Weakly Relative Quasi-Injective Modules

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Abstract:

Let R be a commutative ring with unity and let M, N be unitary R-modules. In this research, we give generalizations for the concepts: weakly relative injectivity, relative tightness and weakly injectivity of modules. We call M weakly N-quasi-injective, if for each $f \in$ Hom(N, $\overline{M}$) there exists a submodule X of $\overline{M}$ such that $f(N) \subseteq X \approx M$, where $\overline{M}$ is the quasi-injective hull of M. And we call M N-quasi-tight, if every quotient N / K of N which embeds in $\overline{M}$ embeds in M. While we call M weakly quasi-injective if M is weakly N-quasi-injective for every finitely generated R-module N.

Moreover, we generalize some properties of weakly N-injective, N-tight and weakly injective modules to weakly N-quasi-injective, N-quasi-tight and weakly quasi-injective modules respectively. The relations among these concepts are also studied.

Introduction

The concept of weak relative injectivity of modules was introduced originally in [1]. Since then, the study of this concept has been illustrated extensively.

We introduced in this research the concept of weak relative quasi-injectivity of modules as a generalization of the concept of weak relative injectivity which motivates our principle subject of this research.

This paper contains five sections. In the first section, we introduced the concept of weakly relative quasi-injectivity of modules, where we call an R-module M weakly N- quasi-injective (N is any R-module) if for each $f \in$ Hom(N, $\overline{M}$) implies that f(N) is contained in

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some submodule of $\overline{M}$ which is isomorphic to M, see definition 1.1. We established some properties of such modules. We showed that the class of such modules is not closed under direct summand, see Ex. 1.6. While we could not prove or disprove that this class of modules is closed under direct sum. But we proved a special case of this, see proposition 1.7. Next we proved that this class of modules is closed under essential extension, see proposition 1.15.

The second section is devoted to give some characterizations of weakly relative quasi-injective modules which are very useful in the next sections, see theorem 2.1, theorem 2.2, theorem 2.3, theorem 2.8, theorem 2.9, and theorem 2.10.

In section three, we generalized the concept of relative tightness of modules which appeared in [2] into the concept of relative quasi-tightness of modules, where we called an R-module M to be N-quasi – tight (N is any R-module) if and only if every quotient N / K of N which embeds in $\overline{M}$ embeds in M, see definition 3.2, we related this concept with the concept of relative quasi-injectivity of modules. It truns out that relative quasi-lightness of modules is a necessary condition for relative quasi-injectivity of modules, see proposition 3.4, while the two concepts are equivalent in the class of uniform modules, see corollary 3.7 and corollary 3.8.

We established in section four certain relations between quasi-tight modules and compressible modules in order to relate weak relative quasi-injectivity and compressibility of modules, where an R-module M is called compressible, if for every essential submodule N of M, M embeds in N, see [3]. Some of the results of this section were given in: Theorem 4.2, Corollary 4.3, Corollary 4.4 and corollary 4.5.

In the last section of this paper, we considered those modules which are weakly quasi-injective relative to each finitely generated module we would refer to any such module as being weakly-injective module. We would establish that:

1. An R-module M is weakly quasi-injective;
   - **i.** If and only if M is weakly R^n –quasi – injective for all positive integer n, see theorem 5.3.
   - **ii.** If and only if for all $x_1, x_2, \ldots, x_n \in \overline{M}$, there exists a submodule X of $\overline{M}$ such that $x_i \in X \approx M$ for all $i = 1, 2, \ldots, n$, see corollary 5.5.
2. A ring R is weakly R^n –quasi – injective if and only if for all $x_1, x_2, \ldots, x_n \in \overline{R}$, there exists an element $b \in \overline{R}$ such that $\mathrm{ann}_R(b) = 0$ and $x_i \in R\,b$ for all $i = 1, 2, \ldots, n$, see proposition 5.6.
3. A cyclic R-module is weakly quasi-injective if and only if it is weakly R^2-quasi-injective, see proposition 5.8.

Section One: Weakly Relative Quasi-Injective Modules

We shall introduce in this section the concept of weakly relative qusi-injectivity of modules. The relation between weakly relative quasi-injective modules and certain types of modules are studied. Some properties of weakly relative quasi-injective modules are established.

1.1 Definition

Let M and N be two R-modules. M is called weakly N-quasi-injective, if for each $f \in \mathrm{Hom}(N, \overline{M})$, there exists a submodule X of $\overline{M}$ such that $f(N) \subseteq X \approx M$, where $\overline{M}$ is the quasi-injective hull of M.

1.2 Remark

Let M and N be two R-modules. Then

i. If M is weakly N-injective, then M is weakly N-quasi-injective and the converse is not true in general.

ii. If M is N-quasi-injective, then M is weakly N-quasi-injective and the converse is not true in general.

iii. If M is quasi-injective, then M is weakly N-quasi-injective and the converse is not true in general.

To disprove the validity of the converse of the above remarks consider the following examples respectively:

1.3 Example

i. Let M = Z_2, N = Z and R = Z. Since Z_2 is a quasi-injective Z-module, then Z_2 is Z-quasi-injective and hence weakly Z-quasi-injective. However, Z_2 is not weakly Z-injective, for if, $f: Z \longrightarrow Z_{2^\infty} = E(Z_2)$ (the injective hull of Z_2), is such that $f(a) = \dfrac{a}{2^3} + Z$ for all $a \in Z$, then $f \in Hom(Z, Z_{2^\infty})$ and $f(Z) \approx Z_8$ which is not embed in Z_2.

ii. Let M =Z, N = 2Z, R = Z and $f: 2Z \longrightarrow Q$ is such that $f(2a) = \dfrac{2a}{5} + Z$ for all $a \in Z$, then $f \in Hom(2Z, Q)$. We take X = $(\dfrac{2}{5})$ the submodule of Q generated by $\dfrac{2}{5}$ and consequently $f(2Z) \subseteq (\dfrac{2}{5})$. Hence Z is weakly 2Z-quasi-injective. However, Z is not 2Z-quasi-injective, since $f(2Z) \not\subset Z$.

iii. Let M = Z, N = 2Z and R = Z. Then Z is weakly 2Z-quasi-injective, but Z is not quasi-injective.

1.4 Proposition

Let M and N be two R-modules and let I be an ideal of R such that $I \subseteq ann_R(\overline{M}) \cap ann_R(N)$. Then M is weakly N-quasi-injective R-module if and only if M is weakly N-quasi-injective R / I-module.

Proof: $I \subseteq ann_R(\overline{M}) \cap ann_R(N)$, implies that M and N are R / I-modules. Moreover, $f: N \longrightarrow \overline{M}$ is an R-homomorphism if and only if f is an R / I –homomorphism, and X is an R-submodule of $\overline{M}$ if and only if X is an R / I – submodule of $\overline{M}$. Hence the details of the proof are followed directly by using the definition 1.1.

1.5 Remark

A direct summand of weakly relative quasi-injective module is not weakly relative quasi-injective in general, as it is shown in the following example.

1.6 Example

Let M =Z $\oplus$ Q, N = Q and R = Z. Let $f \in Hom(Q, \overline{Z \oplus Q}) = Hom(Q, Q \oplus Q)$. If $f = 0$, the proof is obvious.

If $f \neq 0$, then f is a monomorphism, for if $x \in$ Q and $f(x) = 0$ with $x = \frac{a}{b}$ and $a, b \in$ Z, $a \neq 0$, $b \neq 0$, then $0 = f(\frac{a}{b}) = a f(\frac{1}{b})$ implies that $f(\frac{1}{b}) = 0$.

Now, $f(1) = f(\frac{b}{b}) = b f(\frac{1}{b}) = 0$. Hence f(Q) = 0, which is a contradiction. So f is a monomorphism. Therefore f(Q) = 0 ⊕ A or f(Q) = B ⊕ 0 or f(Q)=$\{(x,x): x \in Q\}$, where A and B are submodules of Q.
If f(Q) = 0 ⊕ A, then f(Q) ⊆ 0 ⊕ Q ⊆ Z ⊕ Q ⊆ Q ⊕ Q. Take X = M = Z ⊕ Q, then f(Q) ⊆ X ≈ M = Z ⊕ Q.
Similarly, if f(Q) = B ⊕ 0.
If f(Q)=$\{(x,x): x \in Q\}$ ≈ Q, take Y = $\{(x,x): x \in Q\}$ ⊆ Q ⊕ Q. It is easy to prove that Z ⊕ Y ≈ Z ⊕ Q. Therefore f(Q) ⊆ Y ⊆ Z ⊕ Y ≈ Z ⊕ Q. Hence Z ⊕ Q is weakly Q-quasi-injective. But it is clear that Z is not weakly Q-quasi-injective.

1.6 Remark

We can not prove and we can not disprove that the class of weakly relative quasi-injective modules is closed under direct sum.

However, we give a special case of this.

1.7 Proposition

Let M and N be two R-modules, such that $\overline{L \oplus M} = \overline{L} \oplus \overline{M}$. If L and M are weakly N-quasi-injective, then L ⊕ M is also weakly N-quasi-injective.

Proof: Let $f \in$ Hom(N, $\overline{L \oplus M}$). Then $f \in$ Hom(N, $\overline{L} \oplus \overline{M}$). But Hom(N, $\overline{L} \oplus \overline{M}$) ≈ Hom(N, $\overline{L}$) ⊕ Hom(N, $\overline{M}$) by [4]. Hence $f = (\alpha,\beta)$ with $\alpha \in$ Hom(N, $\overline{L}$) and $\beta \in$ Hom(N, $\overline{M}$). Therefore there exists submodules X and Y of $\overline{L}$ and $\overline{M}$ respectively, such that α(N) ⊆ X ≈ L and β(N) ⊆ Y ≈ M. On the other hand, α(N) ≈ α(N) ⊕ 0 ⊆ X ⊕ Y ≈ L ⊕ M, and β(N) ≈ 0 ⊕ β(N) ⊆ X ⊕ Y ≈ L ⊕ M. Now, f(N) = (α,β)(N) = (α(N),β(N)) ⊆ X ⊕ Y ≈ L ⊕ M, which completes the proof.

1.8 Remark

If L, M and N are R-modules, such that M is weakly N-quasi-injective and M is weakly L-quasi-injective, then it is not true in general that:

i. M is weakly N ⊕ L - quasi-injective.
ii. M is weakly N + L - quasi-injective.

Consider the following examples:

i. Let M = L = N = Z and R = Z. Then Z as a Z-module is weakly - quasi - injective. But Z is not weakly Z ⊕ Z - quasi- injective. In fact, if we define f: Z ⊕ Z ⟶ Q by $f(a,b) = \frac{a}{2} + \frac{b}{3}$ where $a, b \in$ Z, then it can be easily seen that $f \in$ Hom(Z ⊕ Z,Q) and f(Z ⊕ Z) $= ((\frac{1}{2}, \frac{1}{3})) \not\subseteq$ Z.

ii. Let M = Z, N = $(\frac{1}{2})$, L = $(\frac{1}{3})$ and R = Z. Then Z as a Z-module is weakly $(\frac{1}{2})$-quasi-injective and Z is weakly $(\frac{1}{3})$-quasi-injective. But Z is not weakly $(\frac{1}{2}) + (\frac{1}{3})$-quasi-injective. For if, we define $f : (\frac{1}{2}) + (\frac{1}{3}) \longrightarrow Q$ by $f(\frac{a}{2} + \frac{b}{3}) = \frac{a}{2} + \frac{b}{3}$ where $a, b \in$ Z, then it can be easily shown that $f \in \text{Hom}((\frac{1}{2}) + (\frac{1}{3}), Q)$ and $f((\frac{1}{2}) + (\frac{1}{3})) = ((\frac{1}{2}, \frac{1}{3})) \not\subseteq Q$.

1.9 Proposition

Let M be an R-module and N be a submodule of $\overline{M}$. If M is weakly N-quasi-injective, then M is weakly L-quasi-injective for each submodule L of N.

Proof: Let $f \in \text{Hom}(L, \overline{M})$. Consider the following diagram:

$$\begin{array}{ccccc} L & \xrightarrow{i} & N & \xrightarrow{j} & \overline{M} \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \alpha} & & & \\ \overline{M} & & & & \end{array}$$

where i and j are the inclusion homomorphisms and the homomorphism α which makes the diagram commutative exists because $\overline{M}$ is quasi-injective. Therefore $\alpha\, i\, j = f$.
let $\beta = \alpha|_N : N \longrightarrow \overline{M}$. So there exists a submodule X of M such that $\beta(N) \subseteq X \approx M$. But $f(L) \subseteq \beta(N)$, thus $f(L) \subseteq X \approx M$ and hence M is weakly L-quasi-injective.

1.10 Corollary

Let L and N be two submodules of an R-module M such that $L \subseteq N$. If M is weakly N-quasi –injective, then M is weakly L-quasi-injective.

1.11 Corollary

Let M be an R-module and N be a submodule of $\overline{M}$. If L is a submodule of $\overline{M}$ and M is weakly N-quasi-injective, then M is weakly $N \cap L$ - quasi-injective.

In the following two results, we explain the behavior of weakly-quasi-injectivity under homomorphism.

1.12 Proposition

Let H, N and M be R-modules and let $g : N \longrightarrow H$ be an epimorphism. If M is weakly N-quasi-injective, then M is weakly H-quasi-injective.

Proof: Let $f \in \text{Hom}(H, \overline{M})$. Then $f \circ g \in \text{Hom}(N, \overline{M})$. So there exists a submodule X of $\overline{M}$ such that $f(g(N)) \subseteq X \approx M$ which means that M is weakly H-quasi-injective.

1.13 Corollary

Let N be a submodule of an R-module M and let g: M $\longrightarrow$ M be an epimorphism. If M is weakly N-quasi-injective, then M is weakly g(N)-quasi-injective.

Recall that, a submodule N of an R-module M is called quasi-invertible if Hom(M / N, M) = 0, [5].

1.14 Proposition

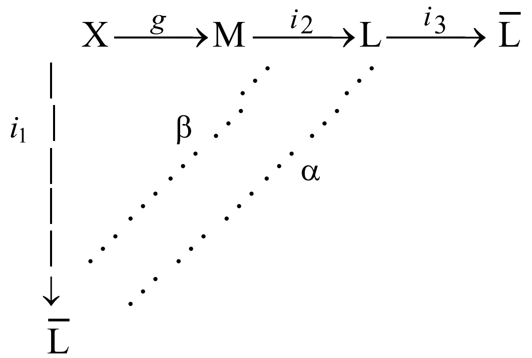
Let M be a torsion-free R-module and let N be a quasi-invertible submodule of M. If M is weakly M / N-quasi-injective, then N is a quasi-invertible submodule of $\overline{M}$.

Proof: Assume that N is not quasi-invertible in $\overline{M}$. Then Hom($\overline{M}$/N, $\overline{M}$) $\neq 0$. Let $f : \overline{M}/N \longrightarrow \overline{M}$ be a non-zero homomorphism. Therefore there exists m + N $\in \overline{M}$/N with $m \in \overline{M}$ and $m \notin$ N such that $0 \neq f(m + N) = x$ for some $x \in \overline{M}$. Let $i : M / N \longrightarrow \overline{M}/N$ be the inclusion homomorphism. Then $f \circ i \in$ Hom(M / N, $\overline{M}$). So there exists a submodule X of $\overline{M}$ such that $f \circ i$ (M / N) $\subseteq X \approx M$.
let $g : X \longrightarrow M$ be an isomorphism, then $g \circ f \circ i \in$ Hom(M / N,M) = 0. Therefore $g \circ f \circ i = 0$ implies that $f \circ i = 0$ and hence f(M / N) = 0. But $m \in \overline{M}$ and $m \notin$ N and M is essential in $\overline{M}$, so there exists $0 \neq r \in$ R such that $r\,m \in$ M. Hence $r\,m$ + N $\in$ M / N and $f(r\,m + N) = 0 = r\,f(m) + N = r\,x$ implies that $r = 0$ which is a contradiction. Therefore N is quasi-invertible in $\overline{M}$.

1.15 Proposition

Let M and N be two R-modules and let L be an essential extension of M. If M is weakly N-quasi-injective, then L is also weakly N-quasi-injective.

Proof: Let $f \in$ Hom(N, $\overline{L}$). But $\overline{L} = \overline{M}$ [by cor.19.8, p.65, [6]]. Hence $f \in$ Hom(N, $\overline{M}$). So there exists a submodule X of $\overline{M}$ such that f(N) $\subseteq X \approx M$. Consider the following diagram:



where $g : X \longrightarrow M$ be an isomorphism and i_1, i_2, i_3 are inclusion homomorphisims. $\overline{L}$ being quasi-injective, so there exists a homomorphism $\alpha : \overline{L} \longrightarrow \overline{L}$ such that $\alpha \circ i \circ g = i_1$ with $i = i_3 \circ i_2$. We claim that ker $\alpha = \{0\}$.

Let $0 \neq \ell \in \overline{L}$ and $\alpha(\ell) = 0$. But M is essential in $\overline{L}$, so there exists $0 \neq r \in$ R such that $0 \neq r\,\ell \in$ M. Hence there exists $x \in$ X such that $g(x) = r\,\ell$.

Now, $x = \alpha \circ i \circ g\,(x) = \alpha\,(r\,\ell) = r\,\alpha(\ell) = 0$. So, $r\,\ell = 0$ which is a contradiction. Therefore α is a monomorphism. Let $\beta = \alpha|_L$. Then $\beta \circ i \circ g = i_1$. Hence X $\subseteq \beta$(L) $\approx$ L. Therefore f(N) $\subseteq$ X $\subseteq \beta$(L), which means that L is weakly N-quasi-injective.

Section Two: Characterizations of Weakly Relative Quasi Injective Modules

We give in this section many interesting characterizations of weakly relative quasi-injective modules which are very useful in the next sections.

First, we shall show that the concept of weakly quasi-injectivity can be given in terms of commutative diagram.

2.1 Theorem

Let M and N be two R-modules. Then M is weakly N-quasi-injective if and only if every element $f \in \mathrm{Hom}(N, \overline{M})$ can be written as a composition $g \circ h$ where $h : N \longrightarrow M$ is a homomorphism and $g : M \longrightarrow \overline{M}$ is a monomorphism. That is the following diagram is commutative.

$$\begin{array}{ccc} N & \xrightarrow[\text{homo.}]{h} & M \\ f \downarrow & \swarrow g & \\ \overline{M} & & \end{array}$$

Proof: Assume that M is weakly N-quasi-injective. Let $f \in \mathrm{Hom}(N, \overline{M})$. Then there exists a submodule X of $\overline{M}$ such that $f(N) \subseteq X \approx M$. So, $f : N \longrightarrow X$ is a homomorphism. Let $\alpha : X \longrightarrow M$ be an isomorphism. We take $h = \alpha \circ f$. Then $h : N \longrightarrow M$ is a homomorphism. Let $g = i \circ \alpha^{-1}$ where $i : X \longrightarrow \overline{M}$ is the inclusion homomorphism. Hence $g : M \longrightarrow \overline{M}$ is a monomorphism.

Now, $g \circ h = (i \circ \alpha^{-1}) \circ (\alpha \circ f) = i \circ f = f$ which proves the "only if" part.

To prove the "if" part:

Let $f \in \mathrm{Hom}(N, \overline{M})$. By hypothesis, there exists a homomorphism $h : N \longrightarrow M$ and a monomorphism $g : N \longrightarrow \overline{M}$ such that $f = g \circ h$. We take $X = g(M)$. Then X is a submodule of $\overline{M}$ and $X \approx M$, moreover, $f(N) = g(h(N)) \subseteq g(M) = X \approx M$. Therefore M is weakly N-quasi-injective.

The following concept is needed for our next result.

Let M and N be two R-modules. M is called N-cyclic, if M is isomorphic to N / K for some submodule K of N, [7].

2.2 Theorem

Let M and N be two R-modules. Then M is weakly N-quasi-injective if and only if for any N-cyclic submodule X of $\overline{M}$ there exists a submodule L of $\overline{M}$ such that $X \subseteq L \approx M$.

Proof: Assume that M is weakly N-quasi-injective. Let X be an N-cyclic submodule of $\overline{M}$. So, $X \approx N / K$ for some submodule K of N. Then we have:

$$N \xrightarrow{\pi} N/K \xrightarrow{\varphi} X \xrightarrow{i} \overline{M}$$

where π is the natural homomorphism, φ is an isomorphism and i is the inclusion homomorphism. Let $f = i \circ \varphi \circ \pi$. Then $f \in \mathrm{Hom}(N, \overline{M})$, implies that there exists a homomorphism $h : N \longrightarrow M$ and a monomorphism $g : N \longrightarrow \overline{M}$ such that $f = g \circ h$ (by Theorem 2.1).

Now, $g\ h(\mathrm{N}) = f(\mathrm{N}) = i\varphi\ \pi(\mathrm{N}) = i\varphi(\mathrm{N} / \mathrm{K}) = i(\mathrm{X}) = \mathrm{X}$. Therefore $g\ h(\mathrm{N}) = \mathrm{X}$. We take $\mathrm{L} = g(\mathrm{N})$ to obtain that L is a submodule of $\overline{\mathrm{M}}$ and $\mathrm{L} \approx \mathrm{M}$. Moreover $\mathrm{X} = g(\mathrm{N}) \subseteq g(\mathrm{M}) = \mathrm{L}$.
Conversely,

To prove M is weakly N-quasi-injective. Let $f \in \mathrm{Hom}(\mathrm{N}, \overline{\mathrm{M}})$. Then $f(\mathrm{N})$ is a submodule of $\overline{\mathrm{M}}$ and $f(\mathrm{N}) \approx \mathrm{N} / \ker f$. That means $f(\mathrm{N})$ is an N-cyclic submodule of $\overline{\mathrm{M}}$. Therefore there exists a submodule L of $\overline{\mathrm{M}}$ such that $f(\mathrm{N}) \subseteq \mathrm{L} \approx \mathrm{M}$, and hence the result follows.

2.3 Theorem

Let M and N be two R-modules. Then the following statements are equivalent:

1. M is weakly N-quasi-injective.
2. For any submodule K of N, M is weakly N / K-quasi-injective.
3. For any submodule K of N and any homomorphism $f: \mathrm{N} / \mathrm{K} \longrightarrow \overline{\mathrm{M}}$, there exists a monomorphism $g: \mathrm{M} \longrightarrow \overline{\mathrm{M}}$ and a homomorphism $h: \mathrm{N} / \mathrm{K} \longrightarrow \mathrm{M}$ such that $g \circ h = f$.

Proof: (1) $\Rightarrow$ (2)

Let K be a submodule of N and let $f \in \mathrm{Hom}(\mathrm{N} / \mathrm{K}, \overline{\mathrm{M}})$. Let $\pi: \mathrm{N} \longrightarrow \mathrm{N} / \mathrm{K}$ be the natural homomorphism. Then $f \circ \pi \in \mathrm{Hom}(\mathrm{N}, \overline{\mathrm{M}})$ and hence by (1), there exists a submodule X of $\overline{\mathrm{M}}$ such that $f \circ \pi(\mathrm{N}) \subseteq \mathrm{X} \approx \mathrm{M}$. Therefore $f(\mathrm{N} / \mathrm{K}) \subseteq \mathrm{X} \approx \mathrm{M}$ which proves (2).

(2) $\Rightarrow$ (3)

We follow as in the proof of theorem 2.1.

(3) $\Rightarrow$ (1)

Let $f \in \mathrm{Hom}(\mathrm{N}, \overline{\mathrm{M}})$ and let $\mathrm{K} = \ker f$. Then define $\overline{f}: \mathrm{N} / \mathrm{K} \longrightarrow \overline{\mathrm{M}}$ by $\overline{f}(a + \mathrm{K}) = f(a)$ for all $a \in \mathrm{N}$. $\overline{f}$ is a homomorphism. It can be easily shown that $\overline{f}$ is a monomorphism. Hence by (3), there exists a monomorphism $g: \mathrm{M} \longrightarrow \overline{\mathrm{M}}$ and a homomorphism $h: \mathrm{N} / \mathrm{K} \longrightarrow \mathrm{M}$ such that $g \circ h = \overline{f}$.

Now, $f(\mathrm{N}) = \overline{f}(\mathrm{N/K}) = g(h(\mathrm{N/K})) \subseteq g(\mathrm{M})$. We take $\mathrm{X} = g(\mathrm{M})$, implies that $f(\mathrm{N}) \subseteq \mathrm{X} \approx \mathrm{M}$, which proves (1).

The following lemma is needed in order to give some applications of theorem 2.3.

2.4 Lemma

Let K, M and N be R-modules with $\mathrm{N} \approx \mathrm{K}$. If M is weakly N-quasi-injective, then M is weakly K-quasi-injective.

Proof: Is obvious, so it is omitted.

2.5 Corollary

Let K, M and N be R-modules. If M is weakly K-quasi-injective and N is K-cyclic. Then M is weakly N-quasi-injective.

Proof: M being K-quasi-injective, implies that M is weakly K / L-quasi-injective for every submodule L of K (by theorem 2.3). But N is K-cyclic, so $\mathrm{N} \approx \mathrm{K} / \mathrm{L}$ for some submodule L of K. Hence M is weakly N-quasi-injective (by lemma 2.4).

2.6 Corollary

If M is weakly N-quasi-injective R-module and A is a direct summand of N, then M is weakly A-quasi-injective.

Proof: follows easily by using theorem 2.3 and lemma 2.4.

As a consequence of 2.6 we have the following result:

2.7 Corollary

Let M and N be two R-modules such that N is quasi-injective and M is weakly N-quasi-injective. Then M is weakly A-quasi-injective for every closed submodule A of N.

Proof: N being quasi-injective and A is a closed submodule of N implies that A is a direct summand of N [see cor. 16.9, p.64, [6]]. Hence the result follows by 2.6.

The following theorem characterizes weakly-quasi-injectivity relative to the R-module R.

2.8 Theorem

Let M be an R-module. Then M is weakly R-quasi-injective if and only if for each element $x \in \overline{M}$, there exists a submodule X of $\overline{M}$ such that $x \in X \approx M$.

Proof: Assume that M is weakly R-quasi-injective. Let $x \in \overline{M}$. Define $f: R \longrightarrow \overline{M}$ by $f(r) = r\,x$ for each $r \in R$. Clearly f is well-defined R-homomorphism. Thus there exists a submodule X of $\overline{M}$ such that $f(R) \subseteq X \approx M$. But $x = 1\cdot x = f(1) \in f(R)$. Hence $x \in X$ which is what we wanted.

Conversely,

Let $f \in \mathrm{Hom}(R, \overline{M})$. Then $f(1) \in \overline{M}$. Let $x = f(1)$. Hence there exists a submodule X of $\overline{M}$ such that $x \in X \approx M$. It is left to show that $f(R) \subseteq X$. Let $a \in f(R)$, then $a = f(r)$ for some $r \in R$. $a = f(r) = r f(1) = r\,x \in X$. Therefore $f(R) \subseteq X$ and hence M is weakly R-quasi-injective.

As a special case, we shall characterize the weakly quasi-injectivity of the R-module R relative to itself.

2.9 Theorem

R is weakly R-quasi-injective R-module if and only if for each element $a \in \overline{R}$, there exists an element $b \in \overline{R}$ such that $a \in R\,b$ and $\mathrm{ann}_R(b) = 0$.

Proof: Assume that R is weakly R-quasi-injective R-module. Let $a \in \overline{R}$. Define $f: R \longrightarrow \overline{R}$ by $f(r) = r\,a$ for each $r \in R$. It can be easily shown that f is a well-defined R-homomorphism. Hence there exists a submodule X of $\overline{R}$ such that $f(R) \subseteq X \approx R$. Clearly, $f(R) = R\,a$. Thus $R\,a \subseteq X$, implies that $a = 1\cdot a \in X$. Let $\alpha: R \longrightarrow X$ be an isomorphism. So there exists an element $c \in R$ such that $a = \alpha(c)$. Hence $a = \alpha(c\cdot 1) = c\,\alpha(1) = c\,b \in R\,b$ where $b = \alpha(1)$. Therefore $a \in R\,b$.

Now, let $r \in \mathrm{ann}_R(b)$. Then $r\,b = 0$ and hence $0 = r\,\alpha(1) = \alpha(r)$ implies that $r = 0$. Hence $\mathrm{ann}_R(b) = 0$.

Conversely,

Let $f \in \mathrm{Hom}(R, \overline{R})$. Then $f(1) \in \overline{R}$. Let $f(1) = a$. So there exists an element $b \in \overline{R}$ such that $a \in R\,b$ and $\mathrm{ann}_R(b) = 0$. We take $X = R\,b$ implies that $X \subseteq \overline{R}$. But $R\,b \approx R / \mathrm{ann}_R(b) \approx R$. Moreover $f(R) = \{f(r) : r \in R\} = \{r\,f(1) : r \in R\} = R\,a \subseteq R\,b$. Therefore $f(R) \subseteq X \approx R$. This completes the proof.

We shall establish in the following theorem a general case of theorem 2.9.

2.10 Theorem

R be an integral domain. Let M and N be two cyclic torsion-free R-modules. Then M is weakly N-quasi-injective if and only if for each element $x \in \overline{M}$ there exists an element $y \in \overline{M}$ such that $x \in Ry$ and $\mathrm{ann}_R(y) = 0$.

Proof: Assume that M is weakly N-quasi-injective. Let $x \in \overline{M}$. Suppose that $M = (m)$ and $N = (n)$ for some $m \in M$ and $n \in N$. Define $f: N \longrightarrow \overline{M}$ by $f(r\,n) = r\,x$ for all $r \in R$. f is well-defined homomorphism. Therefore there exists a submodule X of $\overline{M}$ such that f (N) $\subseteq$ X $\approx$ M. Let $y = f(n)$. Then $y = x \in \overline{M}$ and $x = 1{\cdot}y \in$ R y. Now, if $t \in ann_R(y)$, then $t\,y = 0$. Let g : X $\longrightarrow$ M be an isomorphism, implies that $0 = g(ty) = t\,g(y)$ and hence $t = 0$. Thus $ann_R(y) = 0$.

Conversely,

Let $f \in$ Hom(N, $\overline{M}$), and let $x = f(n)$. Then $x \in \overline{M}$ and hence there exists an element $y \in \overline{M}$ such that $x \in$ R y and $ann_R(y) = 0$. Let X = (x). Then X is a submodule of $\overline{M}$ and f (N) $\subseteq$ X. We claim that X $\approx$ M. Define h : M $\longrightarrow$ X by $h(r\,m) = r\,x$ for all $r \in$ R. It is clear that h is a well-defined homomorphism. Moreover if $r\,x = 0$, implies that $r\,(t\,y) = 0$ for some $t \in$ R. Thus $r\,t = 0$. If $r = 0$, we have done. If $t = 0$, then $x = 0$ which is a contradiction. Thus h is a monomorphism. Clearly h is an epimorphism. Hence X $\approx$ M and therefore M is weakly N-quasi-injective.

When we weaken the conditions in theorem 2.10, we get the following result:

2.11 Proposition

Let M and N be two cyclic R-modules. If M is torsion-free, then M is weakly N-quasi-injective.

Proof: Let M = (m) and N = (n) for some $m \in$ M and $n \in$ N. Let $f \in$ Hom(N, $\overline{M}$) and let $x = f(n)$. Then $x \in \overline{M}$. Suppose that X = (x). Then X is a submodule of $\overline{M}$ and f (N) $\subseteq$ X. Define g : X $\longrightarrow$ M by $g(r\,x) = r\,m$ for all $r \in$ R. If $r\,x = 0$, we claim that $r = 0$.

We have $x \in \overline{M}$ and M is an essential submodule of $\overline{M}$, so there exists a non-zero element $t \in$ R such that $t\,x \in$ M. Hence $ann_R(t\,x) = 0$. But $ann_R(x) \subseteq ann_R(t\,x)$, so $ann_R(x) = 0$. Hence $r = 0$, thus g is well-defined. It can be easily shown that g is an isomorphism. Therefore X $\approx$ M and hence the result follows.

2.12 Remark

The converse of proposition 2.11, may not be true in general, consider the following example:

2.13 Example

Let M = Z_4, N = Z and R = Z. Then M is weakly N-quasi-injective. But M is not torsion-free R-module.

On the other hand, example 1.6 shows that the condition N is cyclic in proposition 2.11, can not be dropped.

Section Three: Weakly Relative Quasi-Injective Modules and Quasi-Tight Modules

We introduce in this section the concept of relative quasi-tightness of modules and we study the relation of this concept with the concept of relative weakly quasi-injectivity of modules.

3.1 Definition

Let M and N be two R-modules. We say that M is N-quasi-tight if and only if every quotient N / K of N which embeds in $\overline{M}$ embeds in M.

M is called R-quasi-tight if and only if for every ideal I of R, every quotient R / I of R which embeds in $\overline{M}$ embed in M.

3.2 Definition

An R-module M is called quasi-tight if M is N-quasi-tight for every finitely generated R-module N.

3.3 Remark

Every N-tight R-module is N-quasi-tight and the converse is not true in general. Consider the following example:

Let M = Z_2, N = Z, and R = Z. Let K be a submodule of N. If N / K embeds in $\overline{Z_2}$ = Z_2, so M is N-quasi-tight.

Now, let K = 4Z. Thus Z / 4Z ≈ Z_4 embeds in Z_{2^∞} = E(Z_2), but Z_4 can not embeds in Z_2. Whence M is not N-tight.

3.4 Proposition

Let M and N be two R-modules. If M is weakly N-quasi-injective, then M is N-quasi-tight.

Proof: Let K be a submodule of N such that N / K embeds in $\overline{M}$. Then there exists a monomorphism f: N / K ⟶ $\overline{M}$. Let π : N ⟶ N /K be the natural homomorphism. Hence $f\circ\pi \in$ Hom(N, $\overline{M}$), so there exists a submodule X of $\overline{M}$ such that $(f\circ\pi)(N) \subseteq X \approx M$. Hence $f(N / K) \subseteq X \approx M$ which implies that f: N / K ⟶ X is a homomorphism. Let g : X ⟶ M be an isomorphism. Then $g\circ f$: N / K ⟶ M is a monomorphism. Which completes the proof.

3.5 Corollary

Let M be an R-module. If M is weakly R-quasi-injective, then M is R-quasi-tight.

Recall that, if A and B are submodules of an R-module C, such that A is a maximal submodule of C with the property that $A \cap B = 0$, then A is called a complement of B in C, [8].

3.6 Theorem

Let M and N be two R-modules. Then M is weakly N-quasi-injective if and only if for each submodule L of N and for every monomorphism f: N / L ⟶ $\overline{M}$, we have:

i. There exists a monomorphism f': N / L ⟶ M, and

ii. For every complement K of f'(N / L) in M, there exists a submodule K′ of $\overline{M}$ such that $K' \cap f(N / L) = 0$ and K′ ≈ K.

Proof: Assume that M is weakly N-quasi-injective. Let L be a submodule of N and let f: N / L ⟶ $\overline{M}$ be a monomorphism, M being weakly N-quasi-injective implies that M is weakly N / L-quasi-injective (by theorem 2.3) and hence there exists a homomorphism f': N / L ⟶ M and there exists a monomorphism α : M ⟶ $\overline{M}$ such that $\alpha\circ f' = f$ (by theorem 2.1). But f is a monomorphism, therefore f' is also a monomorphism. Thus (i) follow.

To verify (ii), let K be a complement of f'(N / L) in M. Let K′ = α(K). Then K′ is a submodule of $\overline{M}$. We claim that $K' \cap f(N / L) = 0$.

Let $x \in K' \cap f(N / L)$ and $x \neq 0$. Hence there exists $0 \neq y \in K$ such that $x = \alpha(y)$ and there exists $0 \neq z \in N / L$ such that $x = f(z)$. Therefore $\alpha(y) = f(z)$ and hence $\alpha(y) = \alpha(f'(z))$, but α

is a monomorphism, so $y = f'(z)$, implies that $0 \neq y \in K \cap f'(N / L)$ which is a contradiction. Hence $K' \cap f(N / L) = 0$, so (ii) is also hold.
Conversely,
Let us assume that (i) and (ii) are hold. Let L be a submodule of N and let f: $N / L \longrightarrow \overline{M}$ be a monomorphism.
By (i), there exists a monomorphism f': $N / L \longrightarrow M$. Let K be a complement of $f'(N / L)$ in M.
By (ii), there exists a submodule K' of $\overline{M}$ such that $K' \cap f(N / L) = 0$ and $K' \approx K$.
Let $h : K \longrightarrow K'$ be an isomorphism and define $\alpha : f'(N / L) \oplus K \longrightarrow \overline{M}$ by $\alpha(f'(x) + k) = f(x) + h(k)$ for all $x \in N / L$, for all $k \in K$. Then α is well-defined homomorphism, moreover, if $f(x) + h(k) = 0$ for some $x \in N / L$ and $k \in K$, implies that $f(x) = - h(k) \in K' \cap f(N / L) = 0$. So, $f(x) = 0$ and $h(k) = 0$, hence α is a monomorphism. Therefore α is extended to a monomorphism β: $M \longrightarrow \overline{M}$. We claim that $\beta \circ f' = f$. Let $x \in N / L$. Then $\beta(f'(x)) = \alpha(f'(x)) = \alpha(f'(x) + 0) = f(x)$. Hence $\beta \circ f' = f$ and so, M is weakly N-quasi-injective (by theorem 2.1).

3.7 Corollary

Let M and N be two R-modules. If M is uniform and N-quasi-tight, then M is weakly N-quasi-injective.

Proof: Let L be a submodule of N let f: $N / L \longrightarrow \overline{M}$ be a monomorphism. But M is N-quasi-tight, therefore there exists a monomorphism f': $N / L \longrightarrow M$ and hence (i) in theorem 3.6 holds.
Now, if $L = N$, then $f'(N / L) = 0$ and hence M is a complement of $f'(N / L)$ in M and $M \cap f'(N / L) = 0$.
If $L \neq N$, then $f'(N / L)$ is a non-zero submodule of M and since M is uniform implies that 0 is the only complement of $f'(N / L)$ in M, and hence (ii) in theorem 3.6 is also hold. Therefore M is weakly N-quasi-injective (by theorem 3.6).

3.8 Corollary

Let M and N be two R-modules such that M is uniform. Then M is N-quasi-tight if and only if M is weakly N-quasi-injective.
Proof: follows by proposition 3.4 and corollary 3.7.

Section Four: Quasi-Tight Modules and Compressible Modules

In this section, we establish some relations between relative quasi-tight modules and compressible modules in the class of quasi-injective modules.

4.1 Definition

An R-module M is called compressible if for all non-zero submodules N of M, M embeds in N, [9].

In general, an R-module M is compressible if for every essential submodule N of M, M embeds in N, [4].

First, we establish the relationship between relative quasi-tight modules and compressible modules.

4.2 Theorem

Let M be a quasi-injective R-module and let N be any R-module. Then every submodule of M is N-quasi-tight if and only if every quotient N / K of N which embeds in M is compressible.

Proof: Assume that every submodule of M is N-quasi-tight. Let N / K be embeds in M. Hence there exists a monomorphism f: N / K $\longrightarrow$ M. We have to show that N / K is compressible. Let L be an essential submodule of N / K. It can be easily seen that f (L) is essential in f (N / K) and hence $\overline{f(L)} = \overline{f(N/L)}$ [by cor.19.8, p.65, [6]]. Since N / K embeds in $\overline{f(N/L)} = \overline{f(L)}$ and f (L) is N-quasi-tight, we get that N / K embeds in f (L) ≈ L. Thus N / K embeds in L which is what we wanted.

Conversely,

Suppose that every quotient of N which embeds in M is compressible. Let A be a submodule of M. We have to show that A is N-quasi-tight. Let K be a submodule of N and let h : N / K $\longrightarrow \overline{A}$ be a monomorphism. But $\overline{A} \subseteq \overline{M}$ = M, implies that $i \circ h$: N / K $\longrightarrow$ M is a monomorphism where i : $\overline{A} \longrightarrow \overline{M}$ is the inclusion homomorphism. Let B = h(N / k) ∩ A. Then B ≠ 0. We claim that B is essential in h(N / K). For if, B ∩ C = 0 for some non-zero submodule C of h(N / K), then 0 = (h(N / K) ∩ A) ∩ C = A ∩ C which is a contradiction. Therefore 0 ≠ B is essential in h(N / K), which implies that h^{-1}(B) is essential in N / K. But N / K is compressible therefore N / K embeds in h^{-1}(B). On the other hand h^{-1}(B) ≈ B⊆ A. Thus N / K embed in A, as desired.

4.3 Corollary

Let M be a quasi-injective R-module. Then every submodule of M is quasi-tight if and only if every finitely generated submodule of M is compressible.

Proof: Assume that every submodule of M is quasi-tight. Let A be a finitely generated submodule of M. Then A is N-quasi-tight for every finitely generated R-module N. Therefore M is A-quasi-tight and according to theorem 4.2. We get that for each submodule B of A such that A / B embeds in M is compressible. But A is finitely generated implies that A / B is also finitely generated. Hence every finitely generated submodule of M is compressible.

Conversely,

Assume that every finitely generated submodule of M is compressible. To prove that every submodule of M is quasi-tight. Let A be a submodule of M and let N be a finitely generated R-module. Let K be a submodule of N such that N / K embeds in $\overline{A}$. But N / K is a finitely generated R-module which embeds in M, so by hypothesis, N / K is compressible. Therefore A is N-quasi-tight for each finitely generated R-module N (by theorem 4.2). Hence A is A-quasi-tight.

4.4 Corollary

Let M be a quasi-injective R-module. Then every submodule of M is weakly R-quasi-injective if and only if every cyclic submodule of M is compressible.

Proof: Assume that every submodule of M is weakly R-quasi-injective. Then every submodule of M is R-quasi-tight (by corollary 3.5) and according to theorem 4.2, we get that every quotient R / I of R (with I is an ideal of R) which embeds in M is compressible.

Now, let A = (a) be a cyclic submodule of M for some $a \in$ M. Then A ≈ R / $\mathrm{ann}_R(a)$. So, R / $\mathrm{ann}_R(a)$ is compressible. Hence A is compressible.

Conversely,

Assume that every cyclic submodule of M is compressible. Because of the fact that every cyclic submodule of M can be written as a quotient R / I for some ideal I of R, and hence for each ideal I of R, if R / I embeds in M is compressible, therefore every cyclic submodule of M is R-quasi-tight (by theorem 4.2). To prove every submodule of M is weakly R-quasi-injective. Let A be a submodule of M and let $x \in \overline{A}$. Then $(x) \approx R / ann_R(x) \subseteq \overline{A}$. But every cyclic submodule of M is R-quasi-tight, hence $(x) \subseteq A$. We take X = A implies that $x \in X$ = A. Thus A is weakly R-quasi-injective (by theorem 2.8).

4.5 Corollary

Let N be an R-module. If every R-module is N-quasi-tight, then N / K is compressible for every submodule K of N.

Proof: Assume that every R-module is N-quasi-tight. Let K be a submodule of N and let A $= \overline{N/K}$. By hypothesis, we get that every submodule of A is N-quasi-tight, and since A is a quasi-injective R-module, implies that N / K is compressible for every submodule K of N (by theorem 4.2).

Section Five: Weakly Quasi-Injective Modules

In this section, we shall concentrate on considering those modules which are weakly quasi-injective relative to each finitely generated module; we shall refer to any such module as being weakly quasi-injective module.

5.1 Definition

An R-module M is called weakly quasi-injective, if M is weakly N-quasi-injective for every finitely generated R-module N.

Equivalently, M is weakly quasi-injective if and only if for each finitely generated R-module N and for each $f \in$ Hom(N, $\overline{M}$), there exists a submodule X of M such that $f(N) \subseteq X \approx M$.

5.2 Remarks

1. A ring R is called weakly quasi-injective if and only if the R-module R is weakly quasi-injective.
2. Every weakly injective R-module is weakly quasi-injective and the converse is not true in general, see example 1.3.

5.3 Theorem

Let M be an R-module. Then M is weakly quasi-injective if and only if M is weakly R^n-quasi-injective for all positive integer n.

Proof: the "only if" part is obvious.

To prove the "if" part. Let N be a finitely generated R-module. We have to show that M is weakly N-quasi-injective. Suppose that $N = Ra_1 + Ra_2 + \ldots + Ra_n$ where $a_i \in N$ for all $i = 1, 2, \ldots, n$. Define $f: R^n \longrightarrow N$ such that $f(r_1, r_2, \ldots, r_n) = r_1 a_1 + r_2 a_2 + \ldots + r_n a_n$ for all $r_1, r_2, \ldots, r_n \in R$. It can be easily checked that f is well-defined epimorphism. Therefore, $R^n / \ker f$ ⊔ N. But M is weakly R^n-quasi-injective, implies that M is weakly $R^n / \ker f$-quasi-injective (by theorem 2.3). Therefore M is weakly N-quasi –injective.

5.4 Proposition

An R-module M is weakly R^n-quasi-injective if and only if for all $x_1, x_2, \ldots, x_n \in \overline{M}$, there exists a submodule X of $\overline{M}$ such that $x_i \in X \approx M$ for all $i = 1,2, \ldots, n$.

Proof: Assume that M is weakly R^n-quasi-injective. Let $x_1, x_2, \ldots, x_n \in \overline{M}$. Let $N = Rx_1 + Rx_2 + \ldots + Rx_n$. Thus N is a finitely generated R-module. In fact N is a submodule of $\overline{M}$. Let $j: N \longrightarrow \overline{M}$ be the inclusion homomorphism. By theorem 5.4, M is weakly N-quasi-injective, therefore there exists a submodule X of $\overline{M}$ such that $j(N) \subseteq X \approx M$. Hence $x_i \in X$ for all $i = 1, 2, \ldots, n$ which completes the proof of the first part.

Conversely,

We have to show that M is weakly R^n-quasi-injective. Let $f \in \text{Hom}(R^n, \overline{M})$. Suppose that $f(1,0,0,\ldots,0) = x_1$, $f(0,1,0,\ldots,0) = x_2$, …, $f(0,0,0,\ldots,1) = x_n$. Then $x_1, x_2, \ldots, x_n \in \overline{M}$. So, by hypothesis there exists a submodule X of $\overline{M}$ such that $x_i \in X \approx M$ for all $i = 1, 2, \ldots, n$ which implies that $f(R^n) \subseteq X \approx M$ and hence M is weakly R^n-quasi-injective.

5.5 Corollary

An R-module M is weakly quasi-injective if and only if for all $x_1, x_2, \ldots, x_n \in \overline{M}$, there exists a submodule X of $\overline{M}$ such that $x_i \in X \approx M$ for all $i = 1, 2, \ldots, n$.

5.6 Proposition

A ring R is weakly R^n-quasi-injective if and only if for all $x_1, x_2, \ldots, x_n \in \overline{R}$ there exists an element $b \in \overline{R}$ such that $\text{ann}_R(b) = 0$ and $x_i \in Rb$ for all $i = 1,2,\ldots,n$.

Proof: Suppose that R is weakly R^n-quasi-injective. Let $x_1, x_2, \ldots, x_n \in \overline{R}$. By proposition 5.4., there exists a submodule X of $\overline{R}$ such that $x_i \in X \approx R$, for all $i = 1, 2, \ldots, n$. Let $\psi : R \longrightarrow X$ be an isomorphism. Put $b = \psi(1)$. Then $b \in \overline{R}$ and for all $i = 1, 2, \ldots, n$, $x_i = \psi(r_i)$ for some $r_i \in R$ and hence $x_i = r_i\,\psi(1) = r_i\,b$ for all $i = 1, 2, \ldots, n$. Therefore $x_i \in Rb$ for all $i = 1, 2, \ldots, n$. Moreover, if $r\,b = 0$ for some $r \in R$, implies that $r = 0$ and hence $\text{ann}_R(b) = 0$.

Conversely,

We have to show that R is weakly R^n-quasi-injective. Let $f \in \text{Hom}(R^n, \overline{R})$. Let $f(1,0,0,\ldots,0) = x_1$, $f(0,1,0,\ldots,0) = x_2$, …, $f(0,0,0,\ldots,1) = x_n$. Then $x_1, x_2, \ldots, x_n \in \overline{R}$ and hence there exists $b \in \overline{R}$ such that $x_i \in Rb$ for all $i = 1, 2, \ldots, n$ and $\text{ann}_R(b)=0$. Let $X = Rb$. Then X is a submodule of $\overline{R}$, $x_i \in X$ for all $i = 1, 2, \ldots, n$ and $X \approx R$. Therefore R is weakly R^n-quasi-injective (by proposition 5.4).

The following corollary is also a consequence of theorem 5.3 and proposition 5.6.

5.7 Corollary

A ring R is weakly quasi-injective if and only if for all $x_1, x_2, \ldots, x_n \in \overline{R}$ there exists an element $b \in R$ such that $\text{ann}_R(b) = 0$ and $x_i \in Rb$ for all $i = 1, 2, \ldots, n$.

Finally, we give the following characterization.

5.8 Proposition

A cyclic R-module is weakly quasi-injective if and only if it is weakly R^2-quasi-injective.

Proof: the "only if " part is obvious. To prove the "if " part, let M be a cyclic R-module. Suppose that M is weakly R^2-quasi-injective. Let us proceed by induction. Assume that M is weakly R^{n-1}–quasi-injective and let $x_1, x_2, \ldots, x_n \in \overline{M}$. By proposition 5.6, there exists a submodule $Rx \subseteq \overline{M}$ such that $x_1, x_2, \ldots, x_{n-1} \in Rx \approx M$. But M is weakly R^2-quasi-injective, so there exists a submodule X of $\overline{M}$ such that $X \approx M$ and $x, x_n \in X$. Hence $x_1, x_2, \ldots, x_n \in X \approx M$. Therefore M is weakly quasi-injective (by corollary 5.5).

5.9 Corollary

A cyclic R-module is weakly R^n-quasi-injective if and only if it is weakly R^2-quasi-injective.

Proof: follows from theorem 5.3 and proposition 5.8.

References

1. Jain, S. K. and Lopes Permonth, S. R., (1990), "A Survry on the Theory of Weakly-Injective Modules", Computational Algebra, Marcel Dekker, 205-232.
2. Salih, M., (1999), "A note on Tightness", Math. Dep. Birzeit University, P.O.Box 14, Glasqow Math. J., 41: 43-44.
3. Jain, S. K. and Lopez-P, S. R., (1990), "Rings whose Cyclics are Essentially Embeddable in Projective Modules", J. of Algebra, 128(1): 208-220.
4. Kasch F., (1982), "Modules and Rings", Academic press, London, NewYork.
5. Mijbas A. S., (1997), "Quasi-Dedekind Modules", Ph.D. Thesis, University of Baghdad.
6. Faith II, C., (1976), "Algebra, Rings Theory", Springer-Verlay, Berlin Heidelberg, New York.
7. Somchit Chotchasithit, (2002), "When is Quasi-p-Injective Module Continuous, South east Asian Bulletin of Mathematics, 26: 391-394.
8. Goodearl, K. R., (1976), "Ring Theory", Marcel Dekker, New York.
9. Nicholson, W. K. and Desale, G., (1981), "Endoprimitive Rings", J. Algebra, 70: 548-560.

مقاسات شبه – اغمارية نسبية ضعيفة

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الخلاصة

لتكن R حلقة تبادلية بمحايد وكل من M و N مقاسا احادياً على R. أعطينا هذا البحث اعماماً للمفاهيم اغمارية نسبية ضعيفة واحكام الاغلاق النسبية واغمارية ضعيفة للمقاسات. أسمينا M مقاس شبه –اغماري – N ضعيف اذا كان لكل($f \in \mathrm{Hom}(\mathrm{N}, \overline{\mathrm{M}})$ ، إذ $\overline{\mathrm{M}}$ الغلاف الشبه –اغماري للمقاس M، يوجد مقاس جزئي X من $\overline{\mathrm{M}}$ ، إذ ان $f(\mathrm{N}) \subseteq \mathrm{X} \approx \mathrm{M}$.

واسمينا M مقاس شبه–محكم الاغلاق– N اذا كان كل كسر N/K من N يغمر في $\overline{\mathrm{M}}$ يمكن ان يغمر في M.

بينما اسمينا M مقاس شبه–اغماري ضعيف اذا كان M شبه –اغماري – N ضعيف لكل مقاس منته التولد N على الحلقة R. فضلاً عن ذلك عممنا بعض الخواص للمقاسات الاغمارية – N الضعيفة والمحكمة الاغلاق – N والاغمارية الضعيفة الى المقاسات شبه – الاغمارية – N الضعيفة ، وشبه – المحكمة الاغلاق – N ، وشبه – الاغمارية الضعيفة على التوالي. وقمنا بدراسة العلاقة بين هذه المفاهيم.