error / Defect Weights

Every known IVP software package reports an estimate of either the relative error or the maximum relative defect. The weights used to scale either the error or the maximum defect differ among IVP software. Therefore, the IVP component of pythODE [4] allows users to select the weights they wish to use. The default weights depend on whether an estimate of the error or maximum defect is being used. If the error is being estimated, then the IVP component of pythODE uses, in this thesis we modify this package to consist SIVP and named " pythSODE ", defined as:

\[
\left\| y(x) - P(x) \right\|_{\infty} / (1 + \left\| P(x) \right\|_{\infty}) \quad ; \quad 0 \leq x \leq 1 ,
\]

where \( y(x) \) is exact solution and \( P(x) \) is suggested solution of SIVP (1).

If the maximum defect is being estimated, then the SIVP component of pythSODE uses:

\[
\left\| P(x) - P''(x) - f(x, P(x), P'(x)) \right\|_{\infty} / (1 + \left\| f(x, P(x), P'(x)) \right\|_{\infty}) \quad ; \quad 0 \leq x \leq 1 ,
\]

where \( P''(x) \) is the second derivative of \( P(x) \) and \( f(x, P(x), P'(x)) \) is the right-hand side of the SIVP.

The relative estimate of both the error and the maximum defect are slightly modified from the one used in SIVP SOLVER.

4.1 Accuracy of The Computed Values

The convergence orders of the approximations appear to be very stable, indicating that our results are reliable and accurate. In this section, we try to quantify the accuracy we can obtain. To examine the error of the approximation, let

\[
E(n) = \max_{1 \leq i \leq N} | y(x_i) - y_i |
\]

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where \( y(x_i) \) is the exact solution at the \( x_i \) and \( y_i \) suggested solution at \( x_i \). It is known that for a large \( N \), if \( M \) is the order of convergence of the method, then:

\[
E(n) \approx K |n|^M
\]

(12)

where \( K \) is the error constant and \( E(n) \) is the max error. Following the generalized algorithm for the order verification of methods given in [5], we can find \( M \) and \( K \) from the line:

\[
y = M x + \log K
\]

(13)

that best fits the equation:

\[
\log E(n) = \log K + M \log n
\]

(14)

where \( y \) represent \( \log E(n) \) and \( x \) represent \( \log n \).

Applying this algorithm for the following example:

\[
y" + \left( \frac{2}{x} \right) y' - 6y = -4y \ln y , \text{ with the IC: } y(0) = 1, y'(0) = 0
\]

the exact solution is \( y(x) = \exp(x - 2) \), we have from table (4) the best fit line whose graph is shown in figure (2), i.e.,

\( y = -6.6214 x - 1.3330 \)

From equation (13), we conclude that the order of convergence \( |M| \approx 6 \) with error constant \( K = 10^{1.3330} \).

6. Conclusions

Our decision to use semi-analytic technique for the solution of the underlying initial value problem was motivated by its advantageous convergence properties, whereas in presence of a singularity other direct higher order methods (finite difference) show order reductions and become inefficient. Moreover, the standard acceleration techniques like Iterated.

Our reason for choosing the global error estimate instead of monitoring the local error is the non-smoothness of the latter near the singular point and the order reductions it suffers from.

The results show that the convergence order estimates are increased, if we increase \( n \). This is not surprising.

References


Table No.(1a): The result of suggested method for n = 2, 3, 4 & method in [3] for example 3

<table>
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<th>b0</th>
<th>0.990235914910346</th>
<th>0.990235874852203</th>
<th>0.9902358795095490</th>
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<td>0.050589341476757</td>
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<td>Numerical solution (y)</td>
<td>P₀</td>
<td>P₁₀</td>
</tr>
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<td>0.999966681360863</td>
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Table No.(1b): Comparison between suggested method with method in [3] for example 3

<table>
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<th>x₁</th>
<th>Error</th>
<th>Error</th>
<th>Error</th>
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Table No.(2): Comparison of numerical results with the Maple11&DTM solution for different values of $n$

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<th>Maple11</th>
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<th>$n = 30$</th>
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Table No. (3) : Max. error of example 3

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<th>P13&quot;</th>
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<td>$x_{i}$</td>
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<td>Error</td>
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Table No. (4) : The log of max. error of the suggested solution of example

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<tr>
<th>P_{2n+1}</th>
<th>n</th>
<th>Log</th>
<th>Max error</th>
<th>Log of max error</th>
</tr>
</thead>
<tbody>
<tr>
<td>P_{5}</td>
<td>0.301029956563981</td>
<td>3.904271375693158e-004</td>
<td>-3.408460003282497</td>
<td></td>
</tr>
<tr>
<td>P_{7}</td>
<td>0.477121254719662</td>
<td>5.080546580960021e-005</td>
<td>-4.294089562455153</td>
<td></td>
</tr>
<tr>
<td>P_{9}</td>
<td>0.602059991327962</td>
<td>3.669790521687588e-006</td>
<td>-5.435358725358707</td>
<td></td>
</tr>
</tbody>
</table>
Figure No.(1): Comparison between the numerical solution in [3] and \( P_{13} \) for example 3

Figure No.(2): The best fit line for the order of convergence
کفاءة التقنية شبه التحليلية في حل مسائل القيم الابتدائية غير الخطية

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الخلاصة

الفكرة من هذا البحث اقتراح تقنية شبه التحليلية لحل مسائل القيم الابتدائية الشاذة للمعادلات الفاضلية الاعتيدية. ويمنع وجود سهولة بحتة من خلال المقارنات مع الحل المضغوط و حلول تقريبية أخرى لأصناف واسعة متجانسة وغير متجانسة. يثبت الاستخدام غير خطي للعوارض القيم الابتدائية الشاذة

استخدم تقنيات حديثة و حديثة للحول النتائج. لاختبار الشبكة المناسبة أيضا نفهمنا بعض الصفات المعقدة في العلاقة لتمثيل المدخلات ولأفضل الكفاءة في أداء الطريقة المختارة من جهة و تأكيد رتبة التقارب من جهة أخرى.

الكلمات المفتاحية: معادلات تعادلية الاعتيدية، مسائل القيم الابتدائية الشاذة

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