An Efficient Single Stage Shrinkage Estimator for the Scale parameter of Inverted Gamma Distribution

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Abstract

The present paper agrees with estimation of scale parameter θ of the Inverted Gamma (IG) Distribution when the shape parameter α is known (α=1), bypreliminarytestsinglestage shrinkage estimators using suitable shrinkage weight factor and region.
The expressions for the Bias, Mean Squared Error [MSE] for the proposed estimators are derived. Comparisons between the considered estimator with the usual estimator (MLE) and with the existing estimator are performed .The results are presented in attached tables.

Keywords: Inverted Gamma Distribution, Maximum Likelihood Estimator (MLE), Shrinkage EstimatorPretest Region, Bias, Mean Squared Error and Relative Efficiency.

Introduction

"In Reliability studies the models which are used in life testing include the Exponential, Gamma, Lognormal and Inverted Gammadistributions. If the failure is mainly due to aging or wearing out process, then its reasonable in many applicationsto choose one of the above mentioned distribution.In a sense, this distribution is unnecessary,it has the same distribution as the reciprocal of a gamma distribution .However, a catalogue of results for the inverse gamma distribution prevents having to repeatedly apply the transformation theorem in applications";[1],[2],[3],[4].

"TheInverted Gamma distribution is prospective to use in life experiments"; [5], it has probability density function (p.d.f) with two parameters α and θ as below :

$$f(x;\propto,\theta)=\frac{x^{-(\alpha+1)}}{\Gamma_{\alpha}\theta^{\alpha}}e^{-1/x\theta};x\geq 0,\alpha,\theta\geq 0 \text{ ------------------------(1)}$$

Here α and θ arerespectively the shape and scale parameters. In conventional notation, wewrite X~IG (α , θ).

This paper deals with the problem for estimation the unknown scale parameter (θ) of IG distribution with known shape parameter (α) when a prior estimate (θ_o) regarding the actual value (θ) is available using preliminary test single stage shrinkage estimator.

It is Well-known that, the prior knowledge regarding due reasons introduced by Thompson [9] as well as the classical estimator of ($\hat{\theta}_{MLE}$) and using shrinkage weight function [$\psi(\hat{\theta})$] ;$0\leq\psi(\hat{\theta})\leq 1$ results what it isknown as "shrinkage estimator", which though perhaps biased has smaller mean squared error (MSE) than that of $\hat{\theta}_{MLE}$.

Thus "Thompson – Type" shrinkage estimator will be

$$\tilde{\theta}_{ss}=\psi(\hat{\theta})\hat{\theta}_{MLE}+(1-\psi(\hat{\theta}))\,\theta_o;\text{-------------(2)}$$

Now , to test the prior knowledge of weather close to actual value θ and to be comfortable to use this priorknowledge,the preliminary test single stage shrinkage estimator (SSSE) will be used for this mission when using the test estimator of level of significant (∆) for testing the hypotheses

$H_O: \theta=\theta_o$ VS. $H_A: \theta\neq\theta_o$

If H_O correct, thenthe estimator which is defined in (2) will be used.

Conversely , if $H_O$rejected,thendifferentshrinkage weight functions$\Psi_2(\hat{\theta})$;$0\leq \Psi_2(.)\leq 1$will be used and then using the followingshrinkage estimator

$$\tilde{\theta}_{ss}=\Psi_2(\hat{\theta})\hat{\theta}_{MLE}+(1-\Psi_2(\hat{\theta}))\,\theta_0 \quad \text{--------------------------------------(3)}$$

Consequently, thecommon form of preliminary test single stage shrinkage estimator(SSSE) will be

$$\tilde{\theta}_{ss}=\begin{cases}\Psi_1(\hat{\theta})\hat{\theta}_{MLE}+\left(1-\Psi_1\left(\hat{\theta}\right)\right)\theta_0 & \text{if } \hat{\theta}\in R\\ \Psi_2(\hat{\theta})\hat{\theta}_{MLE}+(1-\Psi_2(\hat{\theta}))\theta_o & \text{if } \hat{\theta}\notin R\end{cases}\text{-------------(4)}$$

$Where\ \Psi_i(\theta);\ 0 \le \Psi_i(\theta^\wedge) \le 1; i$=1,2 isa shrinkage weight function specifying the belief of $\theta^\wedge$and (1- $\Psi(\theta^\wedge)$) specifying the belief $\theta_o$and $\Psi_1(\theta^\wedge)$ may be a function of $\theta^\wedge$ or may be a constant (ad hoc basis),while (R) is a pretest region for acceptance of the prior knowledge with level of significance (Δ) .

Numerousauthors have been studied the estimator(4)for estimating parameters,see for example[6],[7]and[8].

The purpose of this paper is to employ the preliminary test single stage (SSSE) defined by (4) for estimating the scale parameter (θ) of two parameters Inverted Gamma (IG) distribution when the shape parameter (α) is known.

The expressions of Bias, Mean Squared Error (MSE) and Relative Efficiency (R.Eff(.)) were derived for the proposed estimator.

Numerical resultsand conclusions due mentioned expressions including some constantswereachieved and put in annexed tables.

Comparisons between the proposed estimators with the classical estimator and with existing estimator are performed.

Maximum Likelihood Estimator of θ

Let x_1, x_2,---, x_n be a randomsample of size n form IG (1,θ), then the natural logarithm of the Likelihood function L(1,θ) can be written as:

$$\ln l\left(\frac{x}{\theta}\right) = -n\ ln\theta - \sum_{i=1}^{n} lnx_i^2 - \frac{1}{\theta}\sum_{i=1}^{n} 1/x_i \text{------------------------------(5)}$$

$$\frac{\partial lnl}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} 1/x_i \text{-----------------------------(6)}$$

Let$\frac{\partial lnl}{\partial\theta} = 0$,then the maximum Likelihood estimator of θ is

$$\theta^\wedge_{MLE} = \frac{\sum_{i=1}^{n} 1/x_i}{n} \text{---------------------------------(7)}$$

The distribution of $\theta^\wedge_{MLE}$is G(nα ,θ/n)

$$E\ (\theta^\wedge_{MLE}) = \alpha\theta\ \ ,\ var\ (\theta^\wedge_{MLE}) = \ \alpha\theta^2/n$$

$$Bias(\theta^\wedge_{MLE}) = E(\theta^\wedge) - \theta$$
$$= \propto\theta - \theta$$
$$= \theta(\propto -1)$$

And,
$$MSE(\theta^\wedge_{MLE}) = \frac{\propto\theta^2}{n} + \ \theta^2(\propto -1)^2$$

Preliminary Test Single Shrinkage Estimator (PTSSSE).

Using the form (4), we proposed the preliminary test single stage shrinkage estimator$\theta_{ss}^{\sim}$ for estimator the scale parameter θ of Inverted Gamma distribution when a prior knowledge θ_o available about θ with known shape α =1 as below:-

$$\theta^{\sim}_{ss} = \begin{cases} \theta_0 & if\ \theta^\wedge \in R \\ k\ \theta^\wedge_{MLE} + (1-K)\theta_O & if\ \theta^\wedge \notin R \end{cases} \text{-----(8)}$$

i.e. we put $\psi_1(\theta^\wedge) = 0\ \ \ and\ \ \psi_2(\theta^\wedge) = k$in equation (4)and R is the pretest region

$i.e.;\ R = [\frac{\theta_O}{2\theta}\chi^2_{\frac{\Delta}{2},2n}\ ,\ \frac{\theta_O}{2\theta}\chi^2_{1-\frac{\Delta}{2},2n}]$----------------------------------(9)

For simplicity , assume R =[a,b] , a<b
i.e.
a=$\frac{\theta_O}{2\theta}\chi^2_{\frac{\Delta}{2},2n}$, b=$\frac{\theta_O}{2\theta}x^2_{1-\frac{\Delta}{2},2n}$ -------------------------(10)

Where$\chi^2_{\frac{\Delta}{2},2n}$ and $x^2_{1-\frac{\Delta}{2},2n}$
are respectively the lower and upper 100(Δ/2) percentile point of chi-square distribution with (2n) degree of freedom.
The expression for the bias of the estimator $\theta_{ss}{}^{\sim}$ is as follow:-

Bias ($\theta^{\sim}_{ss}$) = E ($\theta^{\sim}_{ss}$) - θ

$$=\int_R (\theta_0-\theta)\, f\,(\theta^{\wedge}_{MLE})\, d\theta^{\wedge}_{MLE} + \int_{R^-}[k(\theta^{\wedge}-\theta_O\ ) + (\theta_O\ -\theta)]\ f\,(\theta^{\wedge}_{MLE})d\theta^{\wedge}_{MLE}$$

WhereR^- is the complement region of R in real space and f(θ^) isa (P D F) of ($\theta^{\wedge}_{MLE}$) which has the following forms.

$$f(\theta^{\wedge}_{MLE}) = \frac{(\theta^{\wedge})^{n-1}e^{-n\theta^{\wedge}/\theta}}{\Gamma(n)(\frac{\theta}{n})^n};\qquad \theta>0\ ,\ 0<\theta^{\wedge}_{MLE}<\infty \quad \text{------------------(11)}$$

We conclude

$$Bias\big(\theta^{\sim}_{ss}\,,|\,\theta,R\big) = \theta[(\zeta-1)(1-k)-(\frac{k}{n})\,J_1\,(a^*,b^*)+kJ_O\,(a^*,b^*)]$$
$$=\left[\theta(\zeta-1)(1-k)-k(\frac{1}{n}J_1(a^*,b^*)-J_O(a^*,b^*))\right]$$
$$=\theta\left\{(h-k)\left[\frac{1}{n}J_1(a^*,b^*)-J_O(a^*,b^*)\right]+(\zeta-1)(1-k)\right\}\text{--------(12)}$$

Where$J_l(a^*,b^*) = \int_{a^*}^{b^*} y^l\ f(y)dy\ ;\ l = 0\,,1\,,2\,.$

$and\ \zeta = \frac{\theta_o}{\theta}$, $a^*=\frac{\zeta}{2}\chi^2_{\frac{\Delta}{2},2n}$, $b^*=\frac{\zeta}{2}\,x^2_{1-\frac{\Delta}{2},2n}$ andy $= \frac{n\theta^{\wedge}}{\theta}$----------(13)

The bias ratio B(.) of the estimator ($\theta^{\sim}_{ss}$) is defined below

$B(.) = Bias((\theta^{\sim}_{ss}|\theta,R)\ /\theta$ -----------(14)

And the expression for mean square error (MSE) of $\theta^{\sim}_{ss}$ is given asbelow:

$$\text{MSE}\,(\theta_{ss}^{\sim}|\theta,\text{R}) = \theta^2\Big\{(h^2-k^2)\Big[\frac{1}{n^2}J_2(a^*,b^*) - \frac{2}{n}\zeta J_1(a^*,b^*) + \zeta^2 J_0(a^*,b^*)\Big] + 2(\zeta-1)(h-K)\Big[\frac{1}{n}J_1(a^*,b^*) - \zeta J_0(a^*,b^*)\Big] + k^2\Big[\frac{1}{n} + (\zeta-1)^2\Big] - 2k(\zeta-1)^2 + (\zeta-1)^2\Big\} \text{---(15)}$$

In this paper we use the shrinkage weight factor (k) as the same as of Thompson – type as below

$$k = \frac{(\theta_\circ - \theta)^2}{(\theta_\circ - \theta)^2 + var(\theta_{MLE}^{\wedge})}$$

And by simple calculation

$$k = \frac{(\zeta-1)^2}{(\zeta-1)^2 + 1/n} \quad \text{-----------------------------------(16)}$$

The Relative Efficiency of estimator $\theta_{ss}^{\sim}$ w.r.t the classical estimator $\theta_{MLE}^{\wedge}$ is defined as below:-

$$R.Eff\,(\theta_{ss}^{\sim}.|\ \theta\,,R) = \frac{MSE(\theta_{MLE}^{\wedge})}{MSE(\theta_{ss}^{\sim}|\ \theta\,,R)} \quad \text{-------(17)}$$

See for example [6],[7],[8]and[9].

Conclusions and Numerical Results

The computations of Relative Efficiency[R.Ef f(.)] and Bias Ratio [B(.)] for the equation(14) and (17) were used for the estimator $\theta_{ss}^{\sim}$.Thesecomputations (using Math. CAD program) were performed for Δ=0.01, 0.05, 0.1,ζ =0.25(0.25)2 and n = 4, 6, 8, 10, 12. These computation are given in attached tables No.(1)and(2) for some samples of these constants.The observation mentioned in the tables leads to the following results.

1.Bias Ratio [B(.)] of $\theta_{ss}^{\sim}$ increases whenζ increase.
2. The R.E f f of $\theta_{ss}^{\sim}$ are adversely proportional with value of Δ especially when ζ = 1(θ_o =θ) this yields Δ=0.01 has higher R. efficiency for all n.
3. Bias ratio of $\theta_{ss}^{\sim}$increases when Δincreases especially when ζ= 1.
4. The Relative Efficiency [R.E ff(.)]decreases when n increases for all Δ and ζ.
5. Relative Efficiency has the highest value at ζ =1 (θ_o =θ) and decreases otherwise.
6. The proportional estimator is better than the classical estimator in the sense of Mean Squared Error.

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Table (1) Showed Bias Ratio [B(.)]of $\theta_{ss}^{\sim}$

Δ	n	B(-)	ζ							
			0.75	0.50	0.25	1	1.25	1.50	1.75	2
0.01	4	B(-)	-0.75	-0.5	-0.25	-1.966415E-9	0.25	0.5	0.75	1
	6	B(-)	-0.75	-0.5	-0.25	-1.670148E-10	0.25	0.5	0.75	0.9999999
	8	B(-)	-0.7499999	-0.5	-0.25	-6.020044E-12	0.25	0.5	0.7499999	0.9999999
	10	B(-)	-0.7499999	-0.5	-0.25	-1.193395E-13	0.25	0.4999999	0.7499999	0.9999999
	12	B(-)	-0.7499999	-0.4999999	-0.25	-1.490223E-15	0.25	0.4999999	0.7499999	0.9999999
0.05	4	B(-)	-0.7499813	-0.4999878	-0.2499944	-1.2290094E-6	0.2499919	0.4999849	0.749978	0.9999711
	6	B(-)	-0.7499719	-0.4999813	-0.2499907	-1.0438422E-7	0.2499904	0.4999808	0.7499712	0.9999615
	8	B(-)	-0.7499625	-0.499975	-0.2499875	-3.7625274E-9	0.2499875	0.499975	0.7499624	0.9999499
	10	B(-)	-0.7499531	-0.4999688	-0.2499844	-7.458721E-11	0.2499844	0.4999687	0.7499531	0.9999375
	12	B(-)	-0.7499437	-0.4999625	-0.2499813	-9.313892E-13	0.2499812	0.4999625	0.7499437	0.999925
0.1	4	B(-)	-0.7497014	-0.4998049	-0.2499109	-1.966415E-5	0.2498698	0.4997585	0.7496477	0.9995381
	6	B(-)	-0.74955	-0.4997002	-0.2498506	-1.6701475E-6	0.2498464	0.4996933	0.7495389	0.9993833
	8	B(-)	-0.7494	-0.4996	-0.2498	-6.0200438E-8	0.2497998	0.4995995	0.749399	0.999198
	10	B(-)	-0.74925	-0.4995	-0.24975	-1.1933954E-9	0.24975	0.4995	0.7492499	0.9989999
	12	B(-)	-0.7491	-0.4994	-0.2497	-1.490223E-11	0.2497	0.4994	0.7491	0.9988

Table (2) Showed Relative Efficiency [R.Eff (.)] of $\theta_{SS}^{\sim}$.

Δ	n	R.Eff(-)	ζ							
			0.75	0.50	0.25	1	1.25	1.50	1.75	2
0.01	4	R.Eff(-)	0.4444445	1.0000001	4.0000003	5.4235342E+14	4.0000004	1.0000001	0.4444445	0.25
	6	R.Eff(-)	0.2962963	0.6666667	2.666667	2.7386475E+14	2.666667	0.6666667	0.2962963	0.1666667
	8	R.Eff(-)	0.2222223	0.5000001	2.0000003	1.5616604E+14	2.0000003	0.5000001	0.2222223	0.125
	10	R.Eff(-)	0.1777778	0.4000001	1.6000003	9.9998908E+13	1.6000003	0.4000001	0.1777778	0.1
	12	R.Eff(-)	0.1481482	0.3333334	1.3333337	6.9444435E+13	1.3333337	0.3333334	0.1481482	0.0833334
0.05	4	R.Eff(-)	0.4444667	1.0000497	4.0001857	1.3884247E+9	4.0002792	1.0000666	0.4444739	0.2500166
	6	R.Eff(-)	0.2963185	0.6667167	2.6668661	7.0109376E+8	2.6668729	0.6667184	0.2963195	0.1666799
	8	R.Eff(-)	0.2222444	0.50005	2.0002	3.9978506E+8	2.0002002	0.5000501	0.2222445	0.1250126
	10	R.Eff(-)	0.1778	0.40005	1.6002	2.5599721E+8	1.6002	0.40005	0.1778	0.1000125
	12	R.Eff(-)	0.1481704	0.3333833	1.3335333	1.7777775E+8	1.3335333	0.3333833	0.1481704	0.0833458
0.1	4	R.Eff(-)	0.4448011	1.0007958	4.0029708	5.4235342E+6	4.0044671	1.0010656	0.4449165	0.2502665
	6	R.Eff(-)	0.2966521	0.6674671	2.6698578	2.7386475E+6	2.6699666	0.6674953	0.2966682	0.1668785
	8	R.Eff(-)	0.2225782	0.5008008	2.0032011	1.5616604E+6	2.0032051	0.5008023	0.2225793	0.1252013
	10	R.Eff(-)	0.1781338	0.400801	1.6032022	9.9998908E+5	1.6032023	0.4008011	0.1781339	0.1002003
	12	R.Eff(-)	0.1485043	0.3341346	1.3365365	6.9444435E+5	1.3365365	0.3341346	0.1485043	0.0835337

مقدر التخلص ذي المرحله الواحده الكفؤ لمعلمة قياس توزيع معكوس كَاما

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الخلاصة

يتعامل هذا البحث مع تقدير معلمةالقياس θ لتوزيع معكوس كَاما ذي المعلمتين عندما تكون معلمة الشكل$\propto$ معلومة وتساوي 1 بطريقة مقدر الاختبار الأولي المتقلص ذي المرحلة الواحدة (SSSE) وذلك باستعمال عامل وزن ومجال مناسب.أشتقت معادلات التحيز، متوسطمربعات الخطأ (MSE) المقدر المقترح .

وأجريت مقارنات بين المقدر المقترح مع المقدر الكلاسيكي (MLE) ومع المقدرات الموجودة التي انجزت وتم عرض هذة النتائج في الجداول المرفقة.

الكلمات المفتاحية : توزيع معكوس كَاما ، مقدرالأمكان الاعظم ، مقدر الاختبار المتقلص ، التحيز ، متوسط مربع الخطأ والكفأة النسبية