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Solutins of Systems for the Linear Fredholm-Volterra Integral Equations of the Second Kind

H. H. Omran
Department of Mathematics, College of Education, Ibn Al-Haitham, University of Baghdad.

Abstract

In this paper, we present some numerical methods for solving systems of linear Fredholm-Volterra integral equations of the second kind. These methods namely are the Repeated Trapezoidal Method (RTM) and the Repeated Simpson's 1/3 Method (RSM). Also some numerical examples are presented to show the efficiency and the accuracy of the presented work.

1- Introduction

Integral equations have received a considerable interest in the mathematical literature, because of their many filed of applications in different areas of sciences (see, for example [1]-[4]). Many authors gave some numerical solutions for different types of Fredholm integral equations and Volterra integral equations (see, for example [3]-[9]).

In this paper, we show how the numerical methods which are based on the Repeated Trapezoidal Method (RTM) and Repeated Simpson's 1/3 Method (RSM) can be used to solve the following system of linear Fredholm-Volterra integral equation of the second kind:

$$u_i(x) = f_i(x) + \sum_{j=1}^{m} \lambda_{ij} \int_a^b L_{ij}(x,y) u_j(y) dy + \sum_{j=1}^{m} \mu_{ij} \int_a^x K_{ij}(x,y) u_j(y) dy, \quad i = 1,2,\ldots,m. \qquad (1.1)$$

where $a \le x \le b$, λ_{ij}, μ_{ij} are real numbers, f_i, L_{ij}, K_{ij}, are given continuous functions and u_i are the unknown functions that must be determined.

If $\mu_{ij} = 0$ for each i,j=1,2,…,m. then equation (1.1) is called system of linear Fredholm integral equations. Also, if $\lambda_{ij} = 0$ for each i,j=1,2,…,m then equation (1.1) is called system of linear Volterra integral equations. The solution exists for these special types of equation (1.1), (see, [10]-[15]).

2- The Repeated Trapezoidal Method:

Consider the system of Fredholm-Volterra integral equation given by equation (1.1). To solve this equation on the finite interval [a, b], we divide it into n smaller intervals of width h, where $h = (b - a)/n$. The r-th point of subdivision is denoted by x_r, such that $x_r = a + rh$, r = 0, 1, …, n.

If we approximate the integrals that appeared in equation (1.1) by the (RTM) which will yield the following system of linear equations:

$$u_{i,0} = f_{i,0} + \frac{h}{2} \sum_{j=1}^{m} \left(L_{ij,0,0} u_{j,0} + 2 \sum_{s=1}^{r-1} L_{ij,0,s} u_{j,s} + L_{ij,0,n} u_n \right),$$

$$u_{i,r} = f_{i,r} + \sum_{j=1}^{m} \left[\frac{h}{2} \left(L_{ij,r,0} + K_{ij,r,0} \right) u_{j,0} + h \sum_{s=1}^{r-1} \left(L_{ij,r,s} + K_{ij,r,s} \right) u_{j,s} + \frac{h}{2} \left(2L_{ij,r,r} + K_{ij,r,r} \right) u_{j,r} + h \sum_{s=r+1}^{n-1} L_{ij,r,s} u_{j,s} + \frac{h}{2} L_{ij,r,n} u_{j,n} \right], \quad r = 0,1,\ldots,n-1$$

$$u_{i,n} = f_{i,n} + \frac{h}{2}\sum_{j=1}^{m}\Big(\big(L_{ij,n,0} + K_{ij,n,0}\big)u_{j,0} + 2\sum_{s=1}^{n-1}\big(L_{ij,n,s} + K_{ij,n,s}\big)u_{j,s} + \big(L_{ij,n,n} + \mu K_{ij,n,n}\big)u_{j,n}\Big),$$

$$i = 1,2,...,m,\ \ r = 0,1,...,n. \qquad (2.2)$$

By solving the linear system given by equation (2.2) which consists of m(n+1) equations and m(n+1) unknowns, the approximated solution of (1.1) is obtained.

3- The Repeated Simpson's 1/3 Method:

Consider the system of the linear Fredholm-Volterra integral equation of second kind given by equation (1.1). Here we use (RSM) to find the solution of equation (1.1). To do this, we divide the finite interval [a, b] into 2n smaller intervals of width h, where h = (b − a)/2n. The solution of (1.1) in the even nods x_{2r} is given by

$$u_i(x_{2r}) = g_i(x_{2r}) + \sum_{j=1}^{m}\int_a^b L_{ij}(x_{2r},y)u_j(y) + \sum_{j=1}^{m}\int_a^{x_{2r}} K_{ij}(x_{2r},y)u_j(y)dy,$$

$$i = 1,2,...,m,\ \ r = 0,1,...,n. \qquad (3.1)$$

and in the odd nods x_{2r+1} is given by:

$$u_i(x_{2r+1}) = g_i(x_{2r+1}) + \sum_{j=1}^{m}\int_a^b L_{ij}(x_{2r+1},y)u_j(y) + \sum_{j=1}^{n}\int_a^{x_{2r+1}} K_{ij}(x_{2r+1},y)u_j(y)dy,$$

$$i = 1,2,...,m,\ \ r = 0,1,...,n. \qquad (3.2)$$

By using the (RSM) formula to approximate the integrals that appeared in equations (3.1) - (3.2) one can get the following system of equations:

$$u_{i,0} = f_{i,0} + \frac{h}{3}\sum_{j=1}^{m}\left(L_{ij,0,0}u_{j,0} + 4\sum_{s=1}^{n-1}L_{ij,0,2s-1}u_{j,2s-1} + 2\sum_{s=1}^{n-1}L_{ij,0,2s}u_{j,2s} + L_{ij,0,n}u_{j,n}\right),\ i = 1,2,...,m,$$

$$u_{i,2r} = f_{i,r} + \frac{h}{3}\sum_{j=1}^{m}\Big[\big(L_{ij,2r,0} + k_{ij,2r,0}\big)u_{j,0} + 4\sum_{s=1}^{r-1}\big(L_{ij,2r,2s-1} + k_{ij,2r,2s-1}\big)u_{j,2s-1} +$$

$$4\sum_{s=1}^{r-1}\big(L_{ij,2r,2s-1} + k_{ij,2r,2s-1}\big)u_{j,2s-1} + \big(2L_{ij,2r,2r} + k_{ij,2r,2r}\big)u_{j,2r} + 4\sum_{s=r+1}^{n}L_{ij,2r,2s-1}u_{j,2s-1} +$$

$$2\sum_{s=r+1}^{n-1}L_{ij,2r,2s}u_{j,2s} + +L_{ij,2r,2n}u_{j,2n}\Big),\ r = 1,2,\ldots,n-1,$$

$$u_{i,2r+1} = g_{i,2r+1} + \frac{h}{3}\sum_{j=1}^{m}\Big(\big(L_{ij,2r+1,0} + K_{ij,2r+1,0}\big)u_{j,0} + 4\sum_{s=1}^{r}\big(L_{ij,2r+1,2s-1} + K_{ij,2r+1,2s-1}\big)u_{j,2s-1} +$$

$$2\sum_{s=1}^{r-1}\big(L_{ij,2r+1,2s} + K_{ij,2r+1,2s}\big)u_{j,2s} + \left(2L_{ij,2r+1,2s} + \frac{5}{2}K_{ij,2r+1,2s}\right)u_{j,2s} +$$

$$\left(4L_{ij,2r+1,2r+1} + \frac{3}{2}K_{ij,2r+1,2r+1}\right)u_{j,2r+1} + 4\sum_{s=r+2}^{n}L_{ij,2r+1,2s-1}u_{j,2r-1} + 2\sum_{s=r+1}^{n-1}L_{ij,2r+1,2s}u_{j,2s} +$$

$$L_{ij,2r+1,2r+1}u_{2j-1}\Big),\ r = 0,1,\ldots,n-1,\ i = 1,2,...,m,$$

$$u_{i,2n} = g_{i,2n} + \frac{h}{3}\sum_{j=1}^{m}\Big(\big(L_{ij,2n,0} + K_{ij,2n,0}\big)u_{j,0} + 4\sum_{s=1}^{n}\big(L_{ij,2n,2s-1} + K_{ij,2n,2s-1}\big)u_{j,2s-1} +$$
$$2\sum_{s=1}^{n-1}\big(L_{ij,2n,2s} + K_{ij,2n,2s}\big)u_{j,2s} + \big(L_{ij,2n,2n} + K_{ij,2n,2n}\big)u_{j,2n}\Big),\ \ i = 1,2,\ldots,m. \qquad (3.3)$$

By solving the linear system given by equation (3.3) which consists of m(n+1) equations and m(n+1) unknowns, the numerical solution of (1.1) is obtained.

4-Numerical examples:

In this section, we present some examples and their absolute errors to show the high accuracy of the solution obtained by (RTM) and (RSM) for solving systems of linear Fredholm-Volterra integral equations of the second kinds. The results for these examples, are computed by using Matlab Version 2007.

Example (1):

Consider the system of Fredholm-Volterra integral equation:

$$u_1(x) = -\frac{5}{6} + \frac{9}{4}x - \frac{3}{2}x^2 - \frac{5}{6}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^6 + \int_0^1 (y-x)u_1(y)\,dy + \int_0^1 (x+5y)u_2(y)\,dy +$$
$$\int_0^x (x+y)u_1(y)\,dy + \int_0^x (xy+1)u_2(y)\,dy, \quad 0 \le x \le 1,$$
$$u_2(x) = \frac{4}{5} - \frac{5}{12}x - \frac{5}{4}x^2 + \frac{5}{6}x^3 + \frac{1}{4}x^4 - \frac{1}{4}x^5 - \frac{1}{5}x^7 + \int_0^1 \frac{(xy-2)}{2}u_1(y)\,dy + \int_0^1 (x^2+y+2)u_2(y)\,dy +$$
$$\int_0^x (x-y^2)u_1(y)\,dy + \int_0^x (x^2y+x)u_2(y)\,dy, \quad 0 \le x \le 1.$$

with the exact solutions: $u_1(x)=x+1$ and $u_2(x)=x^3$.

We solved this system with (RTM) and (RSM). Tables 1 and 2 shows the a selection absolute error of approximated solutions for example (1) for h = 0.1, 0.02, 0.01 and figure (1) shows that the comparison between the exact solutions and the numerical solutions via (RTM) and (STM) for h = 0.1

Example (2):

Consider the system of Fredholm-Volterra integral equation:

$$u_1(x) = (4 - e - \cos(1) - 2\sin(1))x^2 - 2e^x x + 2x\cos(x)2e^x - \sin(x) - 1 +$$
$$\int_0^1 x^2y^2\big(u_1(y)+u_2(y)\big)dy + \int_0^x (x+y)\big(u_1(y)+u_2(y)\big)dy +, \quad 0 \le x \le 1,$$
$$u_2(x) = (\cos(x) - e^x)x^2 + (e^x - \sin(x) + e - 1)x + \sin(x) - \sin(1) + \cos(1) - 1 -$$
$$\int_0^1 (x-y)\big(u_1(y)+u_2(y)\big)dy + \int_0^x xy\big(u_1(y)+u_2(y)\big)dy, \quad 0 \le x \le 1.$$

with the exact solutions: $u_1(x) = e^x$ and $u_2(x)= \sin(x)$

By using (RTM) method and (RSM), the problem can be solved, and a selection absolute error for h = 0.1, 0.02, 0.01, are listed in table (3) and (4). Figure (2) shows comparison between the exact solutions and the numerical solutions of that example (2) for h = 0.1.

5- Conclusions and Recommendations:

The systems of linear Fredholm-Volterra integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the numarical solutions, for this purpose the presented methods can be proposed. From numerical examples it can be seen that the proposed numerical methods are efficient and accurate to estimate the solution of these equations, Also, we show that when the values of h decreases, the absolute errors decrease to small values and the (RSM) then more accurate results than (RTM).

6-References

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Table (1): The Absolute Errors at Some Mesh Points of Example (1) by Using (RTM).

x_r	E_{1r}	E_{2r}	E_{1r}	E_{2r}	E_{1r}	E_{2r}
	h=0.1		h=0.02		h=0.01	
0	5.06286×10^{-3}	5.06877×10^{-4}	2.07180×10^{-4}	2.06630×10^{-5}	5.18322×10^{-5}	5.16887×10^{-6}
0.1	4.70150×10^{-3}	8.77017×10^{-4}	1.92551×10^{-4}	3.56507×10^{-5}	4.84789×10^{-5}	8.91723×10^{-6}
0.2	4.47285×10^{-3}	1.44803×10^{-3}	1.83361×10^{-4}	5.87847×10^{-5}	4.60420×10^{-5}	1.47031×10^{-5}
0.3	4.38597×10^{-3}	2.21049×10^{-3}	1.79978×10^{-4}	8.96918×10^{-5}	4.50450×10^{-5}	2.24331×10^{-5}
0.4	4.46029×10^{-3}	3.16647×10^{-3}	1.83197×10^{-4}	1.28468×10^{-4}	4.58372×10^{-5}	3.21313×10^{-5}
0.5	4.73251×10^{-3}	4.33054×10^{-3}	1.94515×10^{-4}	1.75718×10^{-4}	4.86701×10^{-5}	4.39494×10^{-5}
0.6	5.26697×10^{-3}	5.73390×10^{-3}	2.16546×10^{-4}	2.32730×10^{-4}	5.41830×10^{-5}	5.82094×10^{-5}
0.7	6.17244×10^{-3}	7.43297×10^{-3}	2.53692×10^{-4}	3.01822×10^{-4}	6.34768×10^{-5}	7.54914×10^{-5}
0.8	7.63063×10^{-3}	9.52479×10^{-3}	3.13269×10^{-4}	3.86967×10^{-4}	7.83812×10^{-5}	9.67892×10^{-5}
0.9	9.94575×10^{-3}	1.21739×10^{-2}	4.07468×10^{-4}	4.94878×10^{-4}	1.01943×10^{-4}	1.23782×10^{-4}
1	1.36332×10^{-2}	1.56602×10^{-2}	5.56811×10^{-4}	6.36931×10^{-4}	1.39294×10^{-4}	1.59316×10^{-4}

Table(2): The Absolute Errors Some Mesh Points of Example (1) byUsing (RSM).

x_r	E_{1r}	E_{2r}	E_{1r}	E_{2r}	E_{1r}	E_{2r}
	h=0.1		h=0.02		h=0.01	
0	9.85011×10^{-4}	2.53612×10^{-4}	8.12018×10^{-6}	2.01117×10^{-6}	1.01950×10^{-6}	2.51490×10^{-7}
0.1	7.35843×10^{-4}	3.81895×10^{-5}	5.98137×10^{-6}	1.41866×10^{-7}	9.65248×10^{-7}	2.24867×10^{-7}
0.2	9.02691×10^{-4}	1.85301×10^{-4}	7.48408×10^{-6}	1.45430×10^{-6}	9.40272×10^{-7}	1.81698×10^{-7}
0.3	5.89914×10^{-4}	1.20343×10^{-4}	4.80683×10^{-6}	1.10028×10^{-6}	9.43165×10^{-7}	1.22708×10^{-7}
0.4	9.30264×10^{-4}	5.57167×10^{-5}	7.74846×10^{-6}	3.83134×10^{-7}	9.74053×10^{-7}	4.72401×10^{-8}
0.5	4.91389×10^{-4}	2.62787×10^{-4}	3.97669×10^{-6}	2.19672×10^{-6}	1.03426×10^{-6}	4.65516×10^{-8}
0.6	1.07408×10^{-3}	1.44862×10^{-4}	8.95561×10^{-6}	1.27950×10^{-6}	1.12600×10^{-6}	1.61511×10^{-7}
0.7	4.09887×10^{-4}	3.36511×10^{-4}	3.22345×10^{-6}	2.69014×10^{-6}	1.25191×10^{-6}	3.01138×10^{-7}
0.8	1.35720×10^{-3}	4.42686×10^{-4}	1.12569×10^{-5}	3.73015×10^{-6}	1.41443×10^{-6}	4.68890×10^{-7}
0.9	3.29348×10^{-4}	2.63176×10^{-4}	2.33478×10^{-6}	1.88579×10^{-6}	1.61460×10^{-6}	6.66812×10^{-7}
1	1.80589×10^{-3}	8.64267×10^{-4}	1.47532×10^{-5}	7.12125×10^{-6}	1.84972×10^{-6}	8.92877×10^{-7}

Table(3): The Absolute Errors at Some Mesh Points of Example (2) by Using (RTM).

x_r	E_{1r}	E_{2r}	E_{1r}	E_{2r}	E_{1r}	E_{2r}
	h=0.1		h=0.02		h=0.01	
0	0	4.59400×10^{-2}	0	1.68901×10^{-3}	0	4.21188×10^{-4}
0.1	1.39964×10^{-3}	3.94492×10^{-2}	5.28576×10^{-5}	1.45109×10^{-3}	1.31920×10^{-5}	3.61864×10^{-4}
0.2	4.82389×10^{-3}	3.31494×10^{-2}	1.80143×10^{-4}	1.22046×10^{-3}	4.49441×10^{-5}	3.04359×10^{-4}
0.3	1.01980×10^{-2}	2.71552×10^{-2}	3.79220×10^{-4}	1.00132×10^{-3}	9.45993×10^{-5}	2.49722×10^{-4}
0.4	1.76153×10^{-2}	2.16013×10^{-2}	6.53606×10^{-4}	7.98625×10^{-4}	1.63036×10^{-4}	1.99188×10^{-4}
0.5	2.73715×10^{-2}	1.66865×10^{-2}	1.01421×10^{-3}	6.19689×10^{-4}	2.52974×10^{-4}	1.54580×10^{-4}
0.6	4.00359×10^{-2}	1.27385×10^{-2}	1.48187×10^{-3}	4.76515×10^{-4}	3.69612×10^{-4}	1.18892×10^{-4}
0.7	5.65854×10^{-2}	1.03185×10^{-2}	2.09218×10^{-3}	3.89560×10^{-4}	5.21819×10^{-4}	9.72234×10^{-5}
0.8	7.86479×10^{-2}	1.04044×10^{-2}	2.90402×10^{-3}	3.94267×10^{-4}	7.24273×10^{-4}	9.84091×10^{-5}
0.9	1.08946×10^{-1}	1.47285×10^{-2}	4.01514×10^{-3}	5.52950×10^{-4}	1.00133×10^{-3}	1.37977×10^{-4}
1	1.52124×10^{-1}	2.64217×10^{-2}	5.59065×10^{-3}	9.77200×10^{-4}	1.39412×10^{-3}	2.43730×10^{-4}

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Table(4): The Absolute Errors Some Mesh Points of Example (2) by Using (RSM).

x_r	E_{1r}	E_{2r}	E_{1r}	E_{2r}	E_{1r}	E_{2r}
	h=0.1		h=0.02		h=0.01	
0	0	1.72067×10^{-3}	0	1.42600×10^{-5}	0	1.79214×10^{-6}
0.1	3.94896×10^{-4}	1.48330×10^{-3}	3.24894×10^{-6}	1.22918×10^{-5}	4.22744×10^{-8}	1.50903×10^{-6}
0.2	1.57813×10^{-4}	1.18168×10^{-3}	1.30728×10^{-6}	9.79618×10^{-6}	1.64235×10^{-7}	1.23117×10^{-6}
0.3	7.70606×10^{-4}	1.04351×10^{-3}	6.35692×10^{-6}	8.64332×10^{-6}	3.63302×10^{-7}	9.63412×10^{-7}
0.4	6.19239×10^{-4}	6.82325×10^{-4}	5.12401×10^{-6}	5.66094×10^{-6}	6.43696×10^{-7}	7.11472×10^{-7}
0.5	1.47546×10^{-3}	6.89838×10^{-4}	1.22051×10^{-5}	5.71102×10^{-6}	1.01797×10^{-6}	4.83757×10^{-7}
0.6	1.45303×10^{-3}	2.81265×10^{-4}	1.20207×10^{-5}	2.33981×10^{-6}	1.51008×10^{-6}	2.94049×10^{-7}
0.7	2.66905×10^{-3}	5.22096×10^{-4}	2.21131×10^{-5}	4.32653×10^{-6}	2.16111×10^{-6}	1.65889×10^{-7}
0.8	2.92364×10^{-3}	1.33308×10^{-4}	2.41924×10^{-5}	1.11631×10^{-6}	3.03940×10^{-6}	1.40236×10^{-7}
0.9	4.81510×10^{-3}	8.24729×10^{-4}	3.99044×10^{-5}	6.84114×10^{-6}	4.25899×10^{-6}	2.89457×10^{-7}
1	5.78126×10^{-3}	7.12361×10^{-4}	4.78547×10^{-5}	5.91702×10^{-6}	6.01328×10^{-6}	7.43744×10^{-7}

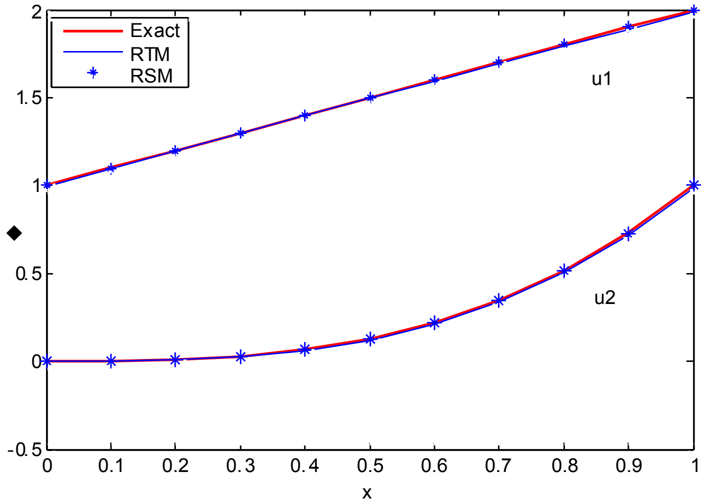

Fig (1): Comparison between the exact solution and numerical solution via (RTM) and (RSM) of example(1) for h=0.1 .

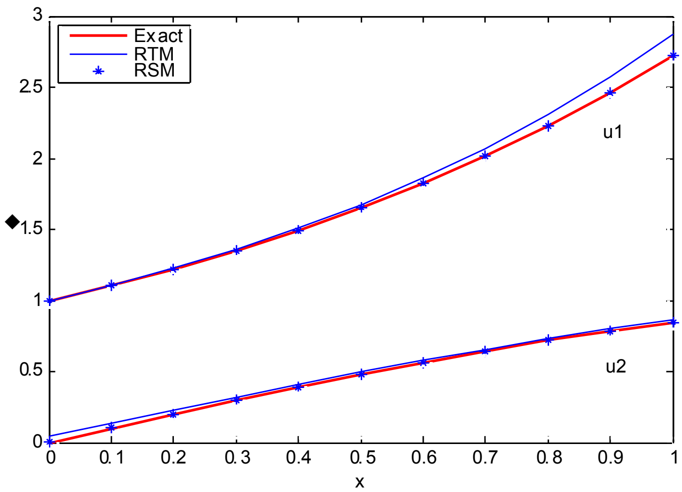

Fig (2): Comparison between the exact solution and numerical solution via (RTM) and (RSM) of example(2) for h=0.1 .

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حلول انظمة معادلات فريدهول– فولتيرا التكاملية الخطية من النوع الثانـي

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قسم الرياضيات،كلية التربية ابن الهيثم ، جامعة بغداد.

الخلاصة

قدمنا بعض الطرائق العددية لحل انظمة معادلات فريدهولم – فولتيرا التكاملية الخطية من النوع الثاني . هذه الطرائق هي طريقة شبه المنحرف المتكررة وطريقة سمبسون 1 / 3 المتكررة. الامثلة العددية اعطيت لتبيان كفايه ودقة العمل المقدم.