Approximation Solutions for System of Linear Fredhom Integral Equations by Using Decomposition Method

H. S. Ali,W. S. Ali*, N. M. Rahmah
College of Engineering, University of Al-Mustansriya
***College of Agriculture, University of Baghdad**

Abstract

In this paper, the Decomposition method was used to find approximation solutions for a system of linear Fredholm integral equations of the second kind. In this method the solution of a functional equations is considered as the sum of an infinite series usually converging to the solution, and Adomian decomposition method for solving linear and nonlinear integral equations. Finally, numerical examples are prepared to illustrate these considerations.

Introduction

The integral equation is an equation in which the unknown function y(x) appears under the integral sign.

The general form of integral equation is given by:[1]

$$h(x)y(x) = f(x) + \int k(x,t)y(t)dt \qquad \ldots(1)$$

Where $h(x)$, $f(x)$ and the kernel $k(x,t)$ are known functions ; $y(x)$ is the function to be determined.

To satisfy linearity condition:

$$L(a_1 f_1(t) + a_2 f_2(t)) = a_1 L(f_1(t)) + a_2 L(f_2(t)) \qquad \ldots(2)$$

Where a_1, a_2 are constants and $L(f(t)) = \int k(x,t) f(t)dt$.

The Integral equation is called homogenous If $f(x) = 0$, otherwise it is called non homogenous.[2]

We can distinguish between two types of integral equations which are:

1. Integral equation of the first kind when $h(x) = 0$ in equation (1).

$$f(x) = \int k(x,t)y(t)dt \qquad \ldots(3)$$

2. Integral equation of the second kind when $h(x) = 1$ in equation (1).

$$y(x) = f(x) + \int k(x,t)y(t)dt \qquad \ldots(4)$$

Integral equations can be classified into different kinds according to the limits of integral[3]:

1. If the limits of equation (1) are constants then the equation is called Fredholm integral equation.

The Fredholm integral equation of the first kind is:-

$$f(x) = \int_a^b k(x,t)y(t)dt \qquad \ldots(5)$$

Where a, b are constants.

2. Fredholm integral equation of the second kind is:-

$$y(x) = f(x) + \int_a^b k(x,t)y(t)dt \qquad \ldots(6)$$

The Decomposition Method Applied to Fredholm Integral Eq

The topic of Adomian decomposition method has been rapidly growing in recent years. The concept of this method was first introduced by G. Adomian in the beginning of 1980's[4]. In this subsection use a decomposition method to find the a approximation solutions for System of linear Fredholm integral equations.

Let us reconsider the following system of linear Fredholm integral equations of the second kind(4)(5)(6).

$$F(t) = G(t) + \int_a^b K(t,s)F(s)ds, \qquad t \in [a,b] \qquad \ldots(7)$$

Where

$$F(t) = (f_1(t), f_2(t), \ldots, f_n(t))^t,$$

$$G(t) = (g_1(t), g_2(t), \ldots, g_n(t))^t,$$

$$K(t,s) = \lfloor k_{i,j}(t,s) \rfloor \qquad \mathrm{i} = 1,2,\ldots,n, \qquad \mathrm{j} = 1,2,\ldots,\mathrm{n}.$$

We suppose that system [7] has a unique solution.

Consider the *i*-th equation of (7):

$$f_i(t) = g_i(t) + \int_a^b \sum_{j=1}^{n} k_{ij}(t,s) f_j(s)ds. \qquad \ldots(8)$$

From (8), we obtain canonical form of Adomian's equation by writing

$$f_i(t) = g_i(t) + N_i(t) \qquad \ldots(9)$$

Where

$$N_i(t) = \int_a^b \sum_{j=1}^{n} k_{ij}(t,s) f_j(s)ds. \qquad \ldots(10)$$

To solve (10) by Adomian's method, let $f_i(t) = \sum_{m=0}^{\infty} f_{im}(t)$, and $N_i(t) = \sum_{m=0}^{\infty} A_{im}$ where A_{im}, $\mathrm{m} = 0,1,\ldots$, are polynomials depending on $f_{10}, f_{11}, \ldots, f_{1m}, \ldots, f_{n0}, \ldots, f_{nm}$ and they are called Adomian polynomials. Hence, (9) can be rewritten as:

$$\sum_{m=0}^{\infty} f_{im}(t) = g_i(t) + \sum_{m=0}^{\infty} A_{im}(f_{10}, f_{11}, \ldots, f_{1m}, \ldots, f_{n0}, f_{n1}, \ldots, f_{nm}). \qquad \ldots(11)$$

From [10] we define:

$$\begin{cases} f_{i0}(t) = g_i(t), & \mathrm{i} = 1,2,\ldots,\mathrm{n}, \\ f_{i,m+1}(t) = A_{im}(f_{10}, f_{11}, \ldots, f_{1m}, \ldots, f_{n0}, \ldots, f_{nm}), & \mathrm{m} = 0,1,2,\ldots \end{cases} \qquad \ldots(12)$$

In practice, all terms of the series $f_i(t) = \sum_{m=0}^{\infty} f_{im}(t)$ can not be determined and so we use an approximation of the solution by the following truncated series:

$$\varphi_{ik}(t) = \sum_{m=0}^{k-1} f_{im}(t), \qquad \text{with } \lim_{k\to\infty} \varphi_{ik}(t) = f_i(t). \qquad \ldots(13)$$

To determine Adomian polynomials, we consider the expansions:

$$f_{i\lambda}(t) = \sum_{m=0}^{\infty} \lambda^m f_{im}(t), \qquad \ldots(14)$$

$$N_{i\lambda}(f_1, f_2, \ldots, f_n) = \sum_{m=0}^{\infty} \lambda^m A_{im}, \qquad \ldots(15)$$

Where, λ is a parameter introduced for convenience. From (15) we obtain:

$$A_{im} = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N_{i\lambda}(f_1, f_2, \ldots, f_n)\right]_{\lambda=0}, \qquad \ldots(16)$$

And from (10), (14) and (16) we have:

$$A_{im}(f_{10},\ldots,f_{1m},\ldots,f_{n0},\ldots,f_{nm}) = \int_a^b \sum_{j=1}^{n} v_{ij}(s,t)\left[\frac{1}{m!}\frac{d^m}{d\lambda^m}\sum_{l=0}^{\infty}\lambda^l f_{jl}\right]_{\lambda=0} ds$$

$$= \int_a^b \sum_{j=1}^{n} v_{ij}(s,t) f_{jm} ds. \qquad \ldots(17)$$

So, (12) for the solution of the system of linear Fredholm integral equations will be as follows:

$$\begin{cases} f_{i0}(t) = g_i(t), \\ f_{i,m+1}(t) = \int_a^b \sum_{j=1}^{n} v_{ij}(s,t) f_{jm}(t) ds, \quad \mathrm{i} = 1,2,\ldots,\mathrm{n}, \quad \mathrm{m} = 0,1,2,\ldots \end{cases} \qquad \ldots(18)$$

Considering (13), we obtain:

$$\varphi_{ik}(t) = g_i(t) + \int_0^t \sum_{j=1}^{n} k_{ij}(t,s) f_{jm}(s) ds, \quad \mathrm{i} = 1,2,\ldots,\mathrm{n}, \quad \mathrm{m} = 0,1,2,\ldots \qquad \ldots(19)$$

In fact (12) is exactly the same as the well known successive approximations method for solving the system of linear Fredholm integral equations defining as:

$$f_{i,m+1}(t) = g_i(t) + \int_0^t \sum_{j=1}^{n} k_{ij}(t,s) f_{jm}(s) ds, \quad \mathrm{i} = 1,2,\ldots, \quad \mathrm{m} = 0,1,2,\ldots \qquad \ldots(20)$$

The initial approximations for the successive approximations method is usually zero function. In the other words, if the initial approximations in this method is selected $g_i(t)$, then the Adomian decomposition method and the successive approximations method are exactly the same.

The following algorithm summarizes the steps for finding the approximation solutions for the second kind of system Fredholm integral equations.

Algorithm (ADSFI)

Input: $(g_i(t), k_i(t,s), f_i(s), a, b, i = 1,\ldots n)$;

Output: series solution of given equation

Step1:

Put $f_{i0}(t) = g_i(t)$ for $i = 0,1,2,\ldots,n$

Step2:

Compute $f_{i,m+1}(t) = \int_a^b \sum_{j=1}^{n} k_{ij}(x,t) f_{jm}(s) ds \qquad i = 0,\ldots,n, \qquad m = 0,1,\ldots$

Step3:

Find the solution $\varphi_{ik}(t) = \sum_{m=0}^{k-1} f_{im}(t), \quad i = 0,\ldots,n, \qquad m = 0,1,\ldots$

End

Example

Consider the following system of linear Fredholm integral equations of the second kind with the exact solutions $f_1(t) = t+1$ and $f_2(t) = t^2+1$.

$$\begin{cases} f_1(t) = \dfrac{t}{18} + \dfrac{17}{36} + \displaystyle\int_0^1 \dfrac{s+1}{3}(f_1(s) + f_2(s))ds, \\ f_2(t) = t^2 - \dfrac{19}{12}t + 1 + \displaystyle\int_0^1 st((f_1(s) + f_2(s))ds. \end{cases}$$

To derive the solutions by using the decomposition method, we can use the following Adomian scheme:

$$\begin{cases} f_{10}(t) = \dfrac{t}{18} + \dfrac{17}{36} = 0.0556t + 0.4722, \\ f_{20}(t) = t^2 - \dfrac{19}{12}t + 1 = t^2 - 1.5833t + 1, \end{cases}$$

And

$$\begin{cases} f_{1,m+1}(t) = \displaystyle\int_0^1 \dfrac{(s+t)}{3}(f_{1m}(s) + f_{2m}(s))ds, \\ f_{2,m+1}(t) = \displaystyle\int_0^1 st(f_{1m}(s) + f_{2m}(s))ds, \end{cases} \qquad \text{m} = 0,1,2,\text{K}$$

For the first iteration, we have:

$$\begin{cases} f_{11}(t) = \displaystyle\int_0^1 \dfrac{(s+t)}{3}(f_{10}(s) + f_{20}(s))ds = \dfrac{25}{72}t + \dfrac{103}{648} = 0.3472t + 0.1590, \\ f_{21}(t) = \displaystyle\int_0^1 st(f_{10}(s) + f_{20}(s))ds = \dfrac{103}{216}t = 0.4769t. \end{cases}$$

Considering (13), the approximated solutions with two terms are:

$$\begin{cases} \varphi_{12}(t) = f_{10}(t) + f_{11}(t) = 0.4028t + 0.6312, \\ \varphi_{22}(t) = f_{20}(t) + f_{21}(t) = t^2 - 1.1065t + 1. \end{cases}$$

Next term are:

$$\begin{cases} f_{12}(t) = \displaystyle\int_0^1 \dfrac{(s+t)}{3}(f_{11}(s) + f_{21}(s))ds = \dfrac{185}{972}t + \dfrac{17}{144} = 0.1903t + 0.1181, \\ f_{21}(t) = \displaystyle\int_0^1 st(f_{11}(s) + f_{21}(s))ds = \dfrac{17}{48}t = 0.3542t. \end{cases}$$

Solution with three terms are:

$$\begin{cases} \varphi_{13}(t) = f_{10}(t) + f_{11}(t) + f_{12}(t) = 0.5931t + 0.7492, \\ \varphi_{23}(t) = f_{20}(t) + f_{21}(t) + f_{22} = t^2 - 0.7523t + 1. \end{cases}$$

In the same way, the components $\varphi_{1k}(t)$ and $\varphi_{2k}(t)$ can be calculated for $k = 3,4,\text{K}$ The solutions with seven terms and eleven terms are given as:

$$\begin{cases} \varphi_{1,7}(t) = f_{10}(t) + f_{11}(t) + \text{K} + f_{1,6}(t) = 0.9129t + 0.9468, \\ \varphi_{2,7}(t) = f_{20}(t) + f_{21}(t) + \text{K} + f_{2,6} = t^2 - 0.1609t + 1. \end{cases}$$

$$\begin{cases} \varphi_{1,11}(t) = f_{10}(t) + f_{11}(t) + \text{K} + f_{1,10}(t) = 0.9813t + 0.9885, \\ \varphi_{2,11}(t) = f_{20}(t) + f_{21}(t) + \text{K} + f_{2,10} = t^2 - 0.0345t + 1. \end{cases}$$

Approximated solutions for some values of t by using Decomposition method and exact values $f_1(t) = t + 1$ and $f_2(t) = t^2 + 1$ of Example, depending on least square error (L.S.E) are presented in Table(1), Fig.1, and Fig.2.

Conclusion

This paper presents the use of the Adomian decomposition method, for the system of linear Fredholm integral equations. As it can be seen, the Adomian decomposition method for a system of linear Fredholm integral equations is equivalent to successive approximations method. Although, the Adomian decomposition method is a very powerful device for solving the integral equations. From solving a numerical example the following points were identified:

1- this method can be used to solve the secant kinds of linear Fredholm integral equation.

2- It is clear that using the decomposition method basis function to approximate when the m order that increases the error would be decrease, as in Fig.1 and Fig.2.

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Table (1): The results of the example by using (ADSFI) algorithm

t	$f_1(t)$ *Exact*	$\varphi_{1,7}(t)$	$\varphi_{1,11}(t)$	$f_2(t)$ *Exact*	$\varphi_{2,7}(t)$	$\varphi_{2,11}(t)$
0	1	0.9468	0.9885	1	1	1
0.1	1.1	1.0381	1.0866	1.01	0.9939	1.0066
0.2	1.2	1.1294	1.1848	1.04	1.0078	1.0331
0.3	1.3	1.2207	1.2829	1.09	1.0417	1.0797
0.4	1.4	1.3120	1.3810	1.16	1.0956	1.1462
0.5	1.5	1.4033	1.4792	1.25	1.1695	1.2328
0.6	1.6	1.4945	1.5773	1.36	1.2634	1.3393
0.7	1.7	1.5858	1.6754	1.49	1.3773	1.4659
0.8	1.8	1.6771	1.7735	1.64	1.5112	1.6124
0.9	1.9	1.7684	1.8717	1.81	1.6652	1.7790
1	2	1.8597	1.9698	2	1.8391	1.9655
L.S.E		0.1113	0.0052		0.0997	0.0046

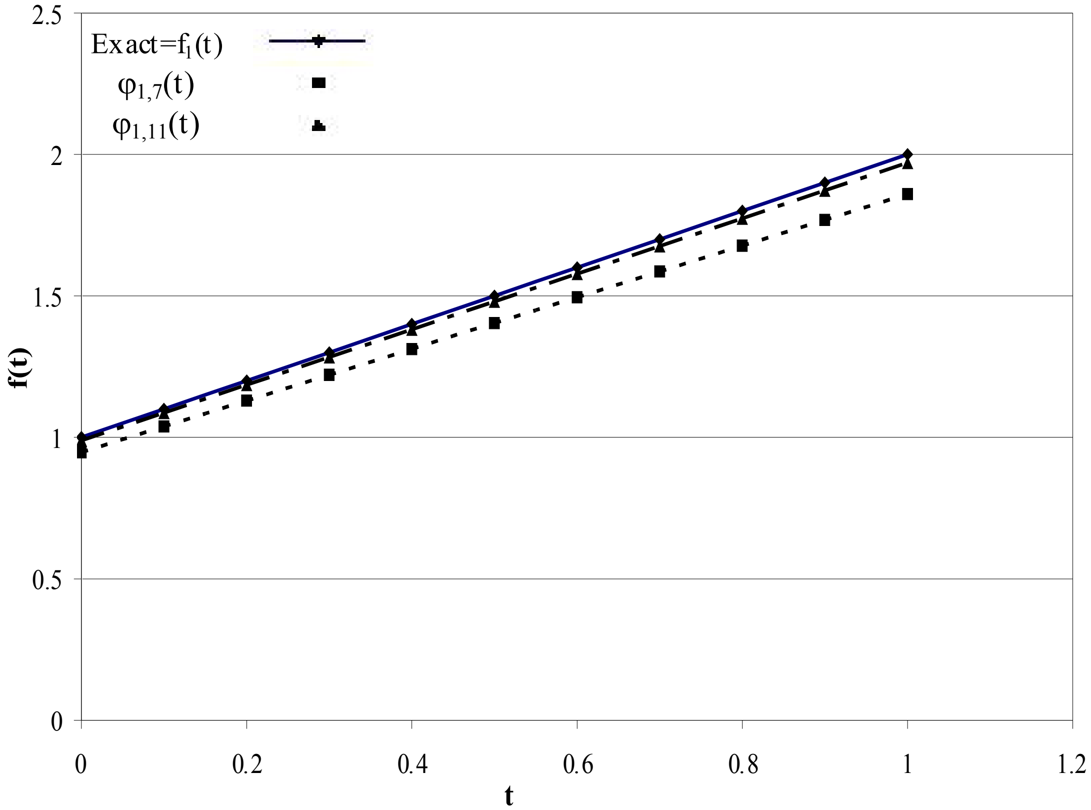

Fig.(1) :Approximations and Exact solution of Fredholm integral equations of Example

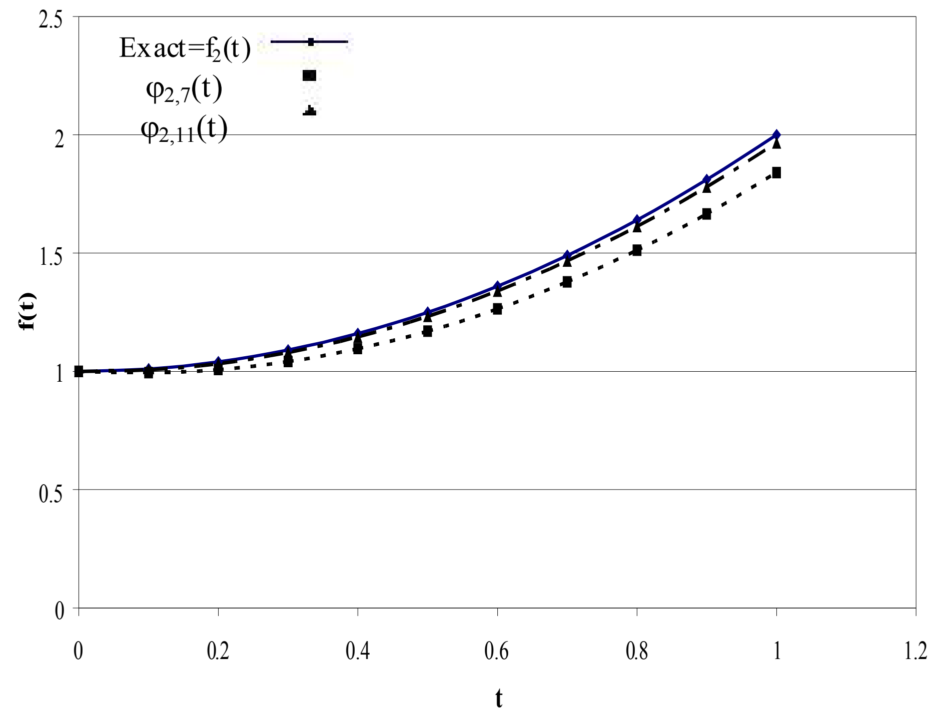

Fig.(2): Approximations and Exact solution of Fredholm integral equations of Example

حلول تقريبية لنظام معادلات فريدهوم التكاملية الخطية باستعمال طريقة الانحلال

حليمة سويدان علي ، وليدة سويدان علي*، نسرين مزهر رحمة
كلية الهندسة ، الجامعة المستنصرية
***كلية الزراعة ، جامعة بغداد**

الخلاصة

في هذا البحث استعمال طريقة الانحلال لإيجاد حلول تقريبية لنظام معادلات فريدهوم التكاملية الخطية من النوع الثاني. والحل في هذه الطريقة يكون دالة تمثل مجموعة متسلسلة غير محددة تتقارب إلى الحل. كما أن طريقة أدموند الانحلالية هي طريقة لحل المعادلات التكاملية الخطية وغير الخطية. والمثال المطبق يوضح هذه الطريقة.