Existence and Uniqueness of The Solution of Nonlinear Volterra Fuzzy Integral Equations

Eman A. Hussain
Ayad W. Ali
Department of Mathematics/College of Science/
University of AL-Mustansiriyah

Abstract

In this paper, we proved the existence and uniqueness of the solution of nonlinear Volterra fuzzy integral equations of the second kind.

Keywords: Volterra fuzzy integral equations.

Introduction

The concept of integration of fuzzy functions has been introduced by Dubois and Prade [1], Goetschel and Voxman [2], Kaleva [3] and others. However, if the fuzzy function is continuous, all the various procedures yield the same result. The fuzzy integral is applied in fuzzy integral equations, such that there is a growing interest in fuzzy integral equations particularly in the past decade. The fuzzy integral equations have been studied by authors of [4,5,6,7] and others.

In this paper, the existence and uniqueness theorem is proved for nonlinear Volterra fuzzy integral equation under the Lipschitz condition and arbitrary kernels by means of the successive iterations involving fuzzy set-valued function of a real variable where values are normal, convex, upper semi continuous and compactly supported fuzzy sets in R^n.

The authors of [8] proved the existence and uniqueness of the solution of linear Volterra fuzzy integral equations of the second kind. P. Prakash and V. Kalaiselvi [9] proved the existence and uniqueness of the solution of nonlinear Volterra fuzzy integral equations with infinite delay of the form

$$\acute{u}(x) = f\big(x, u(x)\big) + \int_{-\infty}^{x} g(x, t, u(t))\, dt, \quad x \in J = (-\infty, \infty) \qquad (1.1)$$

where $f: J \times E^n \to E^n$ and $g: J \times J \times E^n \to E^n$ are levelwise continuous and satisfy the generalized Lipschitz condition. K. Balachandran and K. Kanagarajan in [10] proved the existence and uniqueness of the solution of general nonlinear Volterra-Fredholm fuzzy integral equations of the form

$$u(x) = F(x, u(x), \int_0^x f_1(x, t, u(t)\, dt, \dots, \int_0^x f_m(x, t, u(t)\, dt, \qquad (1.2)$$
$$\int_0^b g_1(x, t, u(t)\, dt, \dots, \int_0^b g_m(x, t, u(t)\, dt)\,, \ 0 \le x \le b,$$

The purpose of this paper is to prove the existence and uniqueness of the solution of nonlinear Volterra fuzzy integral equations of the second kind

$$u(x) = f(x) + \int_a^x K\big(x, t, u(t)\big) dt \qquad (1.3)$$

where f is fuzzy continuous function on $I = [a, b]$, K is continuous fuzzy function over the region $\Delta = I \times I \times E^n = \{(x, t, u(t)) | a \le t \le x \le b,\ u(t) \in E^n\}$ and $u(x)$ is the solution of equation (1.3) to prove its existence and uniqueness.

Preliminaries

By $P_K(R^n)$, we denote the family of all nonempty compact convex subsets of R^n. Let $I = [a, b]$ be a compact interval and denote [3]

$$E^n = \{p : R^n \to [0,1]\}$$

such that p satisfies (i) through (iv) below

i) p is normal i.e. there exists an $x_0 \in R^n$ such that $p(x_0) = 1$,
ii) p is fuzzy convex,
iii) p is upper semi continuous, i.e the $\alpha -$ level sets $[p]^\alpha$ are closed for each $\alpha \in [0,1]$,
iv) $[p]^0 = cl\{x \in R^n | p(x) > 0\}$ is compact.

where the $\alpha -$ level sets $[p]^\alpha$ is defined by $[p]^\alpha = \{x \in R^n | p(x) \ge \alpha\}$ for $0 < \alpha \le 1$ and $[p]^0$ for $\alpha = 0$. Then from (i)-(iv), it follows that $[p]^\alpha \in P_K(R^n)$ for all $0 \le \alpha \le 1$.

If $g: R^n \times R^n \to R^n$ is a function, then using Zadeh's extension principle, we can extend g to $\tilde{g}: E^n \times E^n \to E^n$ by the relation

$$\tilde{g}(p,q)(z) = \sup_{z=g(x,y)} \min\{p(x), q(y)\}$$

for each $p, q \in E^n$, $0 \le \alpha \le 1$ and continuous function g. It is well known that

$$[g(p,q)]^\alpha = g([p]^\alpha, [q]^\alpha)$$

Moreover, we have

$[p+q]^\alpha = [p]^\alpha + [q]^\alpha$, $[kp]^\alpha = k[p]^\alpha$, where $k \in R$.

Define $D: E^n \times E^n \to R^+$ by

$$D(p,q) = \sup_{0\le\alpha\le 1} d([p]^\alpha, [q]^\alpha),$$

where d is the Hausdorf metric defined in $P_K(R^n)$ by

$$d(A,B) = \max\{\sup_{x\in A} \inf_{y\in B} |x-y|, \sup_{y\in B} \inf_{x\in A} |x-y|\}$$

for each $A, B \in P_K(R^n)$, then D is a metric in E^n.

Definition 2.1 [3] A function $F: I \to E^n$ is called strongly measurable, if for all $\alpha \in [0,1]$ the set-valued function $F_\alpha: I \to P_K(R^n)$ is defined by

$$F_\alpha(x) = [F(x)]^\alpha$$

is Lebesgue measurable, where $P_K(R^n)$ has the topology induced by the Hausdorf metric d.

Definition 2.2 [3] A function $F: I \to E^n$ is called integrably bounded if there exists an integrable function h such that $\|y\| < h(x)$for all $y \in F_0(x)$.

Definition 2.3 [3] Let $F: I \to E^n$. The integral of F over I, denoted by $\int_I F(x)\,dx$ or $\int_a^b F(x)\,dx$, is defined levelwise by

$$[\int_I F(x)dx]^\alpha = \int_I F_\alpha(x)dx = \{\int_I f(x)dx \,|f: I \to R^n \text{ is a measurable function for } F_\alpha\}$$

for all $0 \le \alpha \le 1$.

Definition 2.4 [3] A function $F: I \to E^n$ is called levelwise continuous at $t_0 \in I$ if the set-valued function $F_\alpha(x) = [F(x)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorf metric d for all $\alpha \in [0,1]$.

Proposition 2.1 [3] Let $F, G: I \to E^n$ be integrable and $\theta \in R$. Then

1. $\int (F+G) = \int F + \int G$,
2. $\int \theta F = \theta \int F$,
3. $D(F,G)$ is integrable,

4. $D(\int F, \int G) \leq \int D(F,G)$.

Theorem 2.1 [3,11] For any $p, q, r, s \in E^n$ and $\theta \in R$, then the following hold

- (E^n, D) is a complete metric space,
- $D(\theta p, \theta q) = |\theta| D(p,q)$,
- $D(p+r, q+r) = D(p,q)$,
- $D(p+q, r+s) \leq D(p,r) + D(q,s)$.

Definition 2.5 [12] A function $F: I \to E^n$ is called bounded if there exists a constant $M > 0$ such that $D(F(x), \tilde{0}) \leq M$ for all $x \in I$.

Definition 2.6 [7] A function $F: I \to E^n$ is said to be continuous if for arbitrary fixed $x_0 \in I$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $D(F(x), F(x_0)) < \varepsilon$ for each $x \in I$.

Main Results

Theorem 3.1 (Existence and uniqueness)

Assume the following conditions are satisfied

i) $f:[a,b] \to E^n$ is continuous and bounded,

ii) $K: \Delta \to E^n$ is a continuous function,

iii) if $u, v:[a,b] \to E^n$ are continuous, then the Lipschitz condition

$$D\big(K(x,t,u(t)), K(x,t,v(t))\big) \leq L D(u(x), v(x)) \tag{3.1}$$

is satisfied, with $0 < L < \dfrac{1}{b-a}$.

where $\Delta = I \times I \times E^n = \{(x,t,u(t)) | a \leq t \leq x \leq b,\ u(t) \in E^n\}$.

Then there exists a unique fuzzy solution $u(x)$ of (1.3) and the successive iterations

$$\psi_o(x) = f(x)$$
$$\psi_{n+1}(x) = f(x) + \sum_{i=1}^{n+1} \int_a^x K(x,t,u_{i-1}(t))dt, \quad n \geq 0 \tag{3.2}$$

are uniformly convergent to $u(x)$ on $[a,b]$; where

$$u_o(x) = f(x)$$
$$u_n(x) = \int_a^x K(x,t,u_{n-1}(t))dt, \quad n \geq 1 \tag{3.3}$$

First we prove the following Lemma.

Lemma 3.1 If the conditions of Theorem (3.1) are hold and u_n is given by (3.3) then for $n \geq 0$

I) $u_n(x)$ is bounded,

II) $u_n(x)$ is continuous.

Proof:

I) Clearly $u_o(x)=f(x)$ is bounded by part (i) of theorem (3.1). Assume $u_{n-1}(x)$ is bounded. From (3.1), (3.3) and proposition (2.1) we have

$$D(u_n(x),\tilde{0})=D\left(\int_a^x K(x,t,u_{n-1}(t))dt,\tilde{0}\right)\le\int_a^x D\big(K(x,t,u_{n-1}(t)),\tilde{0}\big)dt$$
$$\le L\int_a^x D(u_{n-1}(x),\tilde{0})dt$$
$$\le L\sup_{x\in[a,b]} D(u_{n-1}(x),\tilde{0})\int_a^x dt$$
$$\le (b-a)L\sup_{x\in[a,b]} D(u_{n-1}(x),\tilde{0}),$$

where $\tilde{0}$ is the zero function. Hence by induction $u_n(x)$ is bounded.

II) To prove the continuity of $u_n(x)$, we suppose $a\le x\le\hat{x}\le b$, hence by proposition (2.1) and theorem (2.1) we have

$$D(u_n(x),u_n(\hat{x}))=D\left(\int_a^x K(x,t,u_{n-1}(t))dt,\int_a^{\hat{x}} K(\hat{x},t,u_{n-1}(t))dt\right)$$
$$=D\left(\int_a^x K(x,t,u_{n-1}(t))dt,\int_a^x K(\hat{x},t,u_{n-1}(t))dt+\int_x^{\hat{x}} K(\hat{x},t,u_{n-1}(t))dt\right)$$
$$\le D\left(\int_a^x K(x,t,u_{n-1}(t))dt,\int_a^x K(\hat{x},t,u_{n-1}(t))dt\right)+D\left(\int_x^{\hat{x}} K(\hat{x},t,u_{n-1}(t))dt,\tilde{0}\right)$$
$$\le\int_a^x D\big(K(x,t,u_{n-1}(t)),K(\hat{x},t,u_{n-1}(t))\big)dt+\int_x^{\hat{x}} D\big(K(\hat{x},t,u_{n-1}(t)),\tilde{0}\big)dt$$
$$\le (b-a)\sup_{t\in[a,b]} D\big(K(x,t,u_{n-1}(t)),K(\hat{x},t,u_{n-1}(t))\big)+LD\big(u_{n-1}(x),\tilde{0}\big)\int_x^{\hat{x}} dt$$
$$\le (b-a)\sup_{t\in[a,b]} D\big(K(x,t,u_{n-1}(t)),K(\hat{x},t,u_{n-1}(t))\big)+L(\hat{x}-x)\sup_{x\in[a,b]} D\big(u_{n-1}(x),\tilde{0}\big)$$

Since K is continuous, we obtain

$$D(u_n(x),u_n(\hat{x}))\to 0 \quad \text{as} \quad x\to\hat{x}.$$

Thus $u_n(x)$ is continuous on $[a,b]$.

Proof of Theorem (3.1)

We shall prove that all $\psi_n(x)$, $n\ge 0$ are bounded on $[a,b]$. It is clear that $\psi_o(x)=f(x)$ is bounded by the assumption. Suppose that $\psi_{n-1}(x)$ is bounded. From (3.2) and theorem (2.1) we have

$$D(\psi_n(x),\tilde{0})=D\left(f(x)+\sum_{i=1}^{n}\int_a^x K(x,t,u_{i-1}(t))dt,\tilde{0}\right)$$
$$=D\left(f(x)+\sum_{i=1}^{n-1}\int_a^x K(x,t,u_{i-1}(t))dt+\int_a^x K(x,t,u_{n-1}(t))dt,\tilde{0}\right)$$

$$= D\left(\psi_{n-1}(x) + \int_a^x K(x,t,u_{n-1}(t))dt, \tilde{0} \right)$$

$$\le D\left(\psi_{n-1}(x), \tilde{0}\right) + D\left(\int_a^x K(x,t,u_{n-1}(t))dt, \tilde{0} \right)$$

$$= D(\psi_{n-1}(x), \tilde{0}) + D(u_n(x), \tilde{0})$$

From induction and Lemma (3.1) part (I) we have that $\psi_n(x)$ is bounded. Consequently, $\{\psi_n(x)\}_{n=0}^{\infty}$ is a sequence of bounded functions on $[a,b]$.

In the following, we prove that $\psi_n(x)$ are continuous on $[a,b]$. By Lemma (3.1) part (II) and theorem (2.1) and proposition (2.1) for $a \le x \le \hat{x} \le b$, we have

$$D\left(\psi_n(x), \psi_n(\hat{x})\right) = D\left(f(x) + \sum_{i=1}^{n} \int_a^x K(x,t,u_{i-1}(t))dt,\ f(\hat{x}) + \sum_{i=1}^{n} \int_a^{\hat{x}} K(\hat{x},t,u_{i-1}(t))dt \right)$$

$$\le D\left(f(x), f(\hat{x})\right) + D\left(\sum_{i=1}^{n} \int_a^x K(x,t,u_{i-1}(t))dt, \sum_{i=1}^{n} \int_a^{\hat{x}} K(\hat{x},t,u_{i-1}(t))dt \right)$$

$$= D\left(f(x), f(\hat{x})\right) + D\left(\sum_{i=1}^{n} \int_a^x K(x,t,u_{i-1}(t))dt, \sum_{i=1}^{n} \int_a^{x} K(\hat{x},t,u_{i-1}(t))dt + \sum_{i=1}^{n} \int_x^{\hat{x}} K(\hat{x},t,u_{i-1}(t))dt \right)$$

$$\le D\left(f(x), f(\hat{x})\right) + D\left(\sum_{i=1}^{n} \int_a^x K(x,t,u_{i-1}(t))dt, \sum_{i=1}^{n} \int_a^{x} K(\hat{x},t,u_{i-1}(t))dt \right)$$

$$+ D\left(\sum_{i=1}^{n} \int_x^{\hat{x}} K(\hat{x},t,u_{i-1}(t))dt, \tilde{0} \right)$$

$$\le D\left(f(x), f(\hat{x})\right) + \int_a^x D\left(\sum_{i=1}^{n} K(x,t,u_{i-1}(t)), \sum_{i=1}^{n} K(\hat{x},t,u_{i-1}(t)) \right) dt$$

$$+ \int_x^{\hat{x}} D\left(\sum_{i=1}^{n} K(\hat{x},t,u_{i-1}(t)), \tilde{0} \right) dt$$

$$\le D\left(f(x), f(\hat{x})\right) + (b-a) \sup_{t\in[a,b]} D\left(\sum_{i=1}^{n} K(x,t,u_{i-1}(t)), \sum_{i=1}^{n} K(\hat{x},t,u_{i-1}(t)) \right)$$

$$+ (\hat{x} - x) \sup_{t\in[a,b]} D\left(\sum_{i=1}^{n} K(\hat{x},t,u_{i-1}(t)), \tilde{0}\right)$$

Finally we obtain

$D(\psi_n(x), \psi_n(\hat{x})) \to 0$ as $x \to \hat{x}$.

Therefore the sequence $\{\psi_n(x)\}_{n=0}^{\infty}$ is continuous on $[a,b]$.

To prove uniform convergence of the sequence $\{\psi_n(x)\}_{n=0}^{\infty}$, for $n \ge 0$ we have

$$D\left(\psi_{n+1}(x), \psi_n(x)\right) = D\left(f(x) + \sum_{i=1}^{n+1} \int_a^x K(x,t,u_{i-1}(t))dt, \psi_n(x) \right)$$

$$= D\left(f(x) + \sum_{i=1}^{n} \int_a^x K(x,t,u_{i-1}(t))dt + \int_a^x K(x,t,u_n(t))dt, \psi_n(x) \right)$$

$$= D\left(\psi_n(x) + \int_a^x K(x,t,u_n(t))dt, \psi_n(x) \right)$$

$$= D\left(\int_a^x K(x,t,u_n(t))dt, \tilde{0}\right)$$
$$\leq \int_a^x D\left(K(x,t,u_n(t)), \tilde{0}\right)dt$$
$$\leq \int_a^x LD\left(u_n(x), \tilde{0}\right)dt$$
$$\leq (b-a)L \sup_{x\in[a,b]} D\left(u_n(x), \tilde{0}\right).$$

Hence we obtain

$$\sup_{x\in[a,b]} D(\psi_{n+1}(x), \psi_n(x)) \leq (b-a)L \sup_{x\in[a,b]} D(u_n(x), \tilde{0}) \qquad (3.4)$$

On the other hand, by (3.1) we can obtain for $n \geq 1$,

$$D(u_n(x), \tilde{0}) = D\left(\int_a^x K(x,t,u_{n-1}(t))dt, \tilde{0}\right)$$
$$\leq \int_a^x D\left(K(x,t,u_{n-1}(t)), \tilde{0}\right)dt$$
$$\leq (b-a)LD(u_{n-1}(x), \tilde{0})$$

by the same way we have

$$D(u_{n-1}(x), \tilde{0}) \leq (b-a)LD(u_{n-2}(x), \tilde{0})$$

Thus we obtain

$$D(u_n(x), \tilde{0}) \leq (b-a)LD(u_{n-1}(x), \tilde{0})$$
$$\leq \{(b-a)L\}^2 D(u_{n-2}(x), \tilde{0})$$

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$$\leq \{(b-a)L\}^n D(u_0(x), \tilde{0}) = \{(b-a)L\}^n D(f(x), \tilde{0})$$
$$\leq \{(b-a)L\}^n \sup_{x\in[a,b]} D(f(x), \tilde{0})$$

this implies that

$$\sup_{x\in[a,b]} D(u_n(x), \tilde{0}) \leq Q\{(b-a)L\}^n \qquad (3.5)$$

where $Q = \sup_{x\in[a,b]} D(f(x), \tilde{0})$. For $n \geq 0$, from (3.4) and (3.5) we obtain

$$D(\psi_{n+1}(x), \psi_n(x)) \leq \sup_{x\in[a,b]} D(\psi_{n+1}(x), \psi_n(x)) \leq Q\{(b-a)L\}^{n+1}$$

The series $Q(b-a)L\sum_{n=0}^{\infty}\{(b-a)L\}^n$ is convergent, hence the series $\sum_{n=0}^{\infty} D(\psi_{n+1}(x), \psi_n(x))$ is convergent uniformly on $[a,b]$ by the comparison test, this implies the uniform convergence of the sequence $\{\psi_n(x)\}_{n=0}^{\infty}$. If we denote $u(x)=\lim_{n\to\infty}\psi_n(x)$, then $u(x)$ satisfies (1.3). It is obviously continuous and bounded on $[a,b]$.

At last, we prove the uniqueness of solution. Let $u(x)$ and $v(x)$ be two continuous solutions of (1.3) on $[a,b]$, then

$$0 \le D(u(x),v(x)) = D(u(x)+\psi_n(x), v(x)+\psi_n(x))$$
$$\le D(u(x),\psi_n(x)) + D(v(x),\psi_n(x))$$

and since $\psi_n(x)$ is convergent to solution of (1.3), then

$D(u(x),\psi_n(x)) \to 0,$

$D(u(x),\psi_n(x)) \to 0,$

when $n\to\infty$, then $D(u(x),v(x))=0$ that is $u(x)=v(x)$. This completes the proof.

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وجود ووحدانية الحل لمعادلات فولتيرا التكاملية الضبابية اللاخطية

إيمان علي حسين
أياد ولي علي
قسم الرياضيات / كلية العلوم / الجامعة المستنصرية

الخلاصة

في هذا البحث برهنّا وجود ووحدانية الحل لمعادلات فولتيرا التكاملية الضبابية اللاخطية من النوع الثاني.

الكلمات المفتاحية : معادلات فولتيرا التكاملية الضبابية.