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# On the Riesz Means of Expansion by Riesz Bases Formed by Eigen Functions for the Ordinary Differential Operator of 2m-th Order

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## Abstract:

The aim of this paper is to prove a theorem on the Riesz means of expansions with respect to Riesz bases, which extends the previous results of [1] and [2] on the Schrödinger operator and the ordinary differential operator of 4-th order to the operator of order 2m by using the eigen functions of the ordinary differential operator.

**Some Symbols that used in the paper:**

$\| \ \|$ the uniform norm.

<,> the inner product in $L^2$.

$\partial G$ the set of all boundary elements of G.

$\hat{u}$ the dual function of u.

## 1. Introduction

The theory of non-self adjoint differential operators has great importance in several applications, many mathematicians worked on the equiconvergence theorem for this operators like Ilin, Joó, Komornik and Tahir for the Schrödinger operator see [3], [4] and for Laguerre functions see [5].

By using the method and results of this papers, we shall prove un equiconvergence theorem for the ordinary differential operator of order 2m.

Let $G \subseteq R$ be an arbitrary finite open interval $q(x) \in L^1(G)$ an arbitaray complex function and consider the operator $Lu_r := u_r^{(2m)} + q(x)u_r$, where $m \in \mathbb{N}$ .

Given a complex number $\lambda$, the function $u_{-1}: G \longrightarrow C$, $u_{-1} \equiv 0$ is called an eigenfunction of order – 1 of the operator L with the eigenvalue $\lambda$. A function $u_r: G \longrightarrow C$, $u_r \neq 0$ (r=0,1,…) is said to be an eigenfunction of order r of the operator L with the eigenvalue $\lambda$ if $u_r$ together with its derivative is absolutely continuous on every compact subinterval of G and if for almost all $x \in G$ the equation $Lu_r(x) = \lambda u_r(x) - u_{r-1}(x)$ holds, where $u_{r-1}(x)$ is an eigenfunction of order (r – 1) with the same $\lambda$.

IBN AL- HAITHAM J. FOR PURE & APPL. SCI. VOL.23 (2) 2010

Let us now give a Riesz bases $(u_r(x))\subset L^2(G)$ of the operator L. Let $\lambda_r$ (resp.$0_r$) denote the eigenvalue (resp. the order) of $u_r$ and assume that the following conditions are satisfied:

$$\sup_r 0_r<\infty \qquad ...(1)$$

In case $0_r > 0$,

$$\lambda u_r - Lu_r = u_{r-1} \qquad ...(2)$$

Suppose the biorthogonal system $(v_r)$ of the system $(u_r)$ have the property

$$\sum_{|\mu-\rho_r|\le 1} {}_r\|v_r\|^2_{L^\infty(G)} < V < \infty \qquad ...(3)$$

Now, consider the Riesz mean of the biorthogonal series

$$\sigma_\mu^s(f,x) := \sum_{|\rho_r|<\mu} < f, v_r > u_r(x)(1-\frac{\rho_r^2}{\mu^2})^s \qquad ...(4)$$

($f\in L^1(G)$, $x\in G$, $\mu>0$, $0 \le s <1/2$), where $(v_r)$ is the dual system of $(u_r)$, i.e., $(v_r)\subset L^2(G)$ and $<v_r,u_j>=\delta_{rj}$.

Given any compact interval $K\subset G$, denote by R an arbitrary number from the interval $(0,dist(K,\partial G))$, where $dist(K,\partial G)=\inf\{d(a,b), \forall a\in K, b\in\partial G\}$.

Now fix $x\in K$ arbitrary and define $W_R^s:G\longrightarrow R$ by

$$W_R^s(t) = \begin{cases} a(s)\mu^{1/2-s}|t|^{-s-1/2} J_{s+1/2}(\mu|t|) & \text{if } |t|\le R, \\ 0 & \text{otherwise} \end{cases} \qquad ...(5)$$

where $a(s):=2^s(2\pi)^{-1/2}\Gamma(s+1)$.

Moreover, for any function $f\in L^1(G)$, $x \pm R\in G$ define

$$S_\mu^s(f,x) := \int_{x-R}^{x+R} W_R^s(t)(y-x)f(y)dy \qquad ...(6)$$

Denote by $\theta^s(x,y,\mu)$ the spectral function of the Riesz means (i.e.)

$$\theta^s(x,y,\mu) := \sum_{\rho_r<\mu} u_r(x)\overline{v_r(y)}(1-\frac{\rho_r^2}{\mu^2})^s \qquad ...(7)$$

where $x,y \in G$.

Introduce the operation $D_{R_0}:L^1(G)\longrightarrow R$

$$D_{R_0}[f] := \frac{2}{R_0}\int_{\frac{R_0}{2}}^{R_0} f(R)dR \qquad ...(8)$$

where $R_0\in(0,dist(K,\partial G))$.

## 2. Main Results

Here we will prove the following theorem

**2.1 Theorem:**

Given any compact interval $K\subset G$, for all $0\le s<1/2$, $\mu>0$ and $f\in L^1(G)$ the estimate

$$\sigma_\mu^s(f,x)-S_\mu^s(f,x)=O(1)\mu^{-s} \qquad \ldots(9)$$

Holds uniformly on the compact interval $K\subset G$.

For the proof of this theorem we shall choose the 2m-th roots $\mu_{r,i}$ $(i=1,2,\ldots,2m)$ of $\lambda_r$ such that

$\operatorname{Re}\mu_{r,1}\ge\operatorname{Re}\mu_{r,2}\ge\ldots\ge\operatorname{Re}\mu_{r,m}\ge 0\ge\operatorname{Re}\mu_{r,m+1}\ge\ldots\ge\operatorname{Re}\mu_{r,2m}$

and put $\mu_r=\mu_{r,m}$, $\rho_r=\operatorname{Re}\mu_r$, $\nu_r=|\operatorname{Im}\mu_r|$.

Now, we have from ([3])

$$D=\begin{vmatrix} \hat{u}_r(x) & \hat{u}_r(x-t)+\hat{u}_r(x+t) & \hat{u}_r(x-2R)+\hat{u}_r(x+2R) & \cdots & \hat{u}_r(x-m)+\hat{u}_r(x+m)\\ 1 & 2\operatorname{ch}\omega_1\mu_r t & 2\operatorname{ch}2\omega_1\mu_r R & \cdots & 2\operatorname{ch}m\omega_1\mu_r R\\ 1 & 2\operatorname{ch}\omega_2\mu_r t & 2\operatorname{ch}2\omega_2\mu_r r & \cdots & 2\operatorname{ch}m\omega_2\mu_r R\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 2\operatorname{ch}\omega_m\mu_r t & 2\operatorname{ch}2\omega_m\mu_r R & \cdots & 2\operatorname{ch}m\omega_m\mu_r R \end{vmatrix}$$

By expanding this determinant according to the first row with the definition of $\hat{u}_r$ we get the following equation

$[u_r(x-t)+u_r(x+t)-2u_r(x)\operatorname{ch}\mu_r t]d(\mu_r,R)=$

$$\sum_{\substack{0\le k\le m\\ k\ne 1}} d_k(\mu_r,R,t)[u_r(x+kR)+u_r(x-kR)+\int_{x-mR}^{x+mR} D(\mu_r,R,t,|x-\zeta|)Q(\zeta)d\zeta \qquad \ldots(10)$$

where

$$D(\mu_r R,t,|x-\zeta|)=\begin{cases} \displaystyle\sum_{2\le k\le m} d_k(\mu_r,R,t)\sum_{p=1}^{m}\frac{\omega_p\operatorname{sh}\omega_p\mu_r(kR-|x-\zeta|)}{m\mu_r^{2m-1}}-d_k(\mu_r,R)\sum_{p=1}^{m}\frac{\omega_p\operatorname{sh}\omega_p\mu_r(t-|x-\zeta|)}{m\mu_r^{2m-1}} & ,\ \text{if } |x-\zeta| \\ \displaystyle\sum_{2\le k\le m} d_k(\mu_r,R,t)\sum_{p=1}^{m}\frac{\omega_p\operatorname{sh}\omega_p\mu_r(kR-|x-\zeta|)}{m\mu_r^{2m-1}} & ,\ \text{if } t\le|x-\zeta|\le 2 \\ \displaystyle\sum_{j\le k\le m} d_k(\mu_r,R,t)\sum_{p=1}^{m}\frac{\omega_p\operatorname{sh}\omega_p\mu_r(kR-|x-\zeta|)}{m\mu_r^{2m-1}} & ,\ \text{if } (j-1)R\le|x-\zeta|\le jR,\ 3\le j \end{cases}$$

We want to prove the following estimate

$$\theta^s(x,y,\mu)-W_R^s(|y-x|)=O(1)\mu^{-s} \qquad \ldots(11)$$

By using equation (10), we count the Fourier coefficients of the function $W_R^s(y-x)$ with respect to the system $(u_r)$:

$$< u_r, D_{R_0} W_R^s >= D_{R_0}\int_0^R W_R^s(t)[u_r(x-t)+u_r(x+t)]dt$$

$$= D_{R_0}\int_0^R W_R^s(t)[2u_r \text{ch}\,\mu_r t + \sum_{\substack{0\le k\le m\\ k\ne 1}} \frac{d_k(\mu_r, R, t)}{d(\mu_r, R)}\{u_r(x+kR)+u_r(x-kR)\} +$$

$$\frac{1}{d(\mu_r, R)}\int_{x-mR}^{x+mR} D(\mu_r, R, t, |x-\zeta|)Q(\zeta)d\zeta]dt$$

Then

$$< u_r, D_{R_0} W_R^s >= D_{R_0}\int_0^R W_R^s(t)\cdot 2u_r(x)\text{ch}\,\mu_r t dt + D_{R_0}\int_0^R W_R^s(t)\sum_{\substack{0\le k\le m\\ k\ne 1}} \frac{d_k(\mu_r, R, t)}{d(\mu_r, R)}\{u_r(x+kR)+u_r(x-kR)\}dt +$$

$$D_{R_0}\int_0^R W_R^s(t)\cdot\frac{1}{d(\mu_r, R)}\int_{x-mR}^{x+mR} D(\mu_r, R, t, |x-\zeta|)Q(\zeta)d\zeta dt \qquad ...(12)$$

Now, we want to find $\theta^s(x,y,\mu) - W_R^s(|y-x|)$, by using the definition of $\theta^s(x,y,\mu)$ and the relations $W_R^s(t) = \sum_r \overline{v_r(y)} < u_r, D_{R_0} W_R^s > - O(1)\mu^{-s}$ and $\int_0^\infty W_R^s(t)\cos\rho_r dt = (1-\frac{\rho_r^2}{\mu_r^2})^s$, see ([1],[2]). We have the following

$$\theta^s(x,y,\mu) - W_R^s(|y-x|) = O(1)\mu^{-s} + \sum_r u_r(x)\overline{v_r(y)}\,(1-\frac{\rho_r^2}{\mu_r^2})^s - \sum_r \overline{v_r(y)} < u_r, D_{R_0} W_R^s >$$

$$\theta^s(x,y,\mu) - W_R^s(|y-x|) = O(1)\mu^{-s} + \sum_r u_r(x)\overline{v_r(y)}\,D_{R_0}\int_0^R W_R^s(t)\cos\rho_r t dt -$$

$$\sum_r 2\overline{v_r(y)}u_r(x)D_{R_0}\int_0^R W_R^s(t)\text{ch}\,\mu_r t dt - \sum_r \overline{v_r(y)}\,D_{R_0}\int_0^R W_R^s(t)\sum_{\substack{0\le k\le m\\ k\ne 1}} \frac{d_k(\mu_r, R, t)}{d(\mu_r, R)}\{u_r(x+kR)+$$

$$u_r(x-kR)\}dt - \sum_r \overline{v_r(y)}D_{R_0}\int_0^R W_R^s(t)\cdot\frac{1}{d(\mu_r, R)}\int_{x-mR}^{x+mR} D(\mu_r, R, t, |x-\zeta|)Q(\zeta)d\zeta dt \qquad ...(13)$$

By using the interval transformation $\int_0^\infty = \int_0^R + \int_R^\infty$ , we have

$$\theta^\sigma(x, y, \mu) - w_R^s(|y-x|) = O(1)\mu^{-s} + \sum_r u_r(x)\overline{v_r(y)}[D_{R_0}\int_R^\infty w_R^s(t)\cos\rho_r t dt +$$

$$D_{R_0}\int_0^R w_R^s(t)(\cos\rho_r t - 2\text{ch}\mu_r t)dt] - \sum_r \overline{v_r(y)}D_{R_0}\int_0^R w_R^s(t)\sum_{\substack{0\le k\le m\\ k\ne 1}} \frac{d_k(\mu_r, R, t)}{d(\mu_r, R)}\{u_r(x+kR)+u_r(x-kR)\}dt -$$

$$\sum_r \overline{v_r(y)}D_{R_0}\int_0^R w_R^s(t)\cdot\frac{1}{d(\mu_r, R)}\int_{x-mR}^{x+mR} D(\mu_r, R, t, |x-\zeta|)Q(\zeta)d\zeta dt$$

Now, we want to find the estimates of the integrals in the right hand side, so we will denote to this integrals by $A_1,\ldots,A_4$ respectively.

Firstly, we know from [1]

$$|A_1| \le \mu^{-s} \frac{c(R_0, s)}{1 + |\mu_r - \rho_r|^2}$$

$$|A_2| = \left| D_{R_0} \int_0^R W_R^s(t)(\cos\rho_r t - 2\mathrm{ch}\,\mu_r t)dt \right|$$

By the relations in [4]

$$|\cos\rho_r t| \le \begin{cases} 1 & 0 \le t \le \frac{1}{|\mu_r|} \\ e^{\rho_r t} & t > \frac{1}{|\mu_r|} \end{cases} \quad \text{and} \quad |\mathrm{ch}\,\mu_r t| \le \begin{cases} c|\mu_r|t & 0 \le t \le \frac{1}{|\mu_r|} \\ ce^{\rho_r t} & t > \frac{1}{|\mu_r|} \end{cases}$$

We have the following inequality

$$|\cos\rho_r t - 2\mathrm{ch}\,\mu_r t| \le \begin{cases} 1 + c|\mu_r|t & 0 \le t \le \frac{1}{|\mu_r|} \\ ce^{\rho_r t} & t > \frac{1}{|\mu_r|} \end{cases}$$

Then

$$|A_2| \le D_{R_0} \left[ \int_0^{\frac{1}{|\mu_r|}} |W_R^s(t)|(1 + c|\mu_r|t)dt + \int_{\frac{1}{|\mu_r|}}^R c|W_R^s(t)|e^{\rho_r t}dt \right]$$

$$\le c_0\mu^{-s} c_1 \max_{0 \le t \le \frac{1}{|\mu_r|}} t^{-s} + c_2 \max_{0 \le t \le \frac{1}{|\mu_r|}} t^{-s+1} + c_3 e^{\rho_r R} \max_{\frac{1}{|\mu_r|} \le t \le R} t^{-s}$$

Since $R \le R_0$, we get

$$|A_2| \le c_0\mu^{-s} \left[ c \max_{0 \le t \le \frac{1}{|\mu_r|}} t^{-s+1} + c_3 e^{\rho_r R_0} \max_{\frac{1}{|\mu_r|} \le t \le R} t^{-s} \right]$$

$$\le c_0\mu^{-s}[cR_0^{-s+1} + c_3 e^{\rho_r R_0} R_0^{-s}]$$

$$\le c(R_0, s)e^{\rho_r R_0}\mu^{-s}$$

$$|A_3| = D_{R_0} \int_0^R W_R^s(t) \sum_{\substack{0 \le k \le m \\ k \ne 1}} \frac{d_k(\mu_r, R, t)}{d(\mu_r, R)} \{u_r(x + kR) + u_r(x - kR)\}dt$$

By using the following inequality in [4]

$$\left| \frac{d_0(\mu_r, R, t)}{d(\mu_r, R)} \right| \le ce^{\mathrm{Re}(2\mu_r - \mu_{r,m-1})R} \qquad \ldots(14)$$

and

$$\left|\frac{d_k(\mu_r,R,t)}{d(\mu_r,R)}\right| \le \min(1,|\mu_r t|)\begin{cases} ce^{\operatorname{Re}\mu_{r,m-1}(t-2R)} & \text{if } k=2 \\ ce^{\operatorname{Re}\mu_{r,m-1}t-\operatorname{Re}(\mu_{r,m-k+1}+\mu_{r,m-k+2}+\dots+\mu_{r,m-2}+\mu_{r,m-1})R} & \text{if } 2<k<m \\ ce^{\operatorname{Re}\mu_{r,m-1}(t-2R)-\operatorname{Re}(\mu_{r,1}+\mu_{r,2}+\dots+\mu_{r,m-2})s} & \text{if } k=m \end{cases} \quad \dots(15)$$

where $Q(\mu_r,R)=e^{\operatorname{Re}(m\mu_{r,1}+\dots+\mu_{r,m-1})R}$ and $|d(\mu_r,R)|=Q(\mu_r,R)$.

Hence

$$\left|D_{R_0}\int_0^R W_R^s(t)\sum_{\substack{0\le k\le m\\ k\ne 1}}\frac{d_k(\mu_r,R,t)}{d(\mu_r,R)}dt\right|$$

$$\le D_{R_0}\int_0^R a(s)|\mu|^{\frac{1}{2}-s}|t|^{-s-\frac{1}{2}}\left|J_{s+\frac{1}{2}}(\mu|t|)\right|\sum_{\substack{0\le k\le m\\ k\ne 1}}\left|\frac{d_k(\mu_r,R,t)}{d(\mu_r,R)}\right|dt$$

$$\le D_{R_0}\left[\int_0^{\frac{1}{|\mu_r|}} a(s)|\mu|^{\frac{1}{2}-s}|t|^{-s-\frac{1}{2}}\left|J_{s+\frac{1}{2}}(\mu(t))\right|\sum_{\substack{0\le k\le m\\ k\ne 1}}\left|\frac{d_k(\mu_r,R,t)}{d(\mu_r,R)}\right|dt+\right.$$

$$\left.\int_{\frac{1}{|\mu_r|}}^R a(s)|\mu|^{\frac{1}{2}-s}|t|^{-s-\frac{1}{2}}\left|J_{s+\frac{1}{2}}(\mu(t))\right|\sum_{\substack{0\le k\le m\\ k\ne 1}}\left|\frac{d_k(\mu_r,R,t)}{d(\mu_r,R)}\right|dt\right]$$

$$\left|D_{R_0}\int_0^R W_R^s(t)\sum_{\substack{0\le k\le m\\ k\ne 1}}\frac{d_k(\mu_r,R,t)}{d(\mu_r,R)}dt\right| \le c(s)|\mu|^{\frac{3}{2}-s}\max_{0\le t\le\frac{1}{|\mu_r|}} t^{-s+1}+3c(s)\mu^{\frac{1}{2}-s}e^{\rho_r t}\max_{\frac{1}{|\mu_r|}\le t\le R} t^{-s}$$

$$= c(s)|\mu|^{\frac{3}{2}-s}\max_{0\le t\le\frac{1}{|\mu_r|}} t^{-s+1}+3c(s)\mu^{\frac{1}{2}-s}e^{\rho_r t}\max_{\frac{1}{|\mu_r|}\le t\le R} t^{-s}$$

From (14), (15)

$$\le c(s)\mu^{-s}\left(\mu^{\frac{3}{2}}\max_{0\le t\le\frac{1}{|\mu_r|}} t^{-s+1}+3\mu^{\frac{1}{2}}e^{\rho_r t}\max_{\frac{1}{|\mu_r|}\le t\le R} t^{-s}\right)$$

$$\le c(s)\mu^{-s}\left(\mu^{\frac{3}{2}}R_0^{-s+1}+3\mu^{\frac{1}{2}}e^{\rho_r R_0}R_0^{-s}\right)$$

$$= c(s)\mu^{-s}R_0^{-s}\left(\mu^{\frac{3}{2}}R_0+3\mu^{\frac{1}{2}}e^{\rho_r R_0}\right)$$

$$\le c'(R_0,s)\mu^{-s}e^{\rho_r R_0}$$

Then

$$|A_3|\le c'(R_0,s)\mu^{-s}e^{\rho_r R_0}\|u_r\|_{L^\infty(k_{R_0})}$$

To find the estimate of $A_4$

$$A_4 = D_{R_0}\int_0^R W_R^s(t)\frac{1}{d(\mu_r,R)}\int_{x-mR}^{x+mR} D(\mu_r,R,t,|x-\zeta|)Q(\zeta)\,d\zeta dt$$

$$|A_4| = D_{R_0}\int_0^R |W_R^s(t)|\left|\frac{1}{d(\mu_r,R)}\right|\int_{x-mR}^{x+mR} |D(\mu_r,R,t,|x-\zeta|)Q(\zeta)|\,d\zeta dt$$

$$\le D_{R_0}\int_0^R |W_R^s(t)|\int_{x-mR}^{x+mR} c|\mu_r|^{1-2m}\min\{1,|\mu_r t|\}e^{\rho_r R}\,|Q(\zeta)|\,d\zeta dt$$

From [5].

By using the argument in [6],

$$\int_{x-2s}^{x+2s}|Q(\zeta)|d\zeta \le c\left(\|u\|_{L^\infty(K_{2s})}+\|u^*\|_{L^\infty(K_{2s})}\right)$$

We have

$$|A_4| \le D_{R_0}\int_0^R |W_R^s(t)||\mu_r|^{1-2m}\min\{1,|\mu_r t|\}e^{\rho_r R}\left(\|u_r\|_{L^\infty(K_{mR})}+\|\hat{u}_r\|_{L^\infty(K_{mR})}\right)dt$$

$$= c|\mu_r|^{1-2m}e^{\rho_r R}\left(\|u_r\|_{L^\infty(K_{mR})}+\|\hat{u}_r\|_{L^\infty(K_{mR})}\right)\left[D_{R_0}\int_0^{\frac{1}{|\mu_r|}}|W_R^s(t)||\mu_r|t\,dt + D_{R_0}\int_{\frac{1}{|\mu_r|}}^R |W_R^s(t)|dt\right]$$

$$|A_4| \le c'(s)|\mu|^{1-2m}e^{\rho_r R}\left(\|u_r\|_{L^\infty(K_{mR})}+\|\hat{u}_r\|_{L^\infty(K_{mR})}\right)\left[|\mu|^{\frac{3}{2}-s}\max_{0\le t\le\frac{1}{|\mu_r|}} t^{-s+1} + |\mu|^{\frac{1}{2}-s}\max_{\frac{1}{|\mu_r|}\le t\le R} t^{-s}\right]$$

$$\le c'(s)\mu^{-s}e^{\rho_r R}\left(\|u_r\|_{L^\infty(K_{mR})}+\|\hat{u}_r\|_{L^\infty(K_{mR})}\right)\left[\mu^{\frac{3}{2}}R_0^{-s+1}+\mu^{\frac{1}{2}}R_0^{-s}\right]$$

$$\le c'(R_0,s)\mu^{-s}e^{\rho_r R_0}\left(\|u_r\|_{L^\infty(K_{mR})}+\|\hat{u}_r\|_{L^\infty(K_{mR})}\right)$$

Now, from (3) and ([7],lemma (3)) and Cauchy-Schwartz inequality we obtain the following estimates

$$\sum_r |\overline{v_r(y)}||u_r(x)||A_i| = O(1)\mu^{-s} \quad (i=1,2)$$

$$\sum_r |\overline{v_r(y)}||A_i| = O(1)\mu^{-s} \quad (i=3,4)$$

From this we obtain

$$\theta^s(x,y,\mu) - W_R^s(|y-x|) = O(1)\mu^{-s} \qquad \ldots(16)$$

**Proof of the Theorem (2.1)**

Consider the operator

$$L_\mu(f,x) = \mu^s[\sigma_\mu^s(f,x) - S_\mu^s(f,x)]$$

From $L^1(G)$ into C(K) for any compact K⊂G we know that

$$\sigma_{\mu}^{s}(f,x)=\sum_{|\rho_r|<\mu} < f, v_r > u_r(x)\left(1-\frac{\rho_r^2}{\mu^2}\right)^s$$

$$S_{\mu}^{s}(f,x)=\sum_{r} < f, v_r >< u_r, W_R^s >$$

Then

$$\sigma_{\mu}^{s}(f,x)-S_{\mu}^{s}(f,x)=< f, \theta^s - W_R^s >=\sum_{r} < f, v_r >< u_r, \theta^s - W_R^s >$$

By using Cauchy-Schwartz inequality and the previous result in (16), we obtain

$$\|L_{\mu}(f,x)\|_{C(K)} \le M$$

For any fixed K⊂G. Further there exists $H\subset L^1(G)$ s.t. $\overline{H}=L^1(G)$ and the relation $L_{\mu}(h,x)\to 0$ holds uniformly in x on the K for any h∈H. Hence the result follows by the Banach-Steinhaus theorem.

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# حول متوسطات ريس للتوسيعات بواساطة قواعد ريس المصاغة باستخدام الدوال الذاتية للمؤثر التفاضلي الاعتيادي من الرتبة 2m

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## الخلاصة

هدف هذا البحث برهنة نظرية حول متوسطات ريس للتوسيعات بالنسبة الى قواعد ريس التي توسع النتائج السابقة لـ (لوي وطاهر) المتحققة على مؤثر شرودنكر والمؤثر التفاضلي الاعتيادي من الرتبة الرابعة الى مؤثر تفاضلي من الرتبة 2m باستخدام الدوال الذاتية للمؤثر التفاضلي الاعتيادي.