مجموعات من فضاءات جزئية في مستوي اسقاطي PG(2,q) حول حقل كالوا GF(q)

ميساء جليل محمد
قسم الرياضيات ، كلية التربية- ابن الهيثم ، جامعة بغداد

الخلاصة

في هذا البحث تم درست انواع من مجموعات لفضاءات جزئية في المستوي الاسقاطي PG(2,q) حول حقل كالوا GF(q) . وبيان بعض العلاقات التي تربط بين هذه المجموعات بعضها مع البعض من خلال الامثلة والمبرهنات .

Sets of Subspaces of a Projective Plane PG(2,q) Over Galois Field GF(q)

M. J. Mahammad
Department of Mathematics,College of Education Ibn-Al-Haitham , University of Baghdad

Abstract

In this thesis, some sets of subspaces of projective plane PG(2,q) over Galois field GF(q) and the relations between them by some theorems and examples can be shown.

1. Introduction

A recurring them of this work is the characterization of algebraic varieties in PG(2,q) as finite sets of points with certain combinatorial properties of . this work. Section one which contains some definitions of nucleus point, t-fold nucleus point, Blocking set, t-fold Blocking set, Unital set, (k,n)-arc, flag, strong representive system and, the set of type $(0,1,2,q + 1)$.

Section two, contains some theorems about these subsets and the relation between them and some examples which about some of these subsets.

2.1 Definition "Projective Plane" [2]

A projective plane PG(2,q) over Galois field GF(q) is a two-dimensional projective space, which consists of points and lines with relation between them, in PG(2,q) there are $q^2 + q + 1$ points, and $q^2 + q + 1$ lines, every line contains $1 + q$ points and every point is on $1 + q$ lines, any point in PG(2,q) has the form of a triple (a_1,a_2,a_3) where $a_1, a_2, a_3 \in GF(q)$; such that $(a_1,a_2,a_3) \neq (0,0,0)$. Two points (a_1,a_2,a_3) and (b_1,b_2,b_3) represent the same point if there exists $\lambda \in GF(q)\backslash\{0\}$, such that $(b_1,b_2,b_3) = \lambda\,(a_1,a_2,a_3)$. Similarly any line in PG(2,q) has the form of a triple $[a_1,a_2,a_3]$, where $a_1,a_2,a_3 \in GF(q)$; such that $[a_1,a_2,a_3] \neq [0,0,0]$. Two lines $[a_1,a_2,a_3]$ and $[b_1,b_2,b_3]$ represent the same line if there exists $\lambda \in GF(q)\backslash\{0\}$, such that $[b_1,b_2,b_3] = \lambda\,[a_1,a_2,a_3]$.

There exists one point of the form $(1,0,0)$. There exist q points of the form $(x,1,0)$. There exist q^2 points of the form $(x,y,1)$. A point $p(x_1,x_2,x_3)$ is incident with the line $L[a_1,a_2,a_3]$ if and only if $a_1x_1 + a_2x_2 + a_3x_3 = 0$, i.e.

a point represented by (x_1,x_2,x_3) and the line represented by $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, then

$$(x_1,x_2,x_3)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \Leftrightarrow a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

Any projective plane PG(2,q) satisfies the following axioms:

1. Any two distinct lines are intersected in a unique point.
2. Any two distinct points are contained in a unique line.
3. There exist at least four points such that no three of them are collinear.

2.2 Definition "Blocking Set"[2]

A blocking set B of PG(2,q) is a set of points intersecting every line of PG(2,q) in at least one point, so B is blocking set if and only if PG(2,q)\B is blocking set.

2.3 Definition "Minimal Blocking Set" [2]

A Blocking set B is called minimal in PG(2,q) when no proper subset of it is still a blocking set such that, $\forall$ p $\in$ B, B\{p} is not a blocking set.

2.4 Definition "Nuclei Set"[2]

Let S be a set in PG(2,q), let p be a point in PG(2,q) and p $\notin$ S, p is called nucleus point of a set S if every line in PG(2,q) through p intersects S exactly one, the set of nucleus points of S called nuclei set and denoted by N(S). By the following example we explain the definition:

In a projective plane PG(2,4), let S = {5,6,8,9,10,12,13,14,15,16,17,18,21} and let N(S) = {1,2,3,4} then $\forall$ P $\in$ N(S); P is nucleus point since:through point 1, there are 5 lines which are [1 0 0], [0 0 1],[2 0 1], [0 3 1],[3 3 1] such that each one of them intersects S in one point which are {5,9,10,14,18}, respectively. Similarly for the other points {2,3,4}.

2.5 Definition "t-Fold nucleus Point"[4]

A point p $\in$ PG(2,q) is a t-fold nucleus point of a set S $\subseteq$ PG(2,q) if p $\notin$ S and every line through p meets S at least t points of S.

2.6 Definition "Unital Set" [1]

A unital set PG(2,q) of a square order q is a set U of $(q\sqrt{q}+1)$ points such that each line in PG(2,q) meets U either one or $\sqrt{q}+1$ points, i.e., every line in PG(2,q) is a tangent or a secant of U if contain (1) point or $(\sqrt{q}+1)$ points of line. By the following example we explain the definition:

Let U = {1,6,7,10,11,16,17,18,19} be a set in PG(2,4) and U contains 9 points. U is a unital set since: U contains $4\cdot\sqrt{4}+1 = 9$ points and every and line in PG(2,4) meets U in 1 or 3 points, as shown in the table (1,2).

2.7 Definition "(0,1,2,q + 1)-Set"[1]

A set of points in PG(2,q) is called of type (0,1,2,q + 1) if every line in PG(2,q) meets the set in 0,1,2 or q + 1 points.

2.8 Definition "n-secant"[2]

A line L in PG(2,q) is an i-secant of a (K,n)-arc K if : $|K\cap L| = i$, i=0,1,2,......,n.

2.9 Definition "(k,n)-arc" [2]

A (k,n)-arc in PG(2,q) is a set S of k points with property that every line i contains at most n points of S, a (k,n)-arc S is called complete arc if it is not contained in a (k + 1,n)-arc.
A (k,n)-arc in PG(2,q) is maximal-arc if every line in PG(2,q) is a zero secant or an n-secant of the (k,n)-arc.

2.10 Definition "Flag"[1]

A flag in PG(2,q) is an incident point-line pair;
Flag= $\{(p_i,L_j);\ p_i \in L_j,\ i,j=1,2,\ldots,q^2+q+1\}$.

2.11 Definition "Strong Representive System" [1]

A set $S = \{(p_1,L_1),(p_2,L_2),\ldots,(p_s,L_s)\}$ of flags $p_i \in L_j \Leftrightarrow i = j$ is a strong representive system.

The members of strong represent system that $|S| \leq q\sqrt{q}+1$, with equality if and only if S consists of incident point-tangent pairs of a unital, we denoted points of S by P(S) and the lines of S by L(S), P(S) and L(S) are called spacialy and the others are called ordinary S is called maximal if is not part of Larger-strong represent.

2.12 Definition "Complete Nuclei Set" "New"

Let N(S) be a set of all nucleus points of a set S in PG(2,q), N(S), is called complete nuclei set if N(S) = PG(2,q) \ S.

3. The Relation Between The Sets Subspace of PG(2q) Over GF(q)

This section contains theorems to show that some relations between the : Blocking set ,(k,n)-arc, unital set, nuclei set, strong representive system and the set of type (0,1,2,q+1).

Theorem

Let N(S) be complete, then S is a line.

Proof:

For every point p in N(S), p is a nucleus point of S, then every line in PG(2,q) through p meets S in exactly one point. Since there exists q + 1 lines through p, then there exists at least q + 1 points in S. If there exists another point R in S, then there existS another line through P and R in S then there exists another line through P and R, which is a contradiction since there exists exactly q + 1 lines through P. Hence S contains exactly q + 1 points $P_1, \ldots, P_{q+1}$, suppose these points are not collinear, then ∃ at least one point, say P_{q+1}, not collinear with two points say P_1 and P_2, the line PP_{q+1} intersects the line P_1P_2 in one point, say P_{q+2}, P_{q+2} is on the line P_1P_2 , then P_{q+2} is not on S hence the line PP_{q+1} , since the line PP_{q+1} intersects S in only one point P_{q+1}.

P_{q+2} is not in S, then any line through it intersects S in one point but the line P_1P_2 which passing P_{q+2} intersects S in two points which is a contradiction. Hence the points $P_1, P_2, \ldots, P_{q+1}$ are collinear, similarly for any three points $P_1, P_2, \ldots, P_{q+1}$ are collinear.

Theorem [1] "without prove"

Let B be a blocking in PG(2,q), then for every $P \in$ PG(2,q) \B, p is t-fold nucleus point, if and only if B is a t-fold blocking set, $2 \leq t \leq q$.

Proof:

suppose that B is a blocking set and every $P \in$ PG(2,q) \B, P is a t-fold blocking nucleus point of B, then every line in PG(2,q) through P meets B in at least t points, but from definition of blocking set B can not contain any line, so every line in PG(2,q) meets B in at least t points, then B is a t-fold blocking set.

Conversely, suppose B is a t-fold blocking set then every line in PG(2,q) meets B in at least t points, then for every point $P \in$ PG(2,q) \B and every line through P intersects B in at least t-points, then for every $P \in$ PG(2,q) \B , P is a t-fold nucleus point.

Theorem

In PG(2,q), $q = p^2$, every unital set is a (qp+1,p+1)-arc, $q \geq 3$.

Proof:

Let U be a unital set in PG(2,q), $q = p^2$ then every line in PG(2,q), $q = p^2$ intersectS U in either 1 or p + 1 points and U is a set of qp + 1 points and there are no p + 2 points are collinear then U is (pq+1,p+1)-arc. The converse is not true as shown by the following example:

In projective plan PG(2,4), let K = {1,2,3,4,7,8,11,12,13,15,16,21}, K is an (12,4)-arc, then K is not a unital set since:

1. Every unital in PG(2,q) contains (9) points, but (k,4)-arc contains (12) points.
2. Some line in PG(2,4) meets (k,4)-arc in 4 points, but every line in PG(2,4) most meets every unital set in either 1 or 3 points.
3.

Theorem

In PG(2,q), $q = p^2$ every unital set is a blocking set.

Proof:

let U be a unital set in PG(2,q), $q = p^2$ then U contains qp+1 points and every line in PG(2,q), $q = p^2$ intersects U in either 1 or p + 1 points so every line in PG(2,q), $q = p^2$ meets U, but U dose not contain any line; then U is a blocking set.

The converse is not true and showed by the following example;

In projective plan PG(2,4), let B = {1,2,5,8,9,10,12,13,14,15,16,21}, then B is blocking set but it is not a unital set since:

1. Every unital in PG(2,4) contains (9) points, but B contains (12) points.
2. Some line in PG(2,4) meets (k,4)-arc in 4 points, but every line most meets every unital set in either 1 or 3 points.

Theorem

Let U be a unital set in $PG(2,q^2)$ then every point $P \notin U$,P is either nucleus point or (q+1)-fold nucleus point.

Proof:

Let U is a unital set, then every line meets U in either 1 or q + 1 point, for every point P, every line through P meets U in either 1 or q + 1 points, then P is either nucleus or q + 1-fold nucleus point.

Theorem

Let $S = \{(p_i,L_i); p_i \in B$, B is minimal blocking set; $L_i \in PG(2,q)\ \forall i\}$ then S is a strong representive system.

Proof:

Let $S = \{(p_i,L_j); p_i \in B, \forall\ i ;L_j \in PG(2,q)\ \}$ to prove that S is a strong representive system. Suppose that S is not strong representive system then $\exists\ P_0,P_1 \in B$ s.t. $\exists\ L \in PG(2,q)$ such that $(P_0,L), (P_1,L) \in S$, then the line L in PG(2,q) intersects B in $P_0, P_1 \Rightarrow B \setminus \{P_0\}$ or $B \setminus\{P_1\}$ is blocking set, which is contradiction since B is a minimal blocking set and $\forall\ L \in PG(2,q)$, $|L \cap B| = P$ or L is tangent to B in P.

Theorem

Every maximal (k,2)-arc in PG(2,q) with no 0-secant is a set of type (0,1,2,q+1)-set.

Proof:

It is clear that the maximal (k,2)-arc with no 0-secant mean that every line in PG(2,q) is a 2-secant of (k,2)-arc, so every line intersects maximal (k,2)-arc in two points then the maximal (k,2)-arc is a set of type (0,1,2,q+1)-set.

4. Conclusion and Recommandation

In this research, we took of subspaces of PG(2,q) like Blocking, Nuclei, Unital, (0,1,2,q+1) –set, strong representive system and complete nuclei set as new definitions. Then we found some relations between these subsets and explain them by theorems like; if N(S) is complete then S is line, $\forall\ p \in PG(2,q)\setminus B$, p is t-fold nucleus, if and only if B is t-fold blocking set, every unital set is (qp+1,p+1)-arc, $q \geq 3$, $\forall p \notin u$, u is unital set then p is either nucleus or q+1- fold nucleus point and other relations.

Some of definitions were explained by examples and tables like unital set and nucleus point. So as some theorems, this relation will lead to make new sets of subspaces included at this projective space or others.

References

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Table (1.1):The points and lines of PG(2,4)

i	P_i	L_i				
1	1 0 0	1	2	3	4	5
2	0 1 0	2	6	10	14	18
3	1 1 0	4	6	12	17	19
4	2 1 0	5	7	12	14	21
5	3 1 0	5	8	11	17	18
6	0 0 1	1	18	19	20	21
7	1 0 1	2	7	11	15	19
8	2 0 1	1	6	7	8	9
9	3 0 1	2	8	12	16	20
10	0 1 1	5	6	13	15	20
11	1 1 1	3	9	12	15	18
12	2 1 1	5	9	10	16	19
13	3 1 1	4	7	13	16	18
14	0 2 1	3	8	13	14	19
15	1 2 1	3	7	10	17	20
16	2 2 1	4	8	10	15	21
17	3 2 1	4	9	11	14	20
18	0 3 1	1	14	15	16	17
19	1 3 1	2	9	13	17	21
20	2 3 1	3	6	11	16	21
21	3 3 1	1	10	11	12	13

Table (1.2) :The points and Lines of Untial Set

i	P_i	L_i				
1	1 0 0	1				
2	0 1 0		6	10		18
3	1 1 0		6		17	19
4	2 1 0		7			
5	3 1 0			11	17	18
6	0 0 1	1	18	19		
7	1 0 1		7	11		19
8	2 0 1	1	6	7		
9	3 0 1				16	
10	0 1 1		6			
11	1 1 1					18
12	2 1 1			10	16	19
13	3 1 1		7		16	18
14	0 2 1					19
15	1 2 1		7	10	17	
16	2 2 1			10		
17	3 2 1			11		
18	0 3 1	1			16	17
19	1 3 1				17	
20	2 3 1		6	11	16	
21	3 3 1	1	10	11		

Graph: This graph shows that a set S contains exactly q+1 points