End-ψ-Prime Submodules

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Abstract

Let R be a commutative ring with identity and M an unitary R-module. Let $\delta(M)$ be the set of all submodules of M, and $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. We say that a proper submodule P of M is end-ψ-prime if for each $\alpha \in End_R(M)$ and $x \in M$, if $\alpha(x) \in P$, then either $x \in P + \psi(P)$ or $\alpha(M) \subseteq P + \psi(P)$. Some of the properties of this concept will be investigated. Some characterizations of end-ψ-prime submodules will be given, and we show that under some assumtions prime submodules and end-ψ-prime submodules are coincide.

Key Words: Prime submodules, S-prime submodules, ϕ-prime submodules, end-ψ-prime submodules.

1- Introduction

Throughout this paper, R is a commutative ring with identity and M is an unitary R-module. Prime ideals play an essential role in ring theory. One of the natural generalizations of prime ideals which have attracted the interest of several authors in the last two decades is the notion of prime submodule,[1],[2]. These have led to more information on the structure of the R-module. For an ideal I of R and a submodule N of M let $\sqrt{I}$ denote the radical of I, and [N:M] =]{r ∈ R: rM ⊆ N} which is clearly an ideal of R. Then a proper submodule P of M is called a prime submodule if r ∈ R and x ∈ M with rx ∈ P implies that r ∈ [P:M] or x ∈ P,[3]. Equivalently P is a prime submodule of M if and only if [P:M] is a prime ideal of R and the R/[P:M]- module M/P is torsion free where the R-module X is said to be torsion free if the annihilator of any nonzero element of X is zero, [3]. There are several generalizations of the notion of prime submodules ,such as Ebrahimi Atani ,F. Farzalipour ,introduced and studied weakly prime submodules, where a proper submodule P of M is said to be weakly prime submodule of M if r ∈ R and x ∈ M, 0 ≠ rx ∈ P gives that r ∈ [P:M] or x ∈ P,[4]. A submodule P≠M is almost prime submodule if r ∈ R and x ∈ M with rx ∈ P\[P:M]P implies that r ∈ [P:M] or x ∈ P,[5]. So any prime submodule is weakly prime and any weakly prime submodule is an almost prime submodule. Another generalization of prime submodule is the concept of S-prime submodules, where a proper submodule p of M is said to be S-prime submodule of M if f(m)∈ P ,where f∈S=End(M) and m∈P implies that either m∈P or f(M) ⊆ P,[6].Also in [7] studied S-prime (Endo-prime) submodules. Every S-prime submodule is prime but not conversely ,[6],[7]. Khaksari and Jafari in [8] extended the notion of prime submodule to ϕ-prime. Let M be an R-module and δ(M) be the set of all submodules of M and ϕ: δ(M) ⟶ δ(M) ∪ {ϕ} be a function. A proper submodule P of M is said to be ϕ-prime if r ∈ R and x ∈ M, rx ∈ P\ϕ(P) implies that r ∈ [P:M] or x ∈ P. In this paper ,we define and study the notion of end-ψ-prime submodules. Let δ(M) be the set of all submodules of M and ψ: δ(M) ⟶ δ(M) ∪ {ϕ} be a function. A proper submodule P of M is said to be end-ψ-prime if for each α ∈ End_R(M) and x ∈ M, if α(x) ∈ P, then either x ∈ P + ψ(P) or α(M) ⊆ P + ψ(P).

2-Basic Properties of end-ψ-Prime Submodules

First we give the following definition.

Definition (2.1):

Let M be an R-module and δ(M) be the set of all submodules of M. Let ψ:δ(M) ⟶ δ(M) ∪ {ϕ} be a function. A proper submodule N of M is said to be end-ψ-prime if for each α ∈ End_R(M) and x ∈ M, if α(x) ∈ N, then either x ∈ N + ψ(N) or α(M) ⊆ N + ψ(N).

Remarks and Examples (2.2):

(1) It is clear that every S-prime submodule is end-ψ-prime submodule .The convers is not true as the following example shows .Let M = Z_8 as Z- module ,N ={$\bar{0}$,$\bar{4}$ }.Then N is not S-prime submodule of M (since if f($\bar{x}$) = 2$\bar{x}$,∀$\bar{x}$∈Z_8 where f :Z_8 ⟶Z_8 and f($\bar{2}$) = 2.$\bar{2}$ = $\bar{4}$∈N. But $\bar{2}$ ∉N and f(M) ={$\bar{0}$,$\bar{2}$, $\bar{4}$,$\bar{6}$ }⊈ N, hence N is not S- prime submodule of M .But N is end ψ - prime submodule of M .

Proof: Letψ: δ(Z_8) ⟶δ(Z_8) ∪ {ϕ}, where ψ(N)= N + < $\bar{2}$ >, ∀ N⊆ M *and for all f* : Z_8 ⟶Z_8 .If $f(\bar{x})$∈N={ $\bar{0}$, $\bar{4}$ },then either $\bar{x}$ ∈N+ψ(N) =< $\bar{2}$ > *or f* (Z_8) ⊆ N + ψ(N) = <$\bar{2}$>.Therefore N is end-ψ-prime submodule of Z_8.

(2) Let M = Z_4 as Z-module, N = $\{\bar{0},\bar{2}\}$. Then N is an end-ψ-prime submodule of M (since N is S- prime submodule of M,[6]) .
(3) The only end-ψ-prime submodule of a simple module is {0}.Therefore($\bar{0}$) in the simple Z-module Z_p (p is prime number) is end-ψ-prime submodule.
(4) It is clear that not every end-ψ-prime submodule is prime submodule, see example in remark(2.2,1).
(5) If ψ(N) = N or ψ(N) = 0,then every end-ψ-prime submodule is S-prime submodule and hence is prime submodule.
(6) Let M = Z_{12} as Z-module, then N = $\{\bar{0}, \bar{6}\}$ is not end-ψ-prime of M .Since if $f: Z_{12} \longrightarrow Z_{12}$, where $f(\bar{m}) = 2\bar{m}$ for all $\bar{m} \in Z_{12}$ and let $\psi:\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ such that $\psi(N) = N + \{\bar{0}, \bar{6}\}$, $\forall$ $N \subseteq M$. Since $f(\bar{3}) = \bar{6} \in N$, but $\bar{3} \notin N + \psi(N) = \{\bar{0}, \bar{6}\}$ and $f(Z_{12}) = 2Z_{12} \not\subseteq N + \psi(N) = \{\bar{0}, \bar{6}\}$.Therefore N=$\{\bar{0}, \bar{6}\}$ is not end-ψ-prime submodule of Z_{12}.

Recall that an R-module M is called scalar if for every $f \in End(M)$, $\exists$ r ∈ R, r ≠ 0 such that $f(m) = rm$ for all r ∈ R, [9].

The following proposition shows that(scalar R-module) is a sufficient condition for prime submodule to b end-ψ-prime submodule.

Proposition (2.3):

Let M be a scalar R-module, and N is a prime submodule of M. Then N is an end-ψ-prime submodule of M.

Proof: Let $f \in End(M)$, m ∈ M such that $f(m) \in N$. Since M is scalar, $\exists$ r ∈ R, r ≠ 0 such that $f(x) = rx$ for all x ∈ M. Hence $f(m) = rm \in N$. But N is prime, so either m∈N or rM ⊆N. Thus either m ∈ N + ψ(N) or $f(M) \subseteq N + \psi(N)$. Therefore N is end-ψ-prime submodule .

Corollary (2.4):

Let N a prime submodule of a finitely generated multiplication R-module M .Then N is an end-ψ-prime submodule of M .

Proof: By [9,corollary 1.1.11]"Every finitely generated multiplication R-module M is a scalar module " and so by proposition (2.3) we get the result.

Corollary (2.5):

If N is a prime submodule of a cyclic R-module M, then N is end-ψ-prime submodule of M.

Proof: By [6,porosition 2.1.4],we have N is an S-prime submodule of M and by remark (2.2,1),we have the result.

Recall that a submodule N of an R-module M is said to be fully invariant if $f(N) \subseteq N$, for each R-endomorphism f of M ,[10].

By using this concept ,we can give the following result.

Theorem (2.6):

Let N be a proper fully invariant submodule of an R-module M. If $[N+\psi(N): f(M)]=[N+\psi(N): f(K)]$, for all $N+\psi(N) \subsetneq K$ and for all $f \in End(M)$ such that $f(\psi(N)) = \psi(N)$ then N is an end-ψ-prime sumodule of M.

Proof: Let h(m) ∈ N, where h ∈ End(M) and m ∈ M and suppose that m ∉ N + ψ(N), we must prove that h(M) ⊆ N + ψ(N). Now, N +ψ(N) ⊊ N +ψ(N) + Rm hence by assumption [N+ψ(N):h(M)] = [N+ψ(N):h(N +ψ(N)+ Rm)]. But 1 ∈ [N+ψ(N):h(N +ψ(N)+ Rm)] since h(N) +h(ψ(N)) + h(Rm) ⊆ N +ψ(N), therefore 1 ∈ [N+ψ(N):h(M)], which implies that h(M) ⊆ N + ψ(N). Therefore N is an end-ψ-prime of M.

Recall that "An R-module M is called duo if for each submodule N of M ,N is fully invariant ,[11].

The following result follows immediately from theorem (2.6).

Corollary (2.7):

Let M be an duo R-module .If [N++ψ(N): f (M)]=[N+ψ(N): f (K)], for all N+ψ(N)$\subsetneq$ K and for all $f \in$ End(M) such that f (ψ(N)) = ψ(N) , then N is an end-ψ-prime submodule M.

Proposition (2.8):

If N is an end-ψ-prime submodule of R-module M and ψ(N) < N ,then [N +ψ(N) : f (M)] = [N+ψ(N) : f (k)], for all N+ψ(N) $\subsetneq$ K and for all $f \in$ End(M).

Proof: Since N is an end-ψ-prime of M and ψ(N) < N, so by remark (2.2,5) N is S-prime .Hence by [6,prop.(2.1.14)] ,[N: f (M)] = [N: f (k)], for all $f \in$ End(M) and N $\subsetneq$ K .Since ψ(N) < N ,then [N +ψ(N) : f (M)] = [N+ψ(N) : f (k)], for all and N+ψ(N) $\subsetneq$ K. and for all $f \in$ End(M).

Recall that a submodule N of an R –module M is called relatively divisible (S- relatively divisible) denoted by RD(S-RD) if r M ∩ N = r N for each r ∈ R , f(M) ∩ N=f(N) for all f ∈ End(M), [6],[10] respectively .

Recall that a nonzero module M is called quasi-dedekind if Hom (M/N,M) = 0 for all nonzero submodule of M. Equivalently, M is quasi-dedekind if for any $f \in$ End(M), $f \neq 0$, then ker f = {0} (i.e. f is 1-1), [12].

Proposition (2.9):

Let M be a quasi –Dedekind R-module .Then every proper S-RD submodule of M is end-ψ-prime submodule of R-module M .

Proof: By [13,prop1.12],every proper S-RD submodule of M is strongly S-prime and by [14,rem.(1.2,2)],every strongly S-prime submodule is S-prime submodule .This implies every proper submodule of an R-module M is end ψ-prime submodule by [Rem.(2.1,1)] .

More About end-ψ-Prime Submodules

In this section, several fundamental properties of end-ψ-prime submodule are given.

Proposition (3.1):

Let M be an R-module, N < M, I ≤ R. If P is an end-ψ-prime submodule of M such that IN⊆ P , then N ⊆ P + ψ(P), provided I ⊈ [P + ψ(P):M].

Proof: Suppose IN⊆ P and I ⊈ [P + ψ(P):M], Let x ∈ N, we must prove that x ∈ P + ψ(P) for any x ∈ N . Since I ⊈ [P + ψ(P):M], then there exists a ∈ I and a ∉ [P + ψ(P):M]. Define $f : M \longrightarrow M$ by f (m) = a m for all m ∈ M, it is clear that $f \in End_R(M)$ and f (x) = a x ∈ IN ⊆ P. But P is an end-ψ-prime submodule of M and f (M) = aM ⊈ P + ψ(P), so x ∈ P + ψ(P). Thus N ≤ P + ψ(P).

Proposition (3.2):

Let M be an R-module, let $\phi \in$ End(M). If N is fully invariant end-ψ-prime of an R-module M, such that ϕ (M) ⊈ N and $\psi(\phi^{-1}(N)) = \phi^{-1}(\psi(N))$, then ϕ^{-1}(N) is also end-ψ-prime submodule of M. In fact in this case ϕ^{-1}(N) = N.

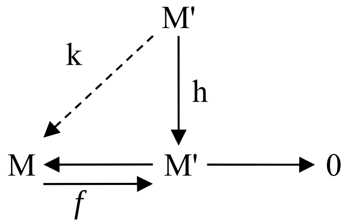
Proof: First, we must show that $\phi^{-1}(N)$ is a proper submodule of M. Suppose that $\phi^{-1}(N) = M$, then $\phi(M) \subseteq N$, which a contradiction to the assumption.
Now, let $f(m) \in \phi^{-1}(N)$, where $f \in End(M)$ and $m \in M$. If $m \notin \phi^{-1}(N) + \psi(\phi^{-1}(N))$, then $\phi(m) \notin N + \phi(\psi(\phi^{-1}(N)))$, which implies that $m \notin N + \psi(N)$, since N is fully invariant submodule of M and $\psi\phi^{-1} = \phi^{-1}\psi$. We only have to show that $f(M) \subseteq \phi^{-1}(N) + \psi(\phi^{-1}(N))$. Since $f(m) \in \phi^{-1}(N)$, then $(\phi \circ f)(m) = \phi(f(m)) \in N$. But N is an end-ψ-prime of M and $m \notin N + \psi(N)$, therefore $(\phi \circ f)(M) \subseteq N + \psi(N)$. This implies $f(M) \subseteq \phi^{-1}(N) + \phi^{-1}(\psi(N)) = \phi^{-1}(N) + \psi(\phi^{-1}(N))$.

Recall that an R-module M is called A-projective (where A is an R-module) if for every $X < A$ and every homoorphism $\phi: M \longrightarrow \frac{A}{X}$ can be lifted to a homomorphism $\psi: M \longrightarrow M$, [14]. If M is A-projective for each R-module A, then M is called projective.

Theorem (3.3):

Let $f: M \longrightarrow M'$ be an epimorphism and let $N < M$ such that $\ker f \leq N$. If N is an end-ψ-prime submodule of a module M, then $f(N)$ is an end-ψ'-prime submodule of a module of M', where M' is M-projective module and $\psi'(f(N)) = f(\psi(N))$.

Proof: First, we must show that $f(N)$ is a proper submodule of a module M'. Suppose $f(N) = M'$. But f is an epimorphism, thus $f(N) = f(M)$ and hence $M = N + \ker f$. This implies that $M = N$. A contradiction.
Now, let $h(m') \in f(N)$, where $h \in End(M')$ and $m' \in M'$ and suppose that $m' \notin f(N) + \psi'(f(N))$, we have to show that $h(M') \subseteq f(N) + \psi'(f(N))$. Since f is an epimorphism and $m' \in M'$, then there exists $m \in M$, such that $f(m) = m' \notin f(N) + \psi'(f(N))$, thus $m \notin N + f^{-1}(\psi'(f(N))) = N + \psi(N)$. Consider the following diagram:



since M' is M-projective module, then there exists a homoorphism $k: M' \longrightarrow M$, such that $f \circ k = h$. Clearly, $k \circ f \in End(M)$. Note that $f(k \circ f(m)) = (f \circ k)(f(m)) = h(m') \in f(N)$ and since $\ker f \subseteq N$, we get $(k \circ f)(m) \in N$. But N is an end-ψ-prime submopdule of M and $m \notin N + \psi(N)$. Therefore $(k \circ f)(M) \subseteq N + \psi(N)$ and hence $k(f(M)) = k(M') \leq N + \psi(N)$. Thus $f(k(M')) \subseteq f(N) + f(\psi(N))$, which implies that $h(M') \subseteq f(N) + \psi'(f(N))$.

Corollary (3.4):

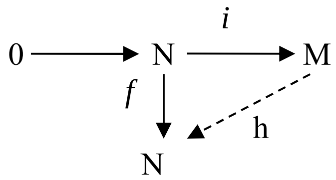
Let M be an R-module, let $K < N < M$ and N is an end-ψ-prime. Then $\frac{N}{K}$ is end-ψ'-prime in $\frac{M}{K}$, provided that $\frac{M}{K}$ is M-projective.

Recall that an R-module M is A-injective (where A is an R-module) if for every $X \leq M$, any homomorphism ϕ: $X \longrightarrow M$ can be extended to a homomorphism ψ: $A \longrightarrow X$, [9], [15]. If M is M-injective M is called quasi-injective, [9].

Proposition (3.5):

Let K be an end-ψ-prime of an R-module M and let $N < M$ which is M-injective and $\psi(K) \subseteq K$. Then either $N \subseteq K$ or $K \cap N$ is an end-ψ-prime in N.

Proof: Suppose that $N \nsubseteq K$, then $K \cap N$ is a proper submodule in N. Let $f(x) \in K \cap N$, where $f \in End(N)$ and $x \in N$. Suppose $x \notin (K \cap N) + \psi'(K \cap N)$, where ψ': $\delta(N) \longrightarrow \delta(N) \cup \{\phi\}$ be a function, then $x \notin K$. We must show that $f(N) \leq (K \cap N) + \psi'(K \cap N)$.
Now, consider the following diagram:



Where i is the inclusion map.

Since N is M-injective, then there exists h: $M \longrightarrow N$, such that $h \circ i = f$, clearly $h \in End(M)$. But $f(X) = (h \circ i)(x) = h(x) \in K$. Since K is an end-$\psi$-prime submodule of M and $x \notin K + \psi(K)$, this implies that $h(M) \subseteq K + \psi(K)$. Also, $f(N) = (h \circ i)(N) = h(N) \subseteq N$ (since $f(N) \subseteq K \cap N$) and $f(N) \subseteq h(N) \subseteq h(M) \subseteq K + \psi(K)$. Therefore, $f(N) \subseteq N \cap (K + \psi(K)) = N \cap K \subseteq N \cap K + \psi'(N \cap K)$.

Corollary (3.6):

Let K be an end-ψ-prime submodule of a quasi-injective R-module M, and let $N < M$. Then either $N \leq K$ or $K \cap N$ is an end-ψ-prime in N.

Proposition (3.7):

Let M be an R-module and let $K < N < M$ and K is fully invariant. If $\frac{N}{K}$ is an end-ψ'-prime submodule of $\frac{M}{K}$ and $\psi'\left(\frac{N}{K}\right) = \frac{N+\psi(N)}{K}$, then N is an end-$\psi$-prime submodule of M.

Proof: Suppose that $f(m) \in N$, where $f \in End(M)$ and $m \in M$. If $m \notin N + \psi(N)$, then we must show that $f(M) \subseteq N + \psi(N)$. Define f^*: $\frac{M}{K} \longrightarrow \frac{M}{K}$ by $f^*(x + K) = f(x) + K$, $\forall\, x \in M$.
To prove f^* is well define, let $x + K = y + K$ where $x, y \in M$, then $x - y \in K$ and hence $f(x - y) \in f(K) \subseteq K$, since K is fully invariant. This implies that $f(x) - f(y) \in K$. Thus $f(x) + K = f(y) + K$.
Now, $f^*(m + K) = f(m) + K \in \frac{N}{K}$. But $\frac{N}{K}$ is an end-ψ-prime of $\frac{M}{K}$ and $m + K \notin \frac{N+\psi(N)}{K} = \psi'\left(\frac{N}{K}\right)$ hence $f^*\left(\frac{M}{K}\right) \subseteq \frac{N}{K} + \psi'\left(\frac{N}{K}\right)$ and thus $\frac{f(M)+K}{K} \leq \frac{N}{K} + \psi'\left(\frac{N}{K}\right)$ and which implies that $\frac{f(M)+K}{K} \leq \frac{N}{K} + \frac{N+\psi(N)}{K} = \frac{N+\psi(N)}{K}$, thus $f(M) + K \leq N + \psi(N)$ and $f(M) \leq N + \psi(N)$.

Proposition (3.8):

Let M be a projective R-module. If N is end-ψ-prime and $\psi(N) = 0 \ \forall \ N < M$, then $\frac{M}{N}$ is a quasi-Dedekind R-module.

Proof: To prove $\frac{M}{N}$ is quasi-dedekind, we shall prove any endomorphism on $\frac{M}{N}$ is either zero mapping or 1-1, let $f : \frac{M}{N} \longrightarrow \frac{M}{N}$ and $f \neq 0$. Since M is projective there exists h: $M \longrightarrow M$ such that $\pi \circ h = f \circ \pi$, where π is the natural projection. Hence for any $m \in M$, $(\pi \circ h)(m) = \pi(h(m)) = h(m) + N = (f \circ \pi)(m) = f(m + N)$.
If $f(x + N) = 0_{\frac{M}{N}} = N$ for some $x + N \in \frac{M}{N}$, then $h(x) + N = N$ and so $h(m) \in N$. Hence either $x \in N + \psi(N)$ or $h(M) \subseteq N$, since N is end-ψ-prime. Thus, either $x + N = N = 0_{\frac{M}{N}}$,or [$\pi(h(M))=0$,then$(f \circ \pi)(M) = f(M/N) = 0$ which is a contradiction] .Therefore f is 1-1- and $\frac{M}{N}$ is quasi-dedekind.

For a partial answer for the converse of Prop.(3.8) we have the following:

Proposition (3.9):

Let $N < M$ such that N is fully invariant such that $\frac{M}{N}$ is a quasi-Dedekind R-module. Then N is end-ψ-prime.

Proof: Let $f \in End(M)$ and $f(m) \in N$ for some $m \in M$. Define g: $\frac{M}{N} \longrightarrow \frac{M}{N}$ by $g(x + N) = f(x) + N, \ \forall \ x \in M$, g is well-defined. If $g = 0$, then $f(M) \subseteq N \subseteq N + \psi(N)$. If $g \neq 0$, then g is 1-1 and hence $g(m + N) = f(m) + N = N$ implies that $m + N = N$; that is $m \in N \subseteq N + \psi(N)$. Thus N is end-ψ-prime.

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المقاسات الجزئية الأولية من النمط End-ψ

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الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد، وليكن M مقاساً معرفاً على الحلقة R. لتكن $\delta(M)$ مجموعة كل المقاسات الجزئية من M ولتكن $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ دالة. في هذا البحث، نقول ان المقاس الجزئي P من M هو مقاس جزئي أولي من النمط End-ψ اذا كان لكل $\alpha \in End(M)$ ، $x \in M$ اذ ان $\alpha(x) \in P$ ، فانه يؤدي الى اما $x \in P + \psi(P)$ أو $\alpha(M) \subseteq P + \psi(P)$. لقد درسنا واعطينا بعض خواص و مميزات هذا النوع من المقاسات الجزئية وبرهنا تحت شروط معينة ان المقاسات الجزئية الاولية وهذا النوع من المقاسات الجزئية يكونان متكافئين 0

الكلمات المفتاحية: المقاسات الجزئية الاولية ، المقاسات الجزئية الاولية من النمط - S ، المقاسات الجزئية الاولية من النمط ϕ، المقاسات الجزئية الاولية من النمط End-ψ.