# Quasi-Fully Cancellation Fuzzy Modules

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## Abstract

In this paper it was presented the idea quasi-fully cancellation fuzzy modules and we will denote it by Q-FCF(M), condition universalistic idea quasi-fully cancellation modules It .has been circulated to this idea quasi-max fully cancellation fuzzy modules and we will denote it by Q-MFCF(M). Lot of results and properties have been studied in this research.

**Key word :** Q-FCF(M), Q-MFCF(M), Direct Sum Q-FCF(M).

## Introduction

Let R be a commutative ring with identity . Let X be a fuzzy module of an R-module M and we will denote it by (X be F(M)) . X is called Q-FCF(M) , if for every fuzzy ideal I of R and two every fuzzy submodules A and B of X , if IA=IB , then A+(F-annI)=B+(F-annI) where F-annI is a fuzzy submodule of X and define by F-annI ={$x_t$ ⊆X: $Ix_t=0_1$} $\forall t \in (0,1]$ see Definition (1.1) .

And X is called Q-MFCF(M) if for every maximal fuzzy ideal of R and for every two fuzzy submodules A and B of X such that IA=IB implies that A+(F-$ann_x$I)=B+(F-$ann_x$I) see (2.1).

Clearly, every Q-FCF(M) is Q-MFCF(M) see (Remark(2.3)(1)) and every fully cancellation fuzzy module is Q-FCF(M). See (Remark(1.3)(1)) but the convers is not true ingenarl see (Remark(1.6))

And , if X is multiplication and naturally cancellation fuzzy module or (fuzzy Principle ideal ) , then X is Q-FCF(M). See Proposition (1.4) and Proposition (1.5 ).

In this chapter , we will study in details the concept of Q-FCF(M).This chapter consists of three parts. In part one we give some basic propositions of Q-MFCF(M).

It turns out that a fuzzy module X is quasi-fully cancellation (quasi-max fully cancellation) if and only if IH ⊆ IK, then H⊆ K+(F-$ann_x$I), where H and K are fuzzy submodules of X and I be a fuzzy ideal (fuzzy maximal ideal) of R. Equivalently , I($h_t$) ⊆ IL, then $h_t$ ⊆ L+(F-$ann_x$I), where $h_t$ ⊆X if and only if (IH:I)=H+(F-$ann_x$I) where L is a fuzzy submodule . And $h_t$ is a fuzzy singleton of R, ∀t ∈ (0,1]. see Proposition (1.7) and Proposition (2.9).And fully cancellation fuzzy module is equivalent quasi-fully cancellation in the class X is torsion free fuzzy module over a fuzzy integral domain R see Proposition (1.4) .

Part two is devoted to study the relation between max-fully cancellation fuzzy module and Q-MFCF(M) but the convers is not true see Remark (2.7) and Example (2.8) . Part three is study the concepts the direct sum of Q-FCF(M) which is mentioned in chapter one section five.

And naturally cancellation fuzzy module X is equivalent to quasi-fully cancellation if X is multiplication fuzzy module.

## § 1. *Quasi-Fully Cancellation Fuzzy Modules*

In this part we give the concept of Q-FCF(M) this concept is generalization of concept quasi-fully cancellation modules[1], and we give a some basic results and properties of this concept. Also, relationships between the class of Q-FCF(M)and other types of modules are established .

" Recall that an R-module M is called quasi-fully cancellation produle. If for every ideal I of R and for every two submodules A,B of M. Such that IA=IB implies A+$ann_M$I=B+$ann_M$I (where $ann_M$I={m∈M, Im=0}.[1]"

Here ,we introduce the principle definition of our work.

**Definition 1.1:**

Let X beF(M). X is called Q-FCF(M) for every fuzzy ideal I of R and for every fuzzy submodules A and B of X , if IA=IB, then A+F-$ann_X$I=B+F-$ann_X$I

Where F-annI is a fuzzy submodule of X. And definition by {$x_t$ ⊆X: I.$x_t$=$0_1$}. ∀ t∈(0,1].

**Proposition 1.2:**

Let X beF(M). such that $(F\text{-}annX)_t$ =F-ann$X_t$ . Then X is a Q-FCF(M)if and only if $X_t$ is a quasi-fully cancellation module, ∀ t∈(0,1].

**Proof:**

Let X be F(M). Let N and K be two submodules of M, and let J be an ideal of R. such that JN=JK.

Now, define: A:M⟶[0,1] , B:M ⟶[0,1] by

$A(x)=\begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1]

$B(x)=\begin{cases} t & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1] and

Define : I:R ⟶[0,1] by

$I(x)=\begin{cases} t & \text{if } x \in J \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1]

It is clear that A and B are fuzzy submodules of X and I be a fuzzy ideal of R.

Also , $A_t$=N, $B_t$=K and $I_t$=J ; $\forall$ t∈(0,1] , JN=JK, then $I_tA_t= I_tB_t$

Therefore $(IA)_t=(IB)_t$ $\forall$ t∈(0,1] .

Thus IA=IB.

ButX is a Q-FCF(M). Then we get A+F-$ann_X$I =B+F-$ann_X$I

Hence $(A+F\text{-}ann_XI)_t=(B+F\text{-}ann_XI)_t$. $\forall$ t∈(0,1]

But $(F\text{-}ann_XI)_t$ =F- $ann_MI_t$ $\forall$ t∈(0,1] see[5,Proposition.(2.2)]

So, $A_t$+F-ann$I_t$=$B_t$+F-ann$I_t$.

Therefore N+F-annJ=K+F-annJ.

Thus $X_t$ is quasi-fully cancellation module.

Another side, let A,B be two fuzzy submodules of X and let I be a fuzzy ideal of R. such that IA=IB. To prove A+F-annI=B+F-annI.

Now, since IA=IB, then $(IA)_t=(IB)_t$ $\forall$ t∈(0,1], so , $I_tA_t= I_tB_t$

But $X_t$ is a quasi-fully cancellation module.

Then $A_t$+F-ann$I_t$=$B_t$+F-ann$I_t$.

But $(F\text{-}annI)_t$=F-sann$I_t$ $\forall$ t∈(0,1] which implies

$A_t+(F\text{-}annI)_t=B_t+(F\text{-}annI)_t$.

So $(A+F\text{-}ann_XI)_t=(B+F\text{-}ann_XI)_t$. , by [2 , Remark (1.1.7)].

Thus A+F-$ann_X$I=B+F-$ann_X$I.

Therefore X is Q-FCF(M).

## Remarks and Examples 1.3:

(1) Every fully cancellation fuzzy module is Q-FCF(M). But the converse is not true ingeneral by the following example:

Let M=$Z_4$ is a Z-module .

LetX : M ⟶[0,1] define by $X(x)=\begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Let A: ($\bar{2}$ ) ⟶[0,1] define by $A(x)=\begin{cases} t & \text{if } x \in (\bar{2}) \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1]

Let B: $Z_4$ ⟶[0,1] define by $B(x)=\begin{cases} t & \text{if } x \in Z_4 \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1]

Define I: (4) ⟶[0,1] define by $I(x)=\begin{cases} t & \text{if } x \in (4) \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1]

It is clear that A and B are fuzzy submodules of X and I is a fuzzy ideal of R

$M=Z_4=X_t$ is not fully cancellation module by [3, Remark and Examples (2.3)(2)].

Since $(4).(\bar{2}) = (4).Z_4=0$, but $(\bar{2}) \neq Z_4$.

Implies that X is not fully cancellation fuzzy module by [4, Proposition (1.2,2)].

Hence $I_t=(4)$, $A_t=(\bar{2})$ and $B_t=Z_4$ .

So $I_tA_t= I_tB_t$ (since $(4).(\bar{2}) =(4).Z_4=(\bar{0})$). Also, $annI_t=ann\ (4)=Z_4$

Then $A_t+annI_t=(\bar{2})+Z_4=Z_4$

Also , $B_t+annI_t=Z_4+Z_4=Z_4$

Then $X_t=M$ is quasi-fully cancellation module. and by proposition (1.2) .

X is Q-FCF(M).

(2) Any fuzzy module of a Z-module Z is Q-FCF(M).

**Proof:**

By[4, Remark and Examples (1.2.3)(1)] we get X is fully cancellation fuzzy module.

And by (1) we interduce X is Q-FCF(M).

(3) Every fuzzy submodule of Q-FCF(M) is quasi-fully cancellation.

**Proof:**

Let X be a Q-FCF(M) of an R-module M. let N,K be two submodules of M and J be an ideal of R. let C be a fuzzy submodule of X .

To prove C is Q-FCF(M).

Define: C: M⟶[0,1] by $C(x)=\begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define A: N⟶[0,1] by $A(x)=\begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$ $\forall\ t\in(0,1]$

Define B:$K$ ⟶[0,1] by $B(x)=\begin{cases} t & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$ $\forall\ t\in(0,1]$

Define I: J⟶[0,1] by $I(x)=\begin{cases} t & \text{if } x \in J \\ 0 & \text{otherwise} \end{cases}$ $\forall\ t\in(0,1]$

It is clear that A,B are fuzzy submodules of C and I is a fuzzy ideal of R.

Also, $I_t=J$ , $A_t=N$ , $B_t=K$ , $C_t=M$.

Since X is Q-FCF(M).

Then $X_t$ is quasi-fully cancellation module.by Proposition (1.2).

Since C is a fuzzy submodule of X.

Then $C_t$ is a submodule of $X_t$ and by [ 3 , Remark and Examples (2.2)],

we get $C_t$ is quasi-fully cancellation module.

Therefore C is Q-FCF(M). by Proposition(1.2)

(4) Let $X_1$and $X_2$ be two fuzzy modules of an R-module $M_1$, $M_2$ respectivly such that $M_1 \cong M_2$.Then $X_1$ is Q-FCF(M)if and only if $X_2$ is Q-FCF(M).

**Proof:**

(⟹) Let $X_1: M_1 \longrightarrow [0,1]$ define by $X_1(x)=\begin{cases} 1 & \text{if } x \in M_1 \\ 0 & \text{otherwise} \end{cases}$

Let $X_2: M_2 \longrightarrow [0,1]$ define by $X_2(x)=\begin{cases} 1 & \text{if } x \in M_2 \\ 0 & \text{otherwise} \end{cases}$

It is clear that $X_1$ and $X_2$ are fuzzy modules of $M_1$ and $M_2$ respectively .

Since $(X_1)_t=M_1$ , $(X_2)_t=M_2$ and $M_1 \cong M_2$ , $\forall$ t∈(0,1] , then $M_2$ is quasi-fully cancellation module by [ 3 ,Remark and Examples (2.2). (5)]

Then $X_2$ is Q-FCF(M) by Proposition (1.2).

Conversely**:** it is clear.

**Proposition 1.4:**

Let X be a multiplication and naturally cancellation fuzzy module of an R-module M then X is Q-FCF(M).

**Proof:**

Since X is multiplication and naturally cancellation fuzzy module, then by[4, Theorem (1.4.3)]. We obtian

X is fully cancellation fuzzy module and by Remark and Examples( ( 1.3 ) (1)).

X is Q-FCF(M).

**Proposition 1.5:**

Let X be a fuzzy torsion free module over a fuzzy integral domain R. If X is quasi-fully cancellation modules, then X is fully cancellation.

**Proof:**

Let X be a fuzzy torsion free module and R be a fuzzy integral domain.

Suppose that IA=IB where A,B are two fuzzy submodules of X and I be a fuzzy ideal of R.

Since X is quasi-fully cancellation, then A+(F-$ann_x$I)=B+(F-$ann_x$I).

Now, let $x_t \subseteq$F-$ann_x$I, then $Ix_t=0_1$ $\forall$ t∈(0,1]

And hence $r_\ell x_t=0_1$ for each fuzzy singleton $r_\ell$ of I, $\forall \ell \in (0,1]$. $r_\ell \neq 0_1$ (since R is integral domain). Then $I \neq 0_1$.

Therefore $x_t=0_1$ (since X is a fuzzy torsion free).

Then F-$ann_x$I=$0_1$.Thus A=B and hence X is a fully cancellation fuzzy module.

**Proposition 1.6:**

Let X be F(M) and let R be a fuzzy principle ideal ring . Then X is a quasi-fully cancellation module.

**Proof:**

Let A and B be two fuzzy submodules of X.

Let I be a fuzzy ideal of a fuzzy principle ideal ring R.

Suppose that IA=IB.

We show that $A+(F\text{-}ann_x I)=B+(F\text{-}ann_x I)$

Since R is a fuzzy principle ideal ring, then $I=(r_\ell)$, where $r_\ell$ be a fuzzy singleton of R, $\forall \ell \in(0,1]$.

Then $(r_\ell)$ A=$(r_\ell)$ B and hence $r_\ell.a_t=r_\ell.b_s$ where $a_t\subseteq A$ and $b_s\subseteq B$ $\forall t, s \in(0,1]$.

Now, $r_\ell.a_t-r_\ell.b_s=0_1$

Then $(ra\text{-}rb)_\lambda=0_\lambda\leq 0_1$ where $\lambda=\min\{\ell,t,s\}$. Therefore $r_\ell(a\text{-}b)_\lambda=0_1$,and hence $r_\ell(a_t\text{-}b_s)=0_1$ .Thus $a_t\text{-}b_s\subseteq F\text{-}ann_x I$. But $a_t=b_s+a_t\text{-}b_s\subseteq B+(F\text{-}ann_x I)$ .

Then $A\subseteq B+(F\text{-}ann_x I)$

Hence $A+(F\text{-}ann_x I) \subseteq B+(F\text{-}ann_x I)$

Similarly:

$B+(F\text{-}ann_x I) \subseteq A+(F\text{-}ann_x I)$

Thus $A+(F\text{-}ann_x I)= B+(F\text{-}ann_x I)$

And hence X is Q-FCF(M).

**Remark 1.7:**

The converse of Remark and examples( (1.3)(1)) is not true in general by the following example:

Let $M=Z_{p\infty}$ and R=Z define by $X:M\longrightarrow[0,1]$ such that

$$X(x)=\begin{cases}1 & \text{if } x\in m\\ 0 & \text{otherwise}\end{cases}$$

It is clear that $\forall\, t\in(0,1]$, $X_t=M$ and M is not fully cancellation module [3, Examples (2.5)(2)].

Thus X is not fully cancellation fuzzy module by Proposition ( 1.2) .

**Proposition 1.8:**

Let X be F(M) and let H,K and L are fuzzy submodules of X. let I be a fuzzy ideal of R. Then the following statements are equivalent:-

1- X is a Q-FCF(M).

2- If IH⊆IK, then H⊆K+(F-ann$_x$I).

3- I($h_t$) ⊆IL, then $h_t$⊆L+(F-ann$_x$I), where $h_t$⊆X. $\forall t \in(0,1]$.

4- (IH:$_x$I)=H+(F-ann$_x$I).

**Proof:**

(1)⇒(2) let X be a Q-FCF(M) and let IH⊆IK.

Then IK⊆IH+IK=I(H+K) by [5,proposition (2.6)] .

Hence K+(F-ann$_x$I)=(H+K)+(F-ann$_x$I) (since X is Q-FCF(M)).

Therefore H⊆K+(F-ann$_x$I).

(2)⇒(3) let I($h_t$) ⊆IL, where $h_t$⊆H and by (2) we have ($h_t$) ⊆L+(F-ann$_x$I).

Thus $h_t$⊆L+(F-ann$_x$I).

(3)⇒(4) let $x_t$ ⊆(IH:$_x$I) , $\forall t \in(0,1]$, then I$x_t$⊆IH and by (3) $x_t$ ⊆H+(F-ann$_x$I).

Therefore (IH:$_x$I) ⊆H+(F-ann$_x$I).

Conversely, let $a_\ell$ ⊆H+(F-ann$_x$I), $\forall \ell \in(0,1]$.

Then $a_\ell$=$h_t$+$m_s$ , where $h_t$⊆H and $m_s$ ⊆ (F-ann$_x$I), $\forall s \in(0,1]$.

Thus I $a_\ell$=I$h_t$ +I$m_s$ . But I$m_s$=$0_1$

Therefore I $a_\ell$=I$h_t$ ⊆IH .

Then $a_\ell$ ⊆ (IH:$_x$I).

Thus H+(F-ann$_x$I) ⊆ (IH:$_x$I). and hence (IH:$_x$I)= H+(F-ann$_x$I).

(4)⇒(1) let IH=IK we want to prove X is Q-FCF(M).

i.e To prove H+(F-ann$_x$I)=K+(F-ann$_x$I)

K⊆(IH:$_x$I) and by (4) we get (IH:$_x$I) = H+(F-ann$_x$I) .

Hence K ⊆H+(F-ann$_x$I).

Therefore K+(F-ann$_x$I) ⊆H+(F-ann$_x$I).

Similarly:

H⊆ (IK:$_x$I) and by (4), we obtain(IK:$_x$I)= K+(F-ann$_x$I), then H⊆K+(F-ann$_x$I).

Therefore H+(F-ann$_x$I)=K+(F-ann$_x$I).

Thus X is Q-FCF(M).

**Proposition 1.9:**

Let X be F(M).Then X is Q-FCF(M) if and only if ((A+(F-ann$_x$I):B)=(IA:$_R$IB) where I be a fuzzy ideal of R and A,B are two fuzzy submodules of X.

**Proof:** (⟹)

Let $x_t$ ⊆ ((A+(F-ann$_x$I):B) , $\forall t \in(0,1]$.

Then $x_tB \subseteq A + (F\text{-ann}_xI)$ and hence $x_t\, b_s \subseteq A+(F\text{-ann}_xI)$ for and $b_s \subseteq B$, $\forall s \in (0,1]$.

Thus $x_tIB \subseteq IA$ . Then $x_t \subseteq (IA:_R IB)$.

Therefore $((A+(F\text{-ann}_xI):B) \subseteq (IA:_R IB)$.

Now, let $x_t \subseteq (IA:_R IB)$, then $x_tIB \subseteq IA$ and by Proposition (1.7),we get $x_tB \subseteq A+(F\text{-ann}_xI)$ and hence $x_t \subseteq ((A+(F\text{-ann}_xI):B)$.

Thus $(IA:_R IB) \subseteq ((A+(F\text{-ann}_xI):B)$.

Therefore $((A+(F\text{-ann}_xI):B)=(IA:_R IB)$.

On the other side: suppose $IB \subseteq IA$, where I be a fuzzy ideal of R and A,B are two fuzzy submodules of X. Then $(IA:_R IB)=R$.

But $((A+(F\text{-ann}_xI):B)=(IA:_R IB)$ .

Then $((A+(F\text{-ann}_xI):B)=R$, it follows that $B \subseteq A+(F\text{-ann}I)$.

Thus X is Q-FCF(M).

## § 2. *Quasi-Max Fully Cancellation Fuzzy Modules*

As we have mentioned in section one, that every fully cancellation fuzzy module is Q-FCF(M) and the converse is not to be true in general.

In this section, we introduce the concept of Q-MFCF(M) and to show that every max-fully cancellation fuzzy module is Q-FCF(M) but the converse is not true . Morevore , we prove that in the class of faithful fuzzy module , the two concepts max-fully cancellation fuzzy module and Q-MFCF(M) are equivalent.

" Recall that an R-module M is called quasi-max fully cancellation module if for every maximal ideal I of R and for every two submodules N and K of M such that IN=IK implies $N+ann_MI=K+ann_MI$ [6]."

We shall fuzzify this concepts as follows:

**Definition 2.1** :

Let X be F(M). X is called Q-MFCF(M) if for every maximal fuzzy ideal of R and for every two fuzzy submodules A and B of X such that IA=IB implies that $A+(F\text{-ann}_xI)=B+(F\text{-ann}_xI)$.

Next , we have the following Proposition.

**Proposition 2.2:**

Let X be F(M) and let I be a maximal fuzzy ideal of *R*. such that $(F\text{-ann}I)_t=F\text{-ann}I_t$ , then X is a Q-MFCF(M) if and only if $X_t$ is a quasi-max fully cancellation module $\forall t \in (0,1]$.

**Proof:**

It is similar of proof of Proposition (1.2) only we take I maximal fuzzy ideal.

**Remarks and examples 2.3:-**

(1) Every Q-FCF(M) is a Q-MFCF(M).

**Proof:** it is clear.

(2) A fuzzy module X of an $Z_6$-module $Z_6$ is Q-MFCF(M).

**Proof:**

Let $M=Z_6$ and $X:M\longrightarrow[0,1]$ such that $X(x)=\begin{cases}1 & \text{if } x\in M\\ 0 & \text{otherwise}\end{cases}$

Define: $I:(\bar{2})\longrightarrow[0,1]$ by $I(r)=\begin{cases}t & \text{if } r\in(\bar{2})\\ 0 & \text{otherwise}\end{cases}$ , $\forall t\in(0,1]$.

Define: $A:(\bar{2})\longrightarrow[0,1]$ by $A(x)=\begin{cases}t & \text{if } x\in(\bar{2})\\ 0 & \text{otherwise}\end{cases}$

Define: $B:Z_6\longrightarrow[0,1]$ by $B(x)=\begin{cases}t & \text{if } t\in Z_6\\ 0 & \text{otherwise}\end{cases}$

It is clear that $A_t=(\bar{2})$ , $B_t=Z_6$ and $I_t=(\bar{2})$ is a maximal ideal and $X_t=M$, $\forall t\in(0,1]$.

Now, $I_tA_t=(\bar{2})(\bar{2})=(\bar{2}).Z_6=I_tB_t=(\bar{2})$

Similarly if $A_t=(0)$, $B_t=(\bar{3})$ since $I_tA_t=(\bar{2})(\bar{0})=(\bar{2}).(\bar{3})=I_tB_t=(\bar{0})$ where $(0)$, $\overline{(2)}$, $(\bar{3})$ are submodules of $M=Z_6$

Then we obtain $(\bar{0})+ann_M(\bar{2})=(\bar{3})+ann_M(\bar{2})=(\bar{3})$

Thus M is Q-MFC(M).Therefore X is Q-MFCF(M). by Proposition (2.2).

(3) The fuzzy module X of an $Z_4$-module $Z_4$ is a Q-FCF(M).

**Proof:**

By Remark and Examples ((1.3)(2)) . We have $Z_4$ is quasi max fully cancellation module and by Proposition (2.2) we get the result.

(4) Let $X_1$ and $X_2$ be a fuzzy modules of an R-module $M_1$,$M_2$ respectively . if $M_1$ is a Q-MFC(M)and $M_1\cong M_2$ , then $X_1$ is a Q-MFCF(M) if and only if $X_2$ is a Q-MFCF(M).

**Proof:**

($\Longrightarrow$) let $X_1:M_1\longrightarrow[0,1]$ define by $X_1(x)=\begin{cases}1 & \text{if } x\in M_1\\ 0 & \text{otherwise}\end{cases}$

Let $X_2:M_2\longrightarrow[0,1]$ define by $X_2(x)=\begin{cases}1 & \text{if } x\in M_2\\ 0 & \text{otherwise}\end{cases}$

Clear that $(X_1)_t=M_1$ ,is a quasi-max fully cancellation module, then $X_1$ is a Q-MFCF(M) by Proposition (2.7).

But $M_1\cong M_2$ , then $M_1$ is a quasi-max fully cancellation module by [6, Remark and Examples (1.3)(4)]

Therefore $X_2$ is a Q-MFCF(M).

**Conversely:** it is clear.

(5) Let X be F(M).and let C be a fuzzy submodule of X, then C is Q-MFCF(M).

**Proof:**

The prove is similar of Remarks and Examples (1.3)(4) ) only we take the ideal of a ring R in maximal fuzzy ideal.

The following Lemmas are needed to prove next Proposition.

**Lemma 2.4:**

Let X be F(M) and let A,B and C are fuzzy submodules of X such that C⊆B.Then C+(B∩A)=(C+A) ∩B.

**Proof:**

First to show that C+(B∩A) ⊆ $B \cap (C + A)$, (since C⊆B and B∩A⊆B)

Then C+(B∩A) ⊆B by [2].

Further C⊆C+A , B∩A⊆C+A.

Thus C+(B∩A) ⊆C+A

Therefore C+(B∩A) ⊆B ∩(C+A).

Conversely: to show that

$B \cap (C + A)$ ⊆ C+ (B∩A)

Let $b_t \subseteq B \cap (C + A)$, ∀t∈(0,1].

Then $b_t = c_\ell + a_s$ for some fuzzy singletons $c_\ell \subseteq C$, $a_s \subseteq A$ $\forall \ell, s \in (0,1]$.

Hence $b_t = (c+a)_t$ where t=min{ $\ell, s$}.

Then b=c+a by[7, Definition (1.1.3)(3)].

Hence a=b-c and $a_s = b_t - c_\ell$

Thus $a_s \subseteq B$ (since $b_t \subseteq B$ and $c_\ell \subseteq C \subseteq B$ )

Hence $a_s \subseteq B \cap A$ and so $b_t = c_\ell + a_s \subseteq$ C+ (B∩A).

Therefore C+(B∩A)=(C+A) ∩B.

**Lemma 2.5:**

Let A be a fuzzy submodule of a fuzzy module X of an R-module M, let I be any fuzzy ideal of R, then $F\text{-}ann_A I = F\text{-}ann_x I \cap A$.

**Proof:**

Since $F\text{-}ann_A I$ is a fuzzy submodule of a fuzzy submodule A. then $F\text{-}ann_x I$ is a fuzzy submodule of X . and $F\text{-}ann_x I \cap A \subseteq F\text{-}ann_A I$

Conversely**:** to show that $F\text{-}ann_A I \subseteq F\text{-}ann_x I \cap A$?

Let $x_t \subseteq$ F-ann$_A$I , where $x_t \subseteq A$ , $\forall t \in (0,1]$.

Then $x_tI=0_1$

$0_1=0_1 \cap A = x_tI \cap A=$F-ann$_A$I $\cap A$

But $x_t \subseteq A \subseteq X \implies x_t \subseteq X$.

Thus F-ann$_x$I $\cap A=0_1= x_tI=$ F-ann$_A$I

Now, we have the following Proposition.

**Proposition 2.6:**

Let A be a fuzzy submodule of a Q-MFCF(M) X, then A is a Q-MFCF(M).

**Proof:** Let B,C are two fuzzy submodules of a fuzzy submodule A and let I be a maximal fuzzy ideal of R. such that IB=IC, then B,C are fuzzy submodules of X.

Since X is a Q-MFCF(M)

Then B+(F-ann$_x$I)=C+(F-ann$_x$I).

But F-ann$_A$I=F-ann$_x$I $\cap$A by Lemma (2.5).

Hence B+ F-ann$_A$I=B+((F-ann$_x$I) $\cap$A)

=(B+(F-ann$_x$I)) $\cap$A by Lemma (2.4).

=(C+(F-ann$_x$I)) $\cap$A

=C+((F-ann$_x$I) $\cap$A) by Lemma (2.4)

=C+( F-ann$_A$I) by Lemma (2.5)

Thus A is a Q-MFCF(M).

**Remark 2.7:**

Every max-fully cancellation fuzzy module is Q-MFCF(M).

**Proof:**

It is clear .

The converse of Remark ( 2.7) is not true in general for example:-

Let $M=Z_6$, $R=Z_6$

Let X :M⟶[0,1] define by $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

$X_t=M$ $\forall t \in (0,1]$, then X is a Q-MFCF(M) by Remark((1.3)(1))

and it is not max-fully cancellation fuzzy module since

Let $I:(\bar{2}) \longrightarrow [0,1]$ define by $I(r) = \begin{cases} t & \text{if } r \in (\bar{2}) \\ 0 & \text{otherwise} \end{cases}$ , $\forall t \in (0,1]$.

And $A:(\bar{0}) \longrightarrow [0,1]$ define by $A(x) = \begin{cases} t & \text{if } x \in (\bar{0}) \\ 0 & \text{otherwise} \end{cases}$

Also, $B:(\bar{3}) \longrightarrow [0,1]$ define by $B(x) = \begin{cases} t & \text{if } x \in (\bar{3}) \\ 0 & \text{otherwise} \end{cases}$, $\forall t \in (0,1]$.

It is clear that A,B are fuzzy suodules of X and I is a fuzzy ideal of R.

$IA=IB \Leftrightarrow I_tA_t=I_tB_t$.

$(\bar{2}).(\bar{0}) = (\bar{2}).(\bar{3}) = (\bar{0})$, but $(\bar{0}) \neq (\bar{3})$

Then $A_t \neq B_t \Leftrightarrow A \neq B$.

Thus X is not max-fully cancellation fuzzy module.

The convers of Remark (2.7) is true under the following conditions.

**Proposition 2.8:**

Let X be F(M) and let $F\text{-ann}_xI=0_1$ be (F-faithful) for every non-empty fuzzy ideal I of R. Then every Q-MFCF(M) is max-fully cancellation fuzzy module.

**Proof:**

Let I be a maximal fuzzy ideal of R. and let A,B be two fuzzy submodules of X such that IA=IB

Since X is Q-MFCF(M) , then $A+(F\text{-ann}_xI)=B+(F\text{-ann}_xI)$. But $(F\text{-ann}_xI)=0_1$. Thus A=B

Then X is max-fully cancellation fuzzy module.

The following is characterization of Q-MFCF(M).

**Theorem 2.9:**

Let X be F(M), let A,B be two fuzzy submodules of X and let I be a maximal fuzzy ideal of R. Then the following statements are equivalent :-

(1) X is a Q-MFCF(M).

(2) If $IA \subseteq IB$ , then $A \subseteq B+(F\text{-ann}_xI)$.

(3) If $I(x_t) \subseteq IB$, then $x_t \subseteq B+(F\text{-ann}_xI)$.where $x_t \subseteq X$.

(4) $(IA:_xI)= B+(F\text{-ann}_xI)$.

**Proof:**

Compare this proof with the proof of Proposition (1.7).

The next result gives another characterization for Q-MFCF(M).

**Proposition 2.10:**

Let X be F(M).Then for any maximal fuzzy ideal I of R the following statements are equivalent:-

(1) X is Q-MFCF(M).

(2) For every fuzzy submodules A,B of X then $((A+(F\text{-ann}_xI)):B)=(IA:IB)$.

**Proof:**

It is similar proof of Proposition (1.8).

## §3. *The Direct Sum of Quasi-Fully Cancellation Fuzzy Modules and Its Generalization*

In this part we study the direct sum of two Q-FCF(M) and we prove some results about it. Also, we study its generalizations .

First, we give the following lemma, which is needed in the next our Proposition.

**Lemma 3.1:**

Let X be F(M) , $M=M_1\oplus M_2$ where $M_1$, $M_2$ are submodules of M, if $X=A_1\oplus A_2$, where $A_1$, $A_2$ are fuzzy submodules of X , then $F\text{-ann}_x I=F\text{-ann}_{A_1}I\oplus F\text{-ann}_{A_2}I$ where I is a fuzzy ideal of R.

**Proof:**

We must prove that $F\text{-ann}_x I \subseteq F\text{-ann}_{A_1}I\oplus F\text{-ann}_{A_2}I$

Let $x_t\subseteq F\text{-ann}_x I$, then $Ix_t=0_1$ and $x_t\subseteq X$, $\forall t\in (0,1]$.

Since $x_t\subseteq X \Longrightarrow x\in X_t=M= M_1\oplus M_2$.

Then $x=x_1+x_2$ for some $x_1\in M_1$, $x_2\in M_2$

Define I: $J\longrightarrow[0,1]$ by

$$I(x)=\begin{cases} t & \text{if } x\in J \\ 0 & \text{otherwise} \end{cases},\quad \forall t\in(0,1].$$

Be a fuzzy ideal of R.

It is clear tat $I_t=J$

Then $J.x=J(x_1+x_2)=0$ (since + on M is a direct sum) it follows

That:- $Jx_1=Jx_2=0$.

$(Jx_1)_t=(Jx_2)_t=0_t \leq 0_1$ $\forall t\in(0,1]$.

$I_t(x_1)_t=I_t(x_2)_t=0_t$ (since $J=I_t$).

$I(x_1)_t=I(x_2)_t=0_t$.

Thus $(x_1)_t\subseteq F\text{-ann}_{A_1}I$ and $(x_2)_t\subseteq F\text{-ann}_{A_2}I$. .

Therefore $x_t=(x_1)_t+(x_2)_t \subseteq F\text{-ann}_{A_1}I\oplus F\text{-ann}_{A_2}I$ .

**Conversely**: let $x_t\subseteq F\text{-ann}_{A_1}I\oplus F\text{-ann}_{A_2}I$

Then $x_t\subseteq ((x_1)_\ell ,(x_2)_s)$ where $(x_1)_\ell \subseteq F\text{-ann}_{A_1}I$ and $(x_2)_s\subseteq F\text{-ann}_{A_2}I$ $\forall \ell, s \in(0,1]$.

Thus $I(x_1)_\ell=0_1$ and $I(x_2)_s=0_1$. Therefore $I(x_1)_\ell + I(x_2)_s=0_1$,

and hence $I((x_1)_\ell + (x_2)_s)=0_1$

Then $[J(x_1+x_2)]_\lambda=0_\lambda \leq 0_1$ where $\lambda=\min\{t,s,\ell\}$ implies, $J(x_1+x_2)=0$

Therefore $Jx=0$ and hence $I(x_t)=0_1$

Hence $x_t \subseteq$ F-ann$_x$I .Therefore F-ann$_x$I= F-ann$_{A1}$I⊕F-ann$_{A2}$I

**Proposition 3.2:**
Let X be F(M) and let X= $A_1 \oplus A_2$, where $A_1$ and $A_2$ be two fuzzy submodules of X. such that F-ann$_{A1}$I⊕F-ann$_{A2}$I=$\lambda_R$ where $\lambda_R(x)=1$; ∀x∈R. Then $A_1$ and $A_2$ are Q-FCF(M).

**Proof:**
(⟹) Let I be a non-empty fuzzy ideal of R and let A and B are fuzzy submodules of X. Suppose that IA=IB we show that A+(F-ann$_x$I)=B+(F-ann$_x$I)

Since (F-ann$A_1$)+ (F-ann$A_2$)= $\lambda_R$ , then by [4,Lemma (1.5.5)] we get , A= $A_1 \oplus A_2$ and B= $B_1 \oplus B_2$ for some fuzzy submodules $A_1$ , $A_2$ of A and for some fuzzy submodules $B_1$, $B_2$ of B.

Now, I( $A_1 \oplus A_2$) =I( $B_1 \oplus B_2$) .

Hence (I$A_1$ , I$A_2$)=(I$B_1$ , I$B_2$). [ 7 , Proposition (3.2.4)]

Which implies that , I$A_1$=I$B_1$ and I$A_2$=I$B_2$

But $A_1$ and $A_2$ are Q-FCF(M).

Thus $A_1$+(F-ann$_{A1}$I)=$B_1$+(F-ann$_{A1}$I) and $A_2$+(F-ann$_{A2}$I)=$B_2$+(F-ann$_{A2}$I)

It follows that , $A_1$+$A_2$+(F-ann$_{A1}$I)+(F-ann$_{A2}$I)= $B_1$+$B_2$+(F-ann$_{A1}$I)+(Fann$_{A2}$I)

Then we have : A+(F-ann$_x$I)=B+(F-ann$_x$I)

Therefore X is Q-FCF(M).

(⟸) it is clear by used Remarks and Examples ((1.3) (4)).

We end this section by the following result.

**Proposition 3.3:**

Let X be F(M) and let X= $A_1 \oplus A_2$ where $A_1$ and $A_2$ are two fuzzy submodules of X, such that F-ann$A_1$+ F-ann$A_2$= $\lambda_R$ where

$\lambda_R(x)=1$, ∀x∈R. Then $A_1$ and $A_2$ are Q-MFCF(M)if and only if X is Q-MFCF(M).

**Proof:**

First side, let A and B be two fuzzy submodules of X and let I be a maximal fuzzy ideal of R.

Since F-ann$A_1$+F-ann$A_2$= $\lambda_R$, where $\lambda_R(x)=1$, ∀x∈R.then by[4, Lemma (1.5.5)] we get , A= $A_1 \oplus A_2$ and B= $B_1 \oplus B_2$, and by similar procedure as in the Proposition ( 3.2 ) . the required result can be obtained .

Another side, it is clear by Remarks and Examples( (2.3)(5)).

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# الموديولات الحذف شبه التامة الضبابية

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## الخلاصة

في هذا البحث تمت دراسة فكرة الموديولات الحذف شبه التامة الضبابية . وقد تم إعطائها الرمز Q-FCF(M). وهي اعمام لحالة الموديولات الحذف شبه التامة الاعتيادية وقد تم تعميم هذه الفكره الى موديولات شبه التامة العظمى الضبابية وقد تم اعطائها الرمز Q-MFCF(M) . نتائج عديدة وخواص كثيرة تمت دراستها في بحثنا هذا.

**الكلمات المفتاحية:** الموديولات الحذف شبه التامة الضبابية ، الموديولات الحذف شبه التامة الاعتيادية ، الموديولات الحذف شبه التامة لعظمى الضبابية .