Semiessential Fuzzy Ideals and Semiuniform Fuzzy Rings

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Abstract
In this paper, we introduce and study semiessential fuzzy ideals of fuzzy rings, uniform fuzzy rings and semiuniform fuzzy rings.

Introduction
Zadah in [1] introduced the notion of a fuzzy subset A of a nonempty set S as a mapping from S into [0,1], Liu in [2] introduced the concept of a fuzzy ring, Martines [3] introduced the notion of a fuzzy ideal of a fuzzy ring.

A non zero proper ideal I of a ring R is called an essential ideal if $I \cap J \neq (0)$, for any non zero ideal J of R, [4].

Inaam in [5] fuzzified this concept to essential fuzzy ideal of fuzzy ring and gave its basic properties.

Nada in [6] introduced and studied notion of semiessential ideal in a ring R, where a non zero ideal I of R is called semiessential if $I \cap P \neq (0)$ for all non zero prime ideals of R, [4].

A ring R is called uniform if every ideal of R is essential. Nada in [6] introduced and studied the notion semiuniform ring where a ring R is called semiuniform ring if every ideal of R is semiessential ideal.

In this paper we fuzzify the concepts semiessential ideal of a ring, uniform ring and semiuniform ring into semiessential fuzzy ideal of fuzzy ring, uniform fuzzy ring and semiuniform fuzzy ring. Where a fuzzy ideal A of a fuzzy ring X is semiessential if $I \cap P \neq (0)$ for any prime fuzzy ideal P of X.

A fuzzy ring X is called uniform (semiuniform) if every fuzzy ideal of X is essential (semiessential) respectively.

In S.1, some basic definitions and results are collected.

In S.2, we study semiesential fuzzy ideals of fuzzy ring, we give some basic properties about this concept.

In S.3, we study the notion of uniform fuzzy rings and semiuniform fuzzy rings. Several properties about them are given.

Throughout this paper, R is commutative ring with unity, and X(0) = 1, for any fuzzy ring.

S.1 Preliminaries
Let R be a commutative ring with identity. A fuzzy subset of R is a function from R into [0,1]. Let A and B be a fuzzy subsets of R we write $A \subseteq B$ if $A(x) \leq B(x)$, for all $x \in R$, (1) and $(A \cap B)(x) = \min \{A(x), B(x)\}$, $\forall\ x \in R$. For each $t \in [0,1]$, the set $\{ x \in R; A(x) \geq t\}$ is called the level subset of R, [7]. If A and B are fuzzy subsets of R, then $\forall\ t \in [0,1]$

1. $(A \cap B)_t = A_t \cap B_t$, [1],
2. $A = B$ iff $A_t = B_t$, [1] .

Let f be a mapping from a set M into a set N, let B be a fuzzy subset of N. The inverse image of B is a fuzzy subset of M defined by $f^{-1}(B)(x) = B(f(x))$, $\forall\ x \in M$, [1].

Let A be a fuzzy subset of a set M. A is called an f-invariant if $A(x) = A(y)$, whenever $f(x) = f(y)$, where $x, y \in M$, [8].

If f is a function from a set M into a set N, let A_1 and A_2 be fuzzy subsets of M and B_1, B_2 be fuzzy subsets of N, then

1. $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$, A_1, A_2 are f-invariant, [8]
2. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$, [8].
3. $f^{-1}(f(A_1)) = A_1$, whenever A_1 is f-invariant, [8]
4. $f(f^{-1}(B_1)) = B_1$, [8].

Moreover the following definitions and properties are needed later

1.1 Definition [2]

Let X be a fuzzy subset of a ring R. X is called a fuzzy ring of R if $X \neq O_1$ and for each $x, y \in R$

1. $X(x - y) \geq \min\{X(x), X(y)\}$,
2. $X(x\,y) \geq \max\{X(x), X(y)\}$.

1.2 Definition [3]

Let X be a fuzzy ring of R, a fuzzy subset A of R is called a fuzzy ideal of X if

1. $A \subseteq X$,
2. $A(x - y) \geq \min\{A(x), A(y)\}$, $\forall\, x, y \in R$,
3. $A(x\,y) \geq \max\{A(x), X(y)\}$, $\forall\, x, y \in R$.

Note that if X is a fuzzy ring of R, then $X(a) \leq X(0)$, $\forall\, a \in R$, [3, proposition 2.7]

If A is a fuzzy ideal of X, then $A(a) \leq A(0)$, $\forall\, a \in R$, [3, proposition 2.9].

1.3 Proposition

A fuzzy subset $X : R \longrightarrow [0,1]$, is a fuzzy ring if X_t is a subring of R, $\forall\, t \in [0, X(0)]$, (3, proposition 2.10 (i)).

Given a fuzzy ring X and a fuzzy set $A : R \longrightarrow [0,1]$ with $A \neq O_1$, then A is a fuzzy ideal of X iff A_t is an ideal of X_t $\forall\, t \in [0, A(0)]$, [3, proposition 2.10 (iii)].

1.4 Definition [9]

A fuzzy ideal P of a fuzzy ring X is called a prime fuzzy ideal if $P \neq \lambda_R$ (where λ_R denotes the characteristic function of R such that $\lambda_R(x) = 1$, $\forall\, x \in R$) and it satisfies:
$\min\{P(x\,y), X(x), X(y)\} \leq \max\{P(x), P(y)\}$ for all $x, y \in R$.
If $X = \lambda_R$, then $P(x\,y) \leq \max\{P(x), P(y)\}$ for all $x, y \in R$.

1.5 Definition [10]

Let X and Y be fuzzy rings of R_1 and R_2 respectively. Then the direct sum of X and Y (denoted by $X \oplus Y$) is defined by:
$X \oplus Y : R_1 \oplus R_2 \longrightarrow [0,1]$ such that
$(X \oplus Y)(a,b) = \min\{X(a), Y(b)\}$, $\forall\, (a,b) \in R_1 \oplus R_2$. If A, B are fuzzy ideals of X and Y respectively, then $A \oplus B : R_1 \oplus R_2 \longrightarrow [0,1]$ defined by :
$(A \oplus B)(a,b) = \min\{A(a), B(b)\}$, $\forall\, (a,b) \in R_1 \oplus R_2$.

1.6 Proposition [9]

Let X , Y be two fuzzy rings of R and $\overline{R}$ respectively and $f : R \longrightarrow \overline{R}$ is a homomorphism, then
If A is a prime fuzzy ideal of X and A is f-invariant, then $f(A)$ is a prime fuzzy ideal of Y.

1.7 Definition [10]

Let A be a fuzzy ideal of a fuzzy ring X of a ring R, A is called an essential fuzzy ideal if $A \cap K \neq O_1$ for each fuzzy ideal K of X, $K \neq O_1$.

S.2 Semiessential Fuzzy Ideals

In this section, we introduce the notion of semiessential fuzzy ideals of fuzzy ring as a generalization of (ordinary) notion semiessential ideals of a ring. We shall give many properties of this concept.

2.1 Definition

Let X be a fuzzy ring of a ring R. Let A be a fuzzy ideal of X such that A $\neq O_{X(0)} = O_1$. A is called a semiessential fuzzy ideal of X if $A \cap B \neq O_1$, for any prime fuzzy ideal B of X.

2.2 Remark

Let X be a fuzzy ring of a ring R. It is clear that if A is an essential fuzzy ideal of X, then A is a semiessential fuzzy ideal of X.

Proof: It is easy, so it is omitted.

The converse of remark 2.2 is not true in general. However an example which will explain this dependens on theorem 2.3, so we shall give it later (see remark 2.5).

2.3 Theorem

Let X be a fuzzy ring of R, let A be a fuzzy ideal of X, if A_t is a semiessential ideal of X_t, $\forall\ t \in (0,1]$. Then A is a semiessential fuzzy ideal of X.

Proof: Let B be a prime fuzzy ideal of X such that $B \neq O_1$. To prove $A \cap B \neq O_1$, since B is a prime fuzzy ideal. Hence B_t is a prime ideal of X_t, $\forall\ t \in (0, X(0)]$ by [11,proposition 1.2.9]. Which implies $A_t \cap B_t \neq (0)$ and $A_t \cap B_t = (A \cap B)_t \neq (0)$. Hence $A \cap B \neq O_1$. Thus A is a semiessential fuzzy ideal of X.

The following remark shows that the converse of this theorem is not true in general.

2.4 Remark

If X is a fuzzy ring of a ring R, A is a semiessential fuzzy ideal of X, then it is not necessarily that A_t is a semiessential ideal of X_t, $\forall\ t \in [0,1]$. As the following example shows:

Example:

Let R = Z_6, define X: $Z_6 \longrightarrow [0,1]$, A: $Z_6 \longrightarrow [0,1]$ by:

$$X(a) = \begin{cases} 1 & \text{if } a = 0 \\ \frac{1}{2} & \text{if } a = 2,4 \\ 0 & \text{otherwise.} \end{cases} \qquad A(a) = \begin{cases} 1 & \text{if } a = 0 \\ \frac{1}{3} & \text{if } a = 2,4 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear X is a fuzzy ring of Z_6, A is a fuzzy ideal of X and $A \neq O_1$.

A is an essential fuzzy ideal of X see (5, remark 2.3). Hence A is semiessential in X. On the other hand, $A_{\frac{1}{2}} = \{0\}$, $X_{\frac{1}{2}} = \{0,2,4\}$. Hence $A_{\frac{1}{2}}$ is not semiessential ideal in $X_{\frac{1}{2}}$.

2.5 Remark

If X is a fuzzy ring of a ring R, A is a semiessential fuzzy ideal of X, then it is not necessarily that A is an essential fuzzy ideal of X, as the following example shows:

Example:

Let R = Z_{12}, define X: $Z_{12} \longrightarrow [0,1]$ by $X(a) = 1$, $\forall\ a \in Z_{12}$, let A: $Z_{12} \longrightarrow [0,1]$ define by:

$$A(x)=\begin{cases}1 & \text{if } x\in(\bar{6})\\ 0 & \text{otherwise.}\end{cases}$$

It is clear A is a fuzzy ideal of a fuzzy ring X, $X_t = Z_{12}$ and $A_t = (\bar{6})$, $\forall\, t > 0$ is a semiessential ideal in Z_{12} since $(\bar{6}) \cap (\bar{3}) = (\bar{6})$ and $(\bar{6}) \cap (\bar{2}) = (\bar{6})$, $(\bar{3})$ and $(\bar{2})$ are prime ideals of Z_{12}. Thus A is a semiessential fuzzy ideal by Theorem 2.3. But A is not an essential fuzzy ideal since there exists fuzzy ideal B of X defined by :

$$B(x)=\begin{cases}1 & \text{if } x\in(\bar{4})\\ 0 & \text{otherwise.}\end{cases}$$

$B \neq O_1$.But $A \cap B = O_1$.

2.6 Remark

Let X be a fuzzy ring of R, let A and B be fuzzy ideals of X such that $A \subseteq B$. If A is a semiessential. Then B is a semiessential fuzzy ideal of X.

Proof: It is clear.

2.7 Corollary

If A and B are fuzzy ideals of a fuzzy ring X of a ring R such that $A \cap B$ is a semiessential fuzzy ideal of X, then A and B are semiessential fuzzy ideals of X.

2.8 Remark

Let A and B be fuzzy ideals of fuzzy ring X of a ring R such that $A \subseteq B$ and B is a semiessential, then it not necessarily that A is semiessential fuzzy ideal of X, as the following example shows:

Example:

Let X: $Z_{12} \longrightarrow [0,1]$, A: $Z_{12} \longrightarrow [0,1]$, B: $Z_{12} \longrightarrow [0,1]$ defined by:

$X(a)= 1$, $\forall\, a \in Z_{12}$, $A(x)=\begin{cases}1 & \text{if } x\in(\bar{4})\\ 0 & \text{otherwise.}\end{cases}$ and $B(x)=\begin{cases}1 & \text{if } x\in(\bar{2})\\ 0 & \text{otherwise.}\end{cases}$.

It is clear that $X_t = Z_{12}$ ($\forall\, t > 0$) and A, B are fuzzy ideals of fuzzy ring X, $B_t = (\bar{2})$ is a semiessential ideal of X_t, $\forall\, t > 0$, since $(\bar{2}) \cap (\bar{2}) = (\bar{2})$ and $(\bar{2}) \cap (\bar{3}) = (\bar{6})$ where $(\bar{2})$ and $(\bar{3})$ are the only prime ideals of $Z_{12} = X_t$, $\forall\, t > 0$. Thus B is a semiessential fuzzy ideal of X. Let $C(x)=\begin{cases}1 & \text{if } x\in(\bar{3})\\ 0 & \text{otherwise.}\end{cases}$, C is a prime fuzzy ideal of X, since $C_t = (\bar{3})$ is a prime ideal of X_t, $\forall\, t > 0$. But $A \cap C = O_1$. Thus A is not a semiessential fuzzy ideal of X.

2.9 Remark

If A and B are semiessential fuzzy ideals of fuzzy ring X of a ring R. Then it not necessarily that $A \cap B$ is semiessential fuzzy ideal of X. We can give the following example :

Example:

Let X: $Z_{36} \longrightarrow [0,1]$ define by $X(a) = 1$, $\forall\, a \in Z_{36}$ and let A: $Z_{36} \longrightarrow [0,1]$, B: $Z_{36} \longrightarrow [0,1]$ defined by:

$$A(x)=\begin{cases}1 & \text{if } x\in(\overline{12})\\ 0 & \text{otherwise.}\end{cases} \quad B(x)=\begin{cases}1 & \text{if } x\in(\overline{18})\\ 0 & \text{otherwise.}\end{cases}$$

A and B are fuzzy ideals of X. But for each $t \in (0,1]$, $A_t = (\overline{12})$, $B_t = (\overline{18})$. It is easy to show that A_t and B_t are semiessential ideals of $X_t = Z_{36}$. Thus A and B are semiessential fuzzy ideals of X by Theorem 2.3. But $A \cap B = O_1$. Thus $A \cap B$ is not a semiessential fuzzy ideal of X.

2.10 Proposition

Let A and B be fuzzy ideals of fuzzy ring X of a ring R such that A is an essential fuzzy ideal and B is a semiessential fuzzy ideal. Then $A \cap B$ is a semiessential fuzzy ideal of X.

Proof: Let P be a non-zero prime fuzzy ideal of X, since B is a semiessential fuzzy ideal of X, then $B \cap P \neq O_1$.Also A is an essential fuzzy ideal of X, we get $(A \cap B) \cap P \neq O_1$. Which implies $A \cap B$ is a semiessential fuzzy ideal of X.

2.11 Proposition

Let X. be a fuzzy ring of R such that $X(a) = 1$, $\forall\ a \in R$. Let I be a semiessential ideal of R. If $A : R \longrightarrow [0,1]$ defined by:

$$A(a) = \begin{cases} 1 & \text{if } a \in I \\ r & \text{if } a \notin I \end{cases}$$

where $r \in (0,1)$. Then A is a semiessential fuzzy ideal of X.

Proof: It is easy, so it omitted.

2.12 Proposition

Let X. be a fuzzy ring of R such that $X(a) = 1$, $\forall\ a \in R$. Let I be a ideal of R. Then I is a semiessential ideal of R if λ_I is a semiessential fuzzy ideal of X where

$$\lambda_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

Proof: It is easy, so it is omitted.

Before studying the direct sum of semiessential fuzzy ideals, we need the following lemma.

2.13 Lemma

Let X and Y be fuzzy rings of rings R_1, R_2 respectively. Let W be a fuzzy ideal of $T = X \oplus Y$, then W is a prime fuzzy ideal of T if there exists A and B prime fuzzy ideals of X, Y respectively such that $W = A \oplus Y$ or $W = X \oplus B$.

Proof: If W is a prime fuzzy ideal in $T = X \oplus Y$. Since W is a fuzzy ideal in $X \oplus Y$, there exists fuzzy ideal A and B of X, Y respectively such that $W = A \oplus B$ by (10,theorem 2.4.1.9). Thus $W_t = A_t \oplus B_t$, $\forall\ t \in (0,1]$. But W is a prime so W_t is prime in $T_t = X_t \oplus Y_t$, $\forall\ t \in (0,1]$. Hence either $W_t = I \oplus Y_t$ or $W_t = X_t \oplus J$ where I, J are prime ideals in X_t, Y_t respectively, by (12, page 53). Therefore $I = A_t$ or $J = B_t$ and hence $W_t = A_t \oplus Y_t$ or $W_t = X_t \oplus B_t$. It follows that $W_t = (A \oplus Y)_t$ or $W_t = (X \oplus B)_t$. Thus $W = A \oplus Y$ or $W = X \oplus B$.

Conversely; If $W = A \oplus Y$ or $W = X \oplus B$, where A and B are prime fuzzy ideals of X, Y respectively. If $W = A \oplus Y$, then $W_t = (A \oplus Y)_t = A_t \oplus Y_t$ but A_t is a prime ideal in X_t, $\forall\ t \in (0,1]$ by (11,proposition 1.2.9). Hence $A_t \oplus Y_t$ is prime in T_t by (12,page 53). That is W_t is a prime ideal in $(X \oplus Y)_t = T_t$. Thus W is a prime fuzzy ideal of $X \oplus Y = T$ by (11, proposition 1.2.9).

Now we can give the following main result.

2.14 Theorem

Let X and Y be fuzzy rings of R_1, R_2 respectively. If A and B are semiessential fuzzy ideals of X, Y respectively. Then $A \oplus B$ is a semiessential fuzzy ideal of $X \oplus Y$.

Proof: To prove A ⊕ B is semiessential fuzzy ideal of X ⊕ Y. Since A ⊕ B is a non-zero fuzzy ideal of X⊕Y by (10, Theorem 2.4.1.9), there exists $(a,b) \in R_1 \oplus R_2$ such that $(A \oplus B)(a,b) = \min\{A(a),B(b)\} \neq 0$. Thus $A(a) \neq 0$ and $B(b)\} \neq 0$.

Now, let W be a non-zero prime fuzzy ideal of X ⊕ Y, hence either W = C ⊕ Y or W = X ⊕ D, for some prime fuzzy ideals C, D of X, Y respectively. Assume W = C ⊕ Y.

If $C = O_1$, then $(A \oplus B) \cap W = (A \oplus B) \cap (C \oplus Y)$

$$= (A \cap C) \oplus (B \cap Y)$$
$$= O_1 \oplus B$$

But $(O_1 \oplus B)(a,b) = \min\{O_1(0),B(b)\}$

$$= \min\{1, B(b)\} = B(b) \neq 0$$

Thus $(A \oplus B) \cap W \neq O_1$.

If $C \neq O_1$, then $A \cap C \neq O_1$, since A is semiessential in X. Hence there exists $a_1 \in R_1$ such that $(A \cap C)(a_1) \neq O_1$, so $\min\{A(a_1),C(a_1)\}\neq 0$, since $(A\oplus B)\cap(C\oplus Y)= (A \cap C) \oplus (B \cap Y) = (A \cap C) \oplus B$.

It follows that $[(A \cap C) \oplus B](a,b) = \min\{A(a_1),C(a_1),B(b)\} \neq 0$.

That $[(A \oplus B) \cap (C \oplus Y) \neq O(X \oplus Y) \neq O_1(0,0)$.

Similarly, if W = X ⊕ D, then $(A\oplus B) \cap W\neq O_1$. Therefore, A ⊕ B is semiessential.

The converse of theorem 2.14 is not true in general as the following example shows;

2.15 Example

Let $X: Z_6 \longrightarrow [0,1]$, $Y: Z_{12} \longrightarrow [0,1]$ define by $X(a) = 1, \forall a \in Z_6$, $Y(b) =1, \forall b \in Z_{12}$, let $A: Z_6 \longrightarrow [0,1]$, $B: Z_{12} \longrightarrow [0,1]$ defined by:

$$A(x) = \begin{cases} 1 & \text{if } x \in \{\bar{0},\bar{3}\} \\ 0 & \text{otherwise.} \end{cases}, \forall x \in Z_6 \quad B(x) = \begin{cases} 1 & \text{if } x \in \{\bar{0},\bar{3},\bar{6},\bar{9}\} \\ 0 & \text{otherwise.} \end{cases}, \forall x \in Z_{12}$$

It is easy to check that A and B are fuzzy ideals of the fuzzy rings X and Y respectively.

$$(A \oplus B)(x,y) = \begin{cases} 1 & \text{if } x \in \{\bar{0},\bar{3}\}, y \in \{\bar{0},\bar{3},\bar{6},\bar{9}\} \\ 0 & \text{otherwise.} \end{cases}$$

$(A \oplus B)_t = A_t \oplus B_t = <\bar{3}> \oplus <\bar{3}>$ is a semiessential ideal in $Z_6 \oplus Z_{12}$. Since the prime ideals in $Z_6 \oplus Z_{12}$ are:

$<\bar{2}> \oplus Z_{12}$, $<\bar{3}> \oplus Z_{12}$, $Z_6 \oplus <\bar{2}>$, $Z_6 \oplus <\bar{3}>$. Hence $((\bar{3}) \oplus (\bar{3}))\cap((\bar{2}) \oplus Z_{12})\neq 0$, $((\bar{3}) \oplus (\bar{3})) \cap ((\bar{3}) \oplus Z_{12}) \neq 0$, $((\bar{3}) \oplus (\bar{3})) \cap (Z_6 \oplus(\bar{2})) \neq 0$ and $((\bar{3}) \oplus (\bar{3})) \cap (Z_6 \oplus(\bar{3})) \neq 0$. Which implies $(A \oplus B)_t$ is semiessential in $(X \oplus Y)_t$, $\forall t$. Thus A ⊕ B is semiessential fuzzy ideal of X ⊕ Y by Theorem 2.3. But A is not semiessential fuzzy ideal of X, since there exists prime fuzzy ideal C of X such that $A \cap C = O_1$, where

$$C(x) = \begin{cases} 1 & \text{if } x \in \{\bar{0},\bar{2},\bar{4}\} \\ 0 & \text{otherwise.} \end{cases} \forall x \in Z_6.$$

Similarly, we can show that B is not semiessential fuzzy ideal of Y.

Next, we have the following proposition about the inverse image of semiessential fuzzy ideals.

2.16 Proposition

Let X and Y be fuzzy rings of rings R_1, R_2 respectively. Let $f : R_1 \longrightarrow R_2$ be a homomorphism. If A is a semiessential fuzzy ideal of Y, then $f^{-1}(A)$ is a semiessential fuzzy ideal of X.

Proof: By (3,proposition 3.3), $f^{-1}(A)$ is a fuzzy ideal of X. Let B be a prime fuzzy ideal of X and B is f-invariant. To prove that $f^{-1}(A) \cap B \neq O_{X(0)} = O_1$.
$f(f^{-1}(A) \cap B) = f(f^{-1}(A) \cap f(B)) = A \cap f(B)$ since A, B are f-invariant. However $f(B)$ is prime fuzzy ideal of Y by proposition 1.6. Therefore, $A \cap f(B) \neq O_{Y(0)}$, since A is semiessential.
On the other hand, $f^{-1}(A \cap f(B)) = f^{-1}(A) \cap f^{-1}(f(B))$
$= f^{-1}(A) \cap B$, since B f-invariant
$\neq O_{X(0)} \neq O_1$.
Thus $f^{-1}(A)$ is a semiessential fuzzy ideal of X.

S.3 Uniform and Semiuniform Fuzzy Rings

Recall that a ring R is called a uniform ring if every non zero ideal I of R is an essential ideal, [4] and a ring R is called semiuniform if every non zero ideal I of R is a semiessential ideal of R, [6].

In this section, we introduce and study the notion of uniform and semiuniform fuzzy rings and give many properties about them.

3.1 Definition

Let X be a fuzzy ring of a ring R. X is called uniform (semiuniform) if every non zero fuzzy ideal A of X is an essential (semiessential) fuzzy ideal of X.

3.2 Proposition

Let X be a uniform fuzzy ring, then X is semiuniform fuzzy ring.

Proof: It is follows by Remark 2.2.

But the converse is not true in general.

3.3 Remark

If X is a semiuniform fuzzy ring. Then it is not necessarily that X is uniform fuzzy ring as the following example shows:

Example:

Let X: $Z_{36} \longrightarrow [0,1]$ define by $X(a) = 1$, $\forall\ a \in Z_{36}$. It is easy to check that X is a fuzzy ring of Z_{36}. For each fuzzy ideal A of X, A_t is an ideal of X_t, $\forall\ t \in [0,1]$ and $X_t = Z_{36}$.
The ideals of Z_{36} are : $(\bar{0}), (\bar{2}), (\bar{3}), (\bar{4}), (\bar{6}), (\overline{12})$ and $(\overline{18})$ which are semiessential ideals in Z_{36}.
So A is a semiessential fuzzy ideal of X by Theorem 2.3. Thus X is a semiuniform fuzzy ring.
But X is not uniform fuzzy ring since there exist fuzzy ideals A and B of X such that

$$A(x) = \begin{cases} 1 & \text{if } x \in (\overline{12}) \\ 0 & \text{otherwise.} \end{cases} \quad B(x) = \begin{cases} 1 & \text{if } x \in (\overline{18}) \\ 0 & \text{otherwise.} \end{cases}, \quad \forall\ x \in Z_{36}.$$

A is not essential fuzzy ideal since $A \cap B = O_1$. Thus X is not uniform fuzzy ring.

3.4 Remark

Let X be a fuzzy ring of a ring R such that X_t is a uniform ring, $\forall\ t \in (0,1]$. Then X is a uniform fuzzy ring.

Proof: It is follow by [5, proposition 2.4].

3.5 Remark

Let X be a fuzzy ring of a ring R such that X_t is a semiuniform ring, $\forall\ t$, then X is a semiuniform fuzzy ring.

Proof: Let A be a non-zero fuzzy ideal of X. To prove A is a semiessential in X.

Suppose there exists a non-zero prime fuzzy ideal P of X such that $A \cap P = O_1$ since $P \neq O_1$, there exist $t_1 \in (0,1]$ such that $P_{t_1} \neq \{0\}$. On the other hand, $(A \cap P) = O_1$. Implies $A_{t_1} \cap P_{t_1} = (O_1)_{t_1} = \{0\}$. But this is a contradiction since $P_{t_1} \neq \{0\}$, P_{t_1} is a prime in X_t, (11, proposition 1.2.9) and A_{t_1} is semiessential in X_t. Hence $A \cap P \neq O_1$ and A is a semiessential fuzzy ideal in X. Thus X is a semiuniform fuzy ring.

The notion of semiessential fuzzy ideal of fuzzy ring can be generalized to semiessential fuzzy submodules similarly we can obtain similar results except the direct sum of essential fuzzy ideals.

References

1. Zadah, L.A., (1965), "Fuzzy Sets", Inform and Control, 8, 338-353.
2. Liu, W.J., (1982), "Fuzzy Invariant Subgroup and Fuzzy Ideals", Fuzzy Sets and Systems, 8, 133-139.
3. Martines, L., (1995), "Fuzzy Subgroups of Fuzzy Groups and Ideals of Fuzzy Rings",The Journal of Fuzzy Math., 3, No. 4, 833-849.
4. Kasch, F., (1982), "Modules and Rings", Academic Press, London, New York.
5. Hadi, M.A. Inaam, (2001), "Some Special Fuzzy Ideals of Fuzzy Rings', J. Math. And Physic, 6, No.2.
6. Al-Daban, K.A.Nada, (2005), "Semiessential Submodules and Semiuniform Modules", M.Sc. Thesis, Tukirt University.
7. Al-Khamees, Y. and Mordeson, (1998), "Fuzzy Principal Ideals and Fuzzy Simple Field Extensions", Fuzzy Sets and Systems, 96, 147-253.
8. Kumar, R., (1991), "Fuzzy Semiprimary Ideals of Ring", Fuzzy Sets and Systems, 42, 263-272.
9. Martines, L., (1999), "Prime and Primary L-Fuzzy Ideals of L-Fuzzy Rings", Fuzzy Sets and Systems, 101, 489-494.
10. Abo-Drab, A.T., (2000), "Almost Quasi-Forbenius Fuzzy Rings", M.Sc. Thesis, University of Baghdad, College of Education, Ibn-AlHaitham.
11. Megeed, N.R., (2000), "Some Results on Ctegories of Rings", Fuzzy Ring and Its Spectrum, M.Sc. Thesis, University of Baghdad, College of Education, Ibn-AlHaitham.
12. Larsen, M.D. and Mc.Carthy P.J., (1971), "Multiplicative Theory of Ideals", Academic Press, New York.

المثاليات شبه الجوهرية الضبابية والحلقات شبه المنتظمة الضبابية

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قسم الرياضيات ، كلية التربية – ابن الهيثم ، جامعة بغداد

الخلاصة

في هذا البحث قدمنا ودرسنا المثاليات شبه الجوهرية الضبابية في حلقة ضبابية، الحلقات المنتظمة الضبابية والحلقات شبه المنتظمة الضبابية.