The Maximum Complete (k,n)-Arcs in the Projective Plane PG(2,4) By Geometric Method

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Abstract

A (k,n)-arc A in a finite projective plane PG(2,q) over Galois field GF(q), $q=p^n$ for same prime number p and some integer $n\geq 2$, is a set of k points, no n+1 of which are collinear.

A (k,n)-arc is complete if it is not contained in a(k+1,n)-arc.

In this paper, the maximum complete (k,n)-arcs, n=2,3 in PG(2,4) can be constructed from the equation of the conic.

1. Introduction

[1], found the complete (k,2)-arcs in the projective planes over Galois field GF(p^n) for some prime number p and some integer n. [2] found an algebraic method for construction of (k,4)-arcs in the projective plane PG(2,4).

In this paper, the maximum complete (k,n)-arcs in PG(2,4), n=2,3 are obtained from the equation of the conic by geometric method.

A projective plane PG(2,q) over GF(q) consists of $1+q+q^2$ points and $1+q+q^2$ of lines, every line contains 1+q points end every point is on 1+q lines. Any point of the plane has the form of triple (x_0,x_1,x_2), where x_0,x_1,x_2 are elements of GF(q) with the exception of a triple consisting of three zero elements, two triples represent the same point if there exists λ in GF(q)\ {0},
s.t. $(y_0,y_1,y_2) = \lambda(x_0,x_1,x_2)$.

Similarly, any line of the plane has the from of a triple $[x_0,x_1,x_2]$, x_0,x_1,x_2 are in GF(q) with the exception of a triple consisting three zero elements. Two lines $[x_0,x_1,x_2]$ and $[y_0,y_1,y_2]$ represent the same line if there exists λ in GP(q)\{0}
s.t. $[x_0,x_1,x_2] = \lambda[y_0,y_1,y_2]$.
The point (x_0,x_1,x_2) is incident with line $[y_0,y_1,y_2]$ if $x_0y_0+x_1y_1+x_2y_2 = 0$.

2. Basic Definitions and Theorems

2.1 Definition [3]

A (k,n)-arc in a finite projective plane as a set of k points no n+1 of which are collinear.

2.2 Definition [4]

A (k,n)-arc is a set of k point, no three of them are collinear, we denote this by k-arc.

2.3 Definition [5]

A (k,n)-arc is said to be complete if it is not contained in a (k+1,n)-arc.
We denote by m(2,p) the maximum number of points in PG(2,p) that a (k,n)-arc can have.

2.4 Theorem [6]

A (k,n)-arc in PG(2,p) is complete if and only if $C_0 = 0$.

Proof: ($\Rightarrow$)

Let a (k,n)-arc K be a complete arc in PG(2,p) suppose that $C_0 \neq 0$, then there is at lest one point say N has index zero and N $\notin$ K. Then K $\cup$ {N} is an arc in PG(2,p).
Hence K $\subseteq$ K $\cup$ {N}, which implies (k,n)-arc K is incomplete (contradiction).

($\Leftarrow$)

Suppose that C_0=0 for the (k,n)-arc K, then there are no points of index zero, then the (k,n)-arc K is complete.

2.5 Definition [4]

A k-arc is called an oval when k = (2,p).

2.6 Definition [7]

Let ℓ be any line in PG(2,p) if ℓ intersects a k-arc in i-points, $|\ell \cap \mathcal{K}| = i$, then ℓ is called an i-secant of k,
Then 2-secont of $\mathcal{K}$ is called a bisecant of $\mathcal{K}$
Then 1-secont of $\mathcal{K}$ is called a unisecant of $\mathcal{K}$
Then 0-secont of $\mathcal{K}$ is called an external of $\mathcal{K}$.

2.7 Definition [3]

Let N be a point in PG(2,p) and N is not on a k-arc, then we say N is a point of index i if there are exactly i-bisecant through N .

2.8 Definition [5]

The set C_i consists of all points of index i
$C_i = |C_i|$ = # the number of points in C_i and # the number of index i.

2.9 Theorem [5]

Let M be a point of the k-arc of PG(2,k) and t(M) be the number of the unisecants of K through M, then t(M) = p+2-k = t.

Proof:

The number of lines in PG(2,p) through any point is p+1 there are exactly p+1 lines through M the point M with any other point of the k-arc determine a bisecant of $\mathcal{K}$ since there are (k-1) points of $\mathcal{K}$ other then M, then there are exactly (k-1) lines determined from M and the other point of $\mathcal{K}$ which are the bisecants of $\mathcal{K}$ through M since any line through M is either a bisecant or a unisecant, then the number of unisecants of $\mathcal{K}$ through M = p+1-(k-1) = p+2-k = t.

2.10 Notation:

T_i = the number of i-secant of a k-arc,
T_2 = the number of bisecant lines of a k-arc.
T_1 = the number of unsecant lines of a k-arc.
T_0 = the number of external lines of a k-arc.

2.11 Definition [4]

Let $\mathcal{K}$ be a k-arc which is an oval an external point to an oval, is a point of intersection two unisecants of K.

2.12 Theorem [5]

$$m(2,p) = \begin{cases} p+1 & \text{for } p \text{ odd} \\ p+2 & \text{for } p \text{ even} \end{cases}$$

2.13 Theorem [5]

The number of external points of an oval k-arc in PG(2,p) is $\frac{p(p+1)}{2}$.

Proof:

If $\mathcal{K}$ is an 0val, then $m(2,p) = \begin{cases} p+1 & \text{for } p \text{ odd} \\ p+2 & \text{for } p \text{ even} \end{cases}$

P is odd $\rightarrow$ k = p+1
t = p+2-k = p+2-(p+1) = 1
T_1 = kt = (p+1)*1 = p+1 = the number of unisecants of k in PG(2,p) ,
Since each two unisecants intersect in an external point, then the number of external points =

$$\binom{p+1}{2} = \frac{(p+1)!}{(p+1-2)!2!} = \frac{(p+1)p(p-1)!}{(p-1)!2*1} = \frac{p(p+1)}{2} .$$

2.14 Definition [3]

An internal point to an oval if it is not any unisecant of the oval which is not on the oval.

2.15 Theorem [5]

The number of the internal points to on oval is $\frac{p(p-1)}{2}$.

Proof:

The number of the external points + the number of the internal points = p^2+p+1- k.

# of external points = $\frac{p(p+1)}{2}$

of the internal points $= p^2 + p + 1 - (p+1) - \frac{p(p+1)}{2}$

$$= \frac{2p^2 + 2p + 2 - 2p - 2 - p^2 - p}{2}$$

$$= \frac{p^2 - p}{2} = \frac{p(p-1)}{2} .$$

2.16 Theorem [7]

In PG(2,p) with p $\geq$ 4, there is a unique conic through a 5-arc .

2.17 Theorem [4,5]

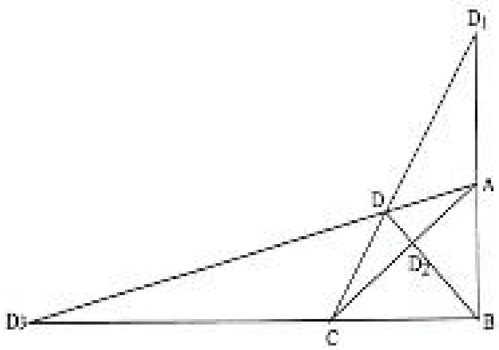
Every conic in PG(2,p) is a(p+1)-arc
The converse of theorem is also satisfied.

2.18 Theorem [3]

In PG(2,p), with p odd, every oval has a conic.

2.19 Definition [8]

A complete quadrangle is a set of four points A, B, C and D in which no three of them are collinear, the points A, B, C and D are called the vertices of the quadrangle, the lines joining any two vertices are called the sides which are AB, AC, BD, BC, AD, CD. Two sides are said to be opposite if they have no vertex in common. The point of intersection of any two opposite sides is called a diagonal point the diagonal points; D_1 = AB $\cap$CD, D2 = AC $\cap$ BD and D_3 = AD$\cap$ BC.



3. Maximum Complete (k,n)-Arcs in PG(2,4)

3.1 The Additions and Multiplications Operations of GF(4) [9]

To find addition and multiplication tables in GF(4), we have the order pairs (x_1,x_2) such that x_1,x_2 in GF(2), as follows: 0$\equiv$(0,0), 1$\equiv$(1,0), 2$\equiv$(0,1), 3$\equiv$(1,1). Put these points in one orbit, (1,0) at the first point and by the principle of $(1,0)A^i$, i=0,1,2,3 and $A=\begin{bmatrix}01\\11\end{bmatrix}$,

$(1,0)A=(0,1)$ and $(1,0)A^2=(1,1)$, So $(1,0)\begin{bmatrix}01\\11\end{bmatrix}\Bigg|\begin{matrix}(0,1)\\(1,1)\end{matrix}$.

Now, in the left of the following table, m is the operation of multiplication and in the right n is the operation of addition, in multiplication side we write the numeration of point as last, and the addition side takes the normal sequence.

m(*)		(+)n = f(m)
1	(1,0)	0
2	(0,1)	1
3	(1,1)	2
Mod 3		

In addition table, we have the following relation:
$(x_1,x_2) + (y_1,y_2) = (z_1,z_2)$ where $z_i = (y_i + x_i)$ mod(2) for i =1,2
In multiplication table we have the following relation:
$((1,0)\,A^{f(m_1)})\,A^{f(m_2)} \Leftrightarrow m_1 * m_2 = m_3$
$=(1,0)\,A^{(f(m_1)+f(m_2))\,(\mathrm{mod}\ 3)}$
$= (x_1,x_2)$

For example : $2*3 = 1 \Leftrightarrow ((1,0)A^1)A^2 = (1,0)A^3 = (1,0)A^0 = (1,0)$, where (1,0) is equal to 1 in multiplication tables.

The additions and multiplications operations of GF(4) are in table (1).

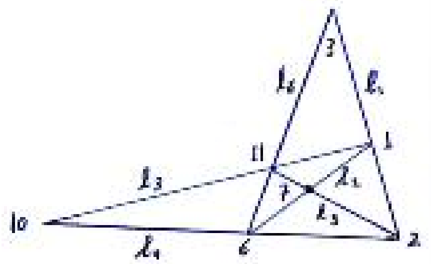
3.2 The Projective Plane PG(2,4)

The projective plane PG(2,4) contains 21 points, 21 lines, 5 points on every line and 5 lines through every point. Let P_i and L_i ,i=1,2,---,21 be points and the lines of PG(2,4) respectively, the points and lines of PG(2,4) are in table (2).

3.3 The Construction of k-arc in PG(2,4)

Let A={1,2,6,11}, be the reference and unit points of PG(2,4) where 1=(1,0,0), 2=(0,1,0), 6=(0,0,1), 11=(1,1,1). A is (4,2)-arc since it contains four points no three of them are collinear. There are six lines from the joining of these points which are

ℓ_1 = [1,2]={1,2,3,4,5}
ℓ_2 = [1,6]={1,6,7,8,9}
ℓ_3 = [1,11]={1,10,11,12,13}
ℓ_4 = [2,6]={2,6,10,14,18}
ℓ_5 = [2,11]={2,7,11,15,19}
ℓ_6 = [6,11]={3,6,11,16,21}



The diagonal points of A are the points {3,7,10} where $\ell_1 \cap \ell_6 = 3$, $\ell_2 \cap \ell_5 = 7$, $\ell_4 \cap \ell_3 = 10$.

The points of PG(2,4) are classified with respect to the lines through the reference and unit points as follows:

1. The number of points on these lines is 19.
2. There exists two points of index zero for A, which are the point 17 and the point 20 not on any of these lines, then the 4-arc A is incomplete.

3.4 The Conic in PG(2,4) Through the Reference and Unit Points .

The general equation of the conic is:

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_1x_2 + a_5x_1x_3 + a_6x_2x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

By substituting the points of the arc A in (1), we get:

$1 = (1,0,0) \longrightarrow a_1 = 0$,

$2 = (0,1,0) \longrightarrow a_2 = 0$,

$6 = (0,0,1) \longrightarrow a_3 = 0$,

$11 = (1,1,1) \longrightarrow a_4 + a_5 + a_6 = 0$.

So (1) becomes:

$a_4 x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 = 0$ ……….. …………………………...(2)

If $a_4 = 0$, then $a_5 x_1 x_3 + a_6 x_2 x_3 = 0$, and hence $(a_5 x_1 + a_6 x_2) x_3 = 0$

Then the conic is degenerated, therefore for $a_4 \neq 0$, similarly $a_5 \neq 0$ and $a_6 \neq 0$.

Dividing equation (2) by a_4, we get:

$x_1 x_2 + \alpha x_1 x_3 + \beta x_2 x_3 = 0$ …………………………………….…(3)

where $x_1x_2 + \frac{a_5}{a_4}x_1x_3 + \frac{a_6}{a_4}x_2x_3 = 0$

where $\alpha = \frac{a_5}{a_4}, \beta = \frac{a_6}{a_4}$, then $\beta = -(1+\alpha)$ since $1 + \alpha + \beta = 0$ (mod.4)

so $x_1 x_2 + \alpha x_1 x_3 - (1+\alpha) x_2 x_3 = 0$ ………………………………..…(4)

where $\alpha \neq 0$ and $\alpha \neq 1$ for if $\alpha = 0$ or $\alpha = 1$, we get a degenerated conics, i.e. $\alpha = 2, 3$.

3.5 The Equation of the Conics of PG(2,4) and the Complete arcs

For any value of α there is a unique conic contains the reference and the unit points.

1. If $\alpha = 2$, then the equation of the conic C_1 is $x_1 x_2 + 2 x_1 x_3 + x_2 x_3 = 0$, the points of C_1 are {1,2,6,11,12,21}, which is not a complete (k,3)-arc, since there exist the points {4,5,7,8,9,14,15,17,18,19,20} which are the points of index zero for C_1.
   Now, we add to C_1 three points of index zero which are {4,7,8}. Then C_1' = {1,2,6,11,12,21,4,7,8} is a complete (9,3)-arc, since $C_0 = 0$ and C_1' is maximum arc.
2. If $\alpha = 3$, then the equation of the conic C_2 is $x_1 x_2 + 3 x_1 x_3 + 2 x_2 x_3 = 0$, the points of C_2 are {1,2,6,11,17}, which is not a complete (k,2)-arc, since there exist one point {20} which is the point of index zero for C_2.
   Now, we add to C_2 one point of index zero {20}. Then C_2' = {1,2,6,11,17,20} is a complete (6,2)-arc, since $C_0 = 0$.

Conclusion:

1. Each of C_1 and C_2 is not complete (k,2)-arc.
2. We add the points of index zero for each of them for completeness.
3. The points of index zero of PG(2,4) with respect to 4-arc A are in the same line = {17,20}.

3.6 The Construction of Complete and Maximum (k,3)-arc in PG(2,4)

We will try to get a complete (k,3)-arc by taking the complete (k,2)-arc, say C_2' and denoted by B, we notice that B={1,2,6,11,17,20,5} is incomplete (k,3)-arc, since there exists the points {7,9,10,12,14,16,19,21} which are the points of index zero for B.

Now, we add two points of index zero which are {9,10}.

Then B′={1,2,6,11,17,20,5,9,10} is a complete (9,3)-arc, since $C_0 = 0$ and B′ is a maximum arc.

3.7 The Construction of Complete (k,4)-arc

We try to get a complete (k,4)-arc by taking the union of two maximum complete (k,3)-arcs, say C_1' and B′ denoted by D, we notice that

D = {1,2,6,11,12,21,4,7,18,17,20,5,9,10} is incomplete (k,4)-arc, since there exists two points {15,16} which are points of index zero for D.

Now, we add the two points of index zero. Then

D′ = {1,2,6,11,12,21,4,7,18,17,20,5,9,10,15,16} is a complete (16,4)-arc since $C_0 = 0$, and D′ is a maximum arc.

Conclusion:

1. There exists one complete and maximum (16,4)-arc.
2. The points of index zero of (k,4)-arc with respect are in the same line.

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Table (1)The addition's and multiplications operations of GF(4)

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

*	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

Table (2) The points and lines of PG(2,4)

i		P_i				L_i		
1	1	0	0	2	6	10	14	18
2	0	1	0	1	6	7	8	9
3	1	1	0	3	6	11	16	21
4	2	1	0	5	6	13	15	20
5	3	1	0	4	6	12	17	19
6	0	0	1	1	2	3	4	5
7	1	0	1	2	7	11	15	19
8	2	0	1	2	9	13	17	21
9	3	0	1	2	8	12	16	20
10	0	1	1	1	10	11	12	13
11	1	1	1	3	7	10	17	20
12	2	1	1	5	9	10	16	19
13	3	1	1	4	8	10	15	21
14	0	2	1	1	18	19	20	21
15	1	2	1	4	7	13	16	18
16	2	2	1	3	9	12	15	18
17	3	2	1	5	8	11	17	18
18	0	3	1	1	14	15	16	17
19	1	3	1	5	7	12	14	21
20	2	3	1	4	9	11	14	20
21	3	3	1	3	8	13	14	19

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الأقواس العظمى الكاملة في المستوى الاسقاطي PG(2,4) بطريقة هندسية

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الخلاصه

الأقواس - (k,n) في مستوي إسقاطي منتهي PG(2,4) حول حقل كالوا PG(q) و $q=p^n$ ، إذ إن p عدد أولي ولعدد صحيح $n \geq 2$, هو مجموعة مكونة من k من النقاط لا يوجد n+1 منها تقع على مستقيم واحد .

القوس – (k,n) يكون كامل إذا لم يكن محتوى في القوس– (k+1,n) .

في هذا البحث سيتم بناء الأقواس – (k,n) العظمى الكاملة و n=2,3,4 في المستوي PG(2,4) من معادلة المخروط .