ψ-Prime Submodules

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Received in:6/ March /2016,Accepted in:5/June /2016

Abstract

Let R be a commutative ring with identity and M be an unitary R-module. Let $\delta(M)$ be the set of all submodules of M, and $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. We say that a proper submodule P of M is ψ-prime if for each $r \in R$ and $x \in M$, if $rx \in P$, then either $x \in P + \psi(P)$ or $r M \subseteq P + \psi(P)$. Some of the properties of this concept will be investigated. Some characterizations of ψ-prime submodules will be given, and we show that under some assumptions prime submodules and ψ-prime submodules are coincide.

Key Words: Prime submodule, weakly prime submodules , ϕ-prime submodules.

1- Introduction

Throughout this paper, R is a commutative ring with identity and M is an unitary R-module. A proper ideal P of a ring R is prime if for all elements a ,b $\in$ R ,ab $\in$ P implies either a $\in$ P or b $\in$ P [1, p.40].In the theory of rings, Prime ideals play important roles. One of the natural generalizations of prime ideals which have attracted the interest of several authors in the last two decades is the notion of prime submodule,[2],[3],[4]. These have led to more information on the structure of the R-module M. For an ideal I of R and a submodule N of M let $\sqrt{I}$ denote the radical of I, and $[N :_R M] = \{r \in R, rM \subseteq N\}$ which is clearly an ideal of R. A proper submodule P of M is called a prime submodule if r $\in$ R and x $\in$ M with rx $\in$ P implies that r $\in$ [P:M] or x $\in$ P,[3]. There are several generalization of the notion of a prime submodules, such as Ebrahimi Atani, F. Farzalipour , introduced and studied weakly prime submodules ,where a proper submodule N of M is said to be weakly prime submodule of M if r $\in$ R and x $\in$ M $0 \neq rx \in N$ gives that r $\in$ [N : M] or x $\in$ N ,[5]. Khaksari and Jafari extended the notion of prime submodule to ϕ-prime.Let M be an R-module and $\delta(M)$ be the set of all submodules of M and $\phi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule P of M is said to be ϕ-prime if r $\in$ R and x $\in$ M, $rx \in P\backslash\phi(P)$ implies that r $\in$ [P:M] or x $\in$ P [6]. In this paper ,we define and study the notion of ψ-prime submodules . Let $\delta(M)$ be the set of all submodules of M and $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule P of M is said to be ψ-prime if for each r $\in$ R and x $\in$ M, if rx $\in$ P, then either $x \in P + \psi(P)$ or $rM \subseteq P + \psi(P)$.

2-Basic Properties of ψ-Prime Submodules

First we give the following definition.

Definition (2.1):

Let M be an R-module and $\delta(M)$ be the set of all submodules of M. Let $\psi:\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule N of M is said to be ψ-prime if for each r $\in$ R and x $\in$ M, if rx $\in$ N, then $x \in N + \psi(N)$ or $rM \subseteq N + \psi(N)$.

Remarks and Examples (2.2)

(1) It is clear that every prime submodule of an R-module M is ψ-prime submodule of M ,but the convers is not true in general for example: Let M = Z_8 as a Z-module, N = $<\bar{4}>$. Then N is not prime submodule of M . But N is ψ-prime submodule of M.

Proof : Let $\psi: \delta(Z_8) \longrightarrow \delta(Z_8) \cup \{\phi\}$, where $\psi(N) = N + <\bar{2}>, \forall N \subseteq M$, then for each r$\in$ Z, $\bar{x} \in Z_8$, if $r\bar{x} \in N$,then $\bar{x} \in N + \psi(N) = N + N + <\bar{2}> = <\bar{2}>$ or $rZ_8 \subseteq N + \psi(N) = N + N + <\bar{2}> = <\bar{2}>$ .Therefore N = $<\bar{4}>$ is a ψ-prime submodule of Z_8.

(2) If $\psi(N) \subseteq N$ or $\psi(N) = 0$, then every ψ-prime submodule of M is a prime submodule.

(3) Let N, W be two submodules of an R- module M and $N \subseteq W$. If N is ψ-prime submodule of M and $\psi(N) \subseteq \psi'(N)$,where $\psi': \delta(W) \longrightarrow \delta(W) \cup \{\phi\}$ and $\psi : \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$,then N is ψ'-prime submodule of W .

Proof : Let r$\in$R , m$\in$W such that rm $\in$N. Since N is ψ-prime submodule of M , so either $m \in N + \psi(N)$ or $rM \subseteq N + \psi(N)$. But $\psi(N) \subseteq \psi'(N)$, so either $m \in N + \psi'(N)$ or $rM \subseteq N + \psi'(N)$, so either $m \in N + \psi'(N)$ or $rW \subseteq N + \psi'(N)$ .Therefore N is ψ'-prime submodule of W .

(4) Given two functions $\psi,\psi': \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$such that $\psi \leq \psi'$ and $\psi(N) \subseteq \psi'(N)$ for each N $\in\delta(M)$. If N is a ψ-prime submodule of M implies N is ψ'-prime submodule of M .

Proof : Let r$\in$R , m$\in$M such that rm $\in$N. Since N is ψ-prime submodule of M , so either $m \in N + \psi(N)$ or $rM \subseteq N + \psi(N)$. But $\psi(N) \subseteq \psi'(N)$, so either $m \in N + \psi'(N)$ or $rM \subseteq N + \psi'(N)$. Therefore N is ψ'-prime submodule of M .

(5) Let N and W be two submodules of an R –module M such that N≅W . If N is ψ-prime submodule of M then it is not necessary that W is ψ-prime submodule of M as the following example explains :

Consider the Z – module Z ,the submodule 2Z is ψ-prime submodule of Z(since it is prime) but 2Z ≅ 30Z and 30Z is not ψ-prime submodule of Z .Since if ψ(N) = N ,∀ N⊆ M and 6.5 =30 ∈ 30Z but 5∉ 30Z +30Z =30Z and 6Z ⊈ 30 Z +30Z = 30Z.

(6) Let M = Z_4 as Z-module, N = $\{\bar{0},\bar{2}\}$ .Since N is prime, then N is ψ-prime submodule of M.

(7) Let M = Z_{12} as a Z – module and N = <$\bar{6}$> and let ψ : δ(Z_{12})⟶δ(Z_{12}) ∪ {ϕ}. If ψ(N) = N+ < $\bar{2}$ >, ∀ N ∈{<$\bar{6}$ > , < $\bar{2}$ > , < $\bar{4}$ >,< $\bar{0}$ > } and ψ(N) = N where N ∈{M , <$\bar{3}$>}.Therefore ψ<$\bar{6}$> = <$\bar{6}$> + <$\bar{2}$> = <$\bar{2}$> and ∀ $\bar{x}$∈ Z_{12}, r∈Z then either $\bar{x}$∈ <$\bar{6}$> + <$\bar{2}$> = <$\bar{2}$> or r Z_{12} ⊆ <$\bar{6}$> + <$\bar{2}$> = <$\bar{2}$> . Hence N is a ψ-prime submodule of M.

(8) {0} is the only ψ-prime submoduleof a simple modules.Therefore ($\bar{0}$) of a simple Z-module Z_p (p is prime) is ψ-prime submodule.

(9) Let M = Z ⊕ Z as a Z-module, N = 2Z⊕(0) and let ψ:δ(M)⟶ δ(M) ∪ {ϕ}.If ψ(N) = N, ∀ N ⊆ M, then N is not ψ-prime submodule of M.

(10) I is a ψ-prime ideal of R if and only if I is a ψ-prime submodule of R.

Now, if N is a prime submodule, then sometimes N is called P – prime submodule , where P = [N : M] ,[4].

For a ψ-prime, we called P- ψ-prime submodule ,where P = [N +ψ(N) : M].

The following theorem gives some characterizations for ψ-prime submodules.

Theorem(2.3):

Let N be a proper submodule of an R- module M and P= [N + ψ(N): M] .Then, the following statements are equivalent:

1. N is ψ-prime submodule of M.
2. For every submodule K of M and for every an ideal I of R such that IK⊆ N, implies that either K⊆ N + ψ(N)or I⊆ P = [N + ψ(N): M] .

Proof: (1) ⟶(2): Let IK⊆ N, where I be an ideal of R and K be a submodule of M. Suppose K⊈ N + ψ(N), then there exists k∈ K such that k∉N + ψ(N). It is clear that for each y∈ I, thus
yk∈N . But N is ψ-prime submodule of M and k∉N + ψ(N), hence y∈ P= [N + ψ(N): M]. Therefore I⊆ P .

(2))⟶(1): Let r ∈ R , m ∈ M such that rm ∈N. Then < r > < m> ⊆ N. So either < m> ⊆ N + ψ(N) or < r> ⊆ P= [N + ψ(N): M] by(2)); i.e., either m∈N + ψ(N) or r ∈ P= [N + ψ (N): M]. Therefore N is ψ-prime submodule of M .

We can give the following result .

Proposition (2.4):

Let N be a proper submodule of an R- module M . If [N + ψ (N): M] = [N+ ψ (N) : K] for each submodule K of M such that K⊋ N + ψ(N) , then N is ψ-prime submodule of M.

Proof: Let r ∈ R , m ∈ M such that rm ∈N and suppose m ∉N + ψ(N). Let K= N + ψ(N)+ <m>, then K⊋ N + ψ(N), m∈K and so r∈ [N:K] ⊆ [N + ψ(N):K] = [N + ψ (N): M].It follows that r∈[N + ψ (N): M] and hence N is ψ-prime.

However , we can give another corollary of proposition (2.4). But first we state and prove the following lemma which we needed .

Lemma (2.5)

Let N be a proper submodule of an R- module M . If $[N + \psi(N): M] = [N + \psi(N) : c]$ for each $c \in M \setminus N + \psi(N)$, then $[N + \psi(N): M] = [N+\psi(N) : K]$ for each submodule K of M such that $K \supsetneq N + \psi(N)$.

Proof: Since $K \subseteq M$ so $[N + \psi(N): M] \subseteq [N+ \psi(N) : K]$ .Let $r \in [N + \psi(N) : K]$,hence $rK \subseteq N + \psi(N)$. But $N + \psi(N) \subsetneq K$, implies that there exists $x \in K$ and $x \notin N + \psi(N)$.Hence $rx \in N+\psi(N)$ and then $r \in [N+\psi(N) : x] =[N+ \psi(N) : M]$, which implies that $r \in [N + \psi(N) : K] \subseteq [N + \psi(N): M]$ .Therefore $[N + \psi(N): M]= [N+ \psi(N): K]$ for each submodule K of M such that $K \supsetneq N + \psi(N)$.

Corollary (2.6):

Let N be a proper submodule of an R- module M . If $[N + \psi(N): M] = [N + \psi(N) : c]$ for each $c \in M \setminus N + \psi(N)$, then N is ψ-prime submodule of M.

Now, the following proposition shows that under the condition $\psi(N) \subseteq N$ for all submodule N of M the convers of proposition (2.4) is true.

Proposition(2.7):

If N is a ψ-prime submodule of an R– module M and $\psi(N) \subseteq N$,then $[N + \psi(N): M]= [N+ \psi(N) : K]$ for each submodule K of M such that $K \supsetneq N + \psi(N)$.

Proof: Since N is a ψ-prime submodule of M and $\psi(N) \subseteq N$,so by (Remark 2.2,(2)) N is a prime submodule .Hence $[N :M] = [N:K]$, for each submodule K of M such that $K \supsetneq N$,[2]. Since $\psi(N) \subseteq N$,then $[N + \psi(N): M]= [N+ \psi(N) : K]$ for each submodule K of M such that $K \supsetneq N + \psi(N)$.

It is well Know if N is a prime submodule of an R – module ,then [N : M] is a prime ideal of R ,see [4]. But for a ψ-prime we have:

Remark(2.8):

(1) If N is ψ-prime submodule of M , then it is not necessarily that [N:M] is a ψ- prime ideal of R, for example: Let $M = Z_8$ as a Z-module ,$N = <\bar{4}>$. Then N is ψ- prime submodule of M by (Remark 2.2,(1)). But $[<\bar{4}>: Z_8] = 4Z$ is not ψ-prime submodule of Z ,where $\psi(I) = I$ for each I an ideal of Z and $\psi: \delta(Z) \longrightarrow \delta(Z) \cup \{\phi\}$ be a function.

Now, the following proposition shows that under the condition $\psi(N) \subseteq N$ for all submodule N of M the above statement is true.

Proposition(2.9):

If N is a ψ-prime submodule of an R– module M and $\psi(N) \subseteq N$,then [N: M] is a ψ-prime ideal of R.

Proof:Since N is a ψ-prime submodule of an R– module M and $\psi(N) \subseteq N$, so N is a prime submodule by (2.2,(2)), then [N:M] is a prime ideal of R and hence is a ψ-prime ideal of R.

Remark (2.10):

If [N :M] is ψ- prime ideal of R , then it is not necessarily that N is ψ-prime submodule of M , for example: Let $M = Z \oplus Z$ as a Z-module, $N = 2Z\oplus(0)$, N is not ψ-prime submodule of M ,by (2.2,9). But $[N :M] = [2Z \oplus (0) : Z \oplus Z] = 0$ is a prime ideal of Z and hence is ψ-prime ideal of Z, where $\psi(I) = I$ for each I an ideal of Z and $\psi: \delta(Z) \longrightarrow \delta(Z) \cup \{\phi\}$ be a function.

Now, we shall give characterization of ψ-prime submoules, but first recall the following: Let R be any ring. A subset S of R is called multiplicatively closed if $1\in S$ and $ab \in S$ for every $a ,b\in S$. We Know that every proper ideal P in R is prime if and only if R-P is multiplicatively closed subset of R,[1,p.42]. And if N is a submodule of an R-module M and S is multiplicatively closed sub set of R, then N(S) ={x∈ M:∃ t ∈ S, such that tx∈ N}be a submodule of M and N⊆ N(S).

Proposition (2.11):

Let N be a proper submodule of an R- module M .If [N + ψ(N):M] is a prime ideal of R and N(S) ⊆ N + ψ(N) for each multiplicatively closed subset of R such that S ∩ [N + ψ(N):M] = ϕ,then N is ψ-prime submodule of M .

Proof: Let r ∈ R , m ∈ M such that rm ∈N and suppose m ∉N + ψ(N), r ∉ [N + ψ(N):M] . Consider the set S ={1,r, r^2,……….},this is multiplicatively closed subset of R and it is clear that S ∩[N + ψ(N):M] = ϕ , since[N + ψ(N): M] is a prime ideal of R. But m ∉N + ψ(N) implies that m ∉N(S) and so r m ∉N which is a contradiction .Therefore either m∈ N + ψ(N) or r∈[N + ψ(N):M] and hence N is ψ-prime submodule of M .

Conversely ,if N is ψ-prime submodule of M, to prove N(S) ⊆ N + ψ(N) . Let x ∈ N(S) , so there exists t ∈S such that tx ∈ N . But N is ψ-prime submodule of M, so either x ∈ N + ψ(N) or t∈[N + ψ(N): M]. But t ∈[N + ψ(N): M] implies that S ∩[N + ψ(N): M] = ϕ which is a contradiction .Thus , x∈N +ψ(N) and hence N(S) ⊆N+ ψ(N).

Proposition (2.12):

If [N + ψ(N):M] is maximal ideal of R , then N is ψ-prime submodule of M .

Proof: Let r ∈ R , m ∈ M such that rm ∈N .If r ∉[N + ψ(N): M] ,then R= < r> +[N + ψ(N): M] .Therefore there exist s∈R and k∈[N + ψ(N): M] such that 1=s r +k and so m= srm +km ∈ N + ψ(N) .Therefore N is ψ-prime submodule of M.

Proposition (2.13):

Let N be a proper submodule of an R- module M such that [K: M] ⊈ [N+ ψ(N):M] for each submodule K of M and containing N+ ψ(N) properly .If [N + ψ(N):M] is a prime ideal of R, then N is ψ-prime submodule of M .

Proof: Suppose [N + ψ(N):M] is a prime ideal of R, to prove N is ψ-prime submodule of M. Let r ∈ R , m ∈ M such that rm ∈N and suppose m ∉N + ψ(N). Let K = N + ψ(N) + <m>, it is clear that N + ψ(N)⊊ K, and so [K: M] ⊈ [N+ ψ(N): M].Then there exists s∈ [K:M] and s ∉ [N + ψ(N):M].Thus, sM ⊆ K and sM ⊈ N+ ψ(N).But, sM ⊆ K implies ,r s M ⊆ r K = r (N + ψ(N) + <m>) ⊆ N + ψ(N) and rs∈ [N + ψ(N):M]. Since [N + ψ(N):M] is a prime ideal of R and s ∉ [N + ψ(N):M], so r∈ [N + ψ(N):M] . Therefore N is ψ-prime submodule of M .

Recall that an R- module M is called mulitplication module if for every submodule N of M ,there exists an ideal I of R such that IM=N ,equivalently ;for every submodule N of M N=[N:M]M, see[7].

Corollary (2.14):

Let N be a proper submodule of a mulitplication R- module M. Then N is ψ-prime submodule of M if [N + ψ(N):M] is a prime ideal of R.

Proof: Suppose [N + ψ(N):M] is a prime ideal of R, to prove N is ψ-prime submodule of M. Let r ∈ R , m ∈ M such that rm ∈N and suppose m ∉N + ψ(N). Let K = N + ψ(N) + < m>,it is clear that N + ψ(N) ⊊K. Since M is mulitplication , so [K: M] ⊈ [N+ ψ(N):M] by[9,remark (2-15),chapter one]. Then there exists s∈ [K:M] and s ∉ [N + ψ(N):M].Thus, sM ⊆ K and sM ⊈ N+ ψ(N).But, sM ⊆ K implies r s M ⊆ r K = r (N + ψ(N) + <m>) ⊆ N + ψ(N) and rs∈ [N + ψ(N):M]. Since [N + ψ(N):M] is a prime ideal of R and s ∉ [N + ψ(N):M], so r∈ [N + ψ(N):M] .Therefore N is ψ-prime submodule of M .

As anther consequence of (2.13) , we have the following result:

Corollary (2.15):

Let N be a proper submodule of a cyclic R- module M. Then N is ψ-prime submodule of M if [N + ψ(N):M] is a prime ideal of R.

Proof: Since M is cyclic ,then M is a mulitplication . Hence the result follows immediately from corollary (2.14).

Recall that an R – module M is said to be bounded module if there exists an element $x \in M$ such that annM = ann(x), where annM = $\{r \in R: rm = 0, \forall\, m \in M\}$, [8].And an R- module M is said to be fully stable if each submodule is stable, where a submodule N of an R – module M is said to be stable if $f(N) \subseteq N$ for each $f \in Hom(N,M)$, [9].

Corollary (2.16):

Let N be a proper submodule of a bounded fully stable R- module M. Then N is ψ-prime submodule of M if [N + ψ(N):M] is a prime ideal of R.

Proof: Since M is bounded fully stable R- module, then M is a cyclic R- module by [10,prop.1.1.4,ch.1]. Hence the result follows immediately from corollary (2.15).

Proposition (2.17):

Let M be an R-module and N,L be two submodules of M .If K be a P- ψ-prime submodule of M such that $N \cap L \subseteq K$, then $L \subseteq K + \psi(K)$ or $[N: M] \subseteq P = [K + \psi(N): M]$.

Proof: Suppose $[N:M] \not\subseteq [K + \psi(N): M] = P$, so there exists $s \in [N: M]$ and $s \notin P = [K + \psi(N): M]$. Let $t \in L$, then $st \in L \cap N$ and so $st \in K$. But K is P- ψ-prime submodule of M and $s \notin [K + \psi(N): M]$. Therefore $t \in K + \psi(K)$, thus $L \subseteq K + \psi(K)$.

Corollary (2.18):

Let A an ideal of a ring R and N be a submodule of M . If K is a P- ψ-prime submodule of M such that $AM \cap N \subseteq K$, then either $AM \subseteq K + \psi(K)$ or $N \subseteq K + \psi(K)$.

proposition (2.19):

Let M be an R-module and N be a submodules of M .If $P = [N + \psi(N): M]$ is a prime ideal of R , then $[N + \psi(N): M] = [N + \psi(N): rM]$, $\forall\, r \notin [N + \psi(N): M]$.

Proof: Since $rM \subseteq M$,so $[N + \psi(N): M] \subseteq [N + \psi(N): rM]$. Let $a \in [N + \psi(N): rM]$, so $ar\, M \subseteq N + \psi(N)$. Which means that $ar \in [N + \psi(N): M]$. But $[N + \psi(N): M]$ is a prime ideal of R, so either $a \in [N + \psi(N): M]$ or $r \in [N + \psi(N): M]$, but $r \notin [N + \psi(N): M]$, so $a \in [N + \psi(N): M]$. Thus $[N + \psi(N): M] = [N + \psi(N): rM]$, $\forall\, r \notin [N + \psi(N): M]$.

Now, we can give the following proposition :

Proposition (2.20):

Let N be a submodule of an R – module M and $P = [N + \psi(N): M]$. If the ideal = $[N + \psi(N): <e>] = P$, for each $e \in M$, $e \notin N + \psi(N)$, then N is a ψ- prime submodule of M .

Proof: Let $r \in R$, $x \in M$ such that $rx \in N$ and suppose $x \notin N + \psi(N)$. Thus $r \in [N + \psi(N): <x>]$. But $[N + \psi(N):_R <x>] = P$, so $r \in P$. Therefore N is a ψ - prime submodule of M .

However, we can give a corollary of proposition (2.20).But first we state and prove the following lemma which is needed .

Lemma (2.21):

Let N be a submodule of an R – module M . If the submodule $[N + \psi(N):_M <r>] = N + \psi(N)$, for each $r \in R$, $r \notin P$, then the ideal $P = [N + \psi(N):_R <e>]$, for each $e \in M$; $e \notin N + \psi(N)$.

Proof: Let $e \in M$; $e \notin N + \psi(N)$. It is clear that $P \subseteq [N + \psi(N):_R <e>]$. Let $r \in [N + \psi(N):_R <e>]$, then $re \in N + \psi(N)$. Suppose $r \notin P = [N + \psi(N):_R M]$ .Since $[N + \psi(N):_M <r>] = N+\psi(N)$ and $e \in [N+ \psi(N) :< r >]$, so $e \in N+\psi(N)$ which a contradicts our assumption. Thus $r \in P$ for each $e \in M$ such that $e \notin N + \psi(N)$.Therefore $P = [N + \psi(N):_R :<e>]$.

Corollary (2.22):

Let N be a submodule of an R – module M and $P = [N + \psi(N): M]$. If the submodule $[N + \psi(N):_M <r>] = N + \psi(N)$, for each $r \in R$, then N is a ψ- prime submodule of M .

Note that, the intersection of two ψ-prime submodules of an R – module M need not be a ψ-prime submodule of M ,for example :

The Z-module Z_6 has two ψ-prime submodules , $N_1 = <\bar{2}>$ and $N_2 = <\bar{3}>$ but $N_1 \cap N_2 = <\bar{0}>$ is not a ψ-prime submodule of Z_6 ,where $\psi(N) = N$, $\forall N \subseteq M$.

However , we have the following proposition:

Proposition (2.23):

Let K is a ψ-prime of an R-module M and let $N < M$ such that $\psi(K) \subseteq K$. Then either $N \subseteq K$ or $K \cap N$ is a ψ'-prime in N, where $\psi' : \delta(N) \longrightarrow \delta(N) \cup \{\phi\}$ and $\psi : \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$.

Proof: Since K is a ψ-prime of an R-module M and $\psi(K) \subseteq K$, so K is a prime by (2.2,(1)). Hence either $N \subseteq K$ or $K \cap N$ is a prime in N,[11]. Therefore either $N \subseteq K$ or $K \cap N$ is a ψ'-prime in N.

proposition (2.24):

Let P be an ideal of a ring R and M be an R – module . Then a proper submodule N of M is a P - ψ - Prime if and only if

1. $P \subseteq [N + \psi(N): M]$, and
2. $cm \notin N$,for all $c \in R \setminus P$,$m \in M \setminus N +\psi(N)$.

Proof: Suppose N is a P - ψ - Prime. To prove that (1) and (2) are hold. It is clear that $P = [N + \psi(N): M]$ .Therefore $P \subseteq [N + \psi(N): M]$.

Now if $c \in R \setminus P$ and $m \in M \setminus N +\psi(N)$, then $c \notin [N + \psi(N):M]$ and $m \notin N + \psi N)$, hence $cm \notin N$.

Conversely, let $c \in R$ and $m \in M$ such that $m \notin N + \psi(N)$ and $c \notin [N + \psi(N):M]$. Since $P \subseteq [N + \psi(N):M]$, then $m \in M \setminus N +\psi(N)$ and $c \notin P$ .Therefore , $c \in R \setminus P$. Hence $cm \notin N$, which implies that N is a P - ψ - Prime.

proposition (2.25):

Let $\phi : M \longrightarrow M'$ be an homomorphism . If N is ψ' - prime submodule of an R-module M', such that $\phi(M) \nsubseteq N$ and $\psi(\phi^{-1}(N)) = \phi^{-1}(\psi'(N))$, then $\phi^{-1}(N)$ is ψ-prime submodule of M, where $\psi:\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ and $\psi' :\delta(M') \longrightarrow \delta(M') \cup \{\phi\}$.

Proof: First, we must show *that* $\phi^{-1}(N)$ is a proper submodule of M. Suppose that $\phi^{-1}(N) = M$, then $\phi(M) \subseteq N$, which a contradiction to the assumption. Let $r \in R$, $m \in M$ such that $rm \in \phi^{-1}(N)$ .Then $r\phi(m) \in N$ and N is ψ' - prime submodule of an R-module M', then either $\phi(m) \in N+ \psi'(N)$ or $rM' \subseteq N+ \psi'(N)$. If $\phi(m) \in N+ \psi'(N)$, then $m \in \phi^{-1}(N) + \phi^{-1}(\psi'(N))$ and hence $m \in \phi^{-1}(N) +\psi(\phi^{-1}(N))$. If $rM' \subseteq N+ \psi'(N)$, then $r\phi(M) \subseteq N+ \psi'(N)$

since $\phi(M) \subseteq M'$. This implies $r\ M \subseteq \phi^{-1}(N) + \phi^{-1}(\psi'(N)) = \phi^{-1}(N) + \psi(\phi^{-1}(N))$.Therefore $\phi^{-1}(N)$ is ψ-prime submodule of M.

Theorem (2.26):

Let $f: M \longrightarrow M'$ be an epimorphism and let $N < M$ such that $\ker f \leq N$. If N is a ψ-prime submodule of an R- module M and $\psi'(f(N)) = f(\psi(N))$, then f (N) is a ψ'-prime submodule of a module of M', where $\psi:\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ and $\psi':\delta(M') \longrightarrow \delta(M') \cup \{\phi\}$.

Proof: First, we must show that f (N) is a proper submodule of a module M'. Suppose f (N) = M'. But f is an epimorphism, thus f (N) = f (M) and hence M = N + ker f. This implies that M = N. A contradiction.

Now, let $r\ m' \in f(N)$, where $r \in R$ and $m' \in M'$,$m'= f(m)$ for some $m \in M$ since f is an epimorphism . Then $r f(m) \in f(N)$,so $f(r m)= f(n)$ for some $n \in N$ and hence $f(r m) - f(n) = 0$ .Thus we get that $rm-n \in \ker f \subseteq N$ which implies that $rm \in N$ .But N is a ψ-prime ,so either $m \in N + \psi(N)$ or $r M \subseteq N + \psi(N)$. If $m \in N + \psi(N)$,then $f(m) \in f(N) + f(\psi(N))$;that is $m' \in f(N) + f(\psi(N)) = f(N) + \psi'(f(N))$. If $r M \subseteq N + \psi(N)$,then $r f(M) \subseteq f(N) + f(\psi(N)) = f(N) + \psi'(f(N)$ implies that $r M' \subseteq f(N) + \psi'(f(N)$.

Corollary (2.27):

Let M be an R-module, let $K < N < M$ and N be a ψ-prime of M. Then N/K is a ψ'-prime submodule of M/K, where $\psi': \delta(M/K) \to \delta(M/K)$.

Proof: Let $\pi:M \longrightarrow M/K$ be the natural mapping ,then the result follows by proposition(2.26).

Proposition (2.28):

Let M be an R-module and let $K < N < M$ .If N is a ψ-prime submodule of M, then N/K is a ψ'-prime submodule of M/K, where $\psi': \delta(M/K) \to \delta(M/K)$ and $\psi'(N/K) = \psi(N)/K$.

Proof: Let $r \in R$ and $m \in N/K$ with $r\ m \in [N/K]$,where $m= m+K$,for some $m \in M$. So we have $rm \in N$,which gives that either $m \in N + \psi(N)$ or $r M \subseteq N + \psi(N)$ .Therefor either $m + K \in (N + \psi(N))/K =[N/K] +[\psi(N)/K] = [N/K] + [\psi'(N/K)]$ or $r [M/K] \subseteq [(N + \psi(N))/K] \subseteq [N/K] +[\psi(N)/K] = [N/K] + [\psi'(N/K)]$ .Hence either $m \in [N/K] + [\psi'(N/K)]$ or $r [M/K] \subseteq [N/K] + [\psi'(N/K)]$ .Therefore N / K is a ψ'-prime submodule of M / K.

Let R_1 ,R_2 be two commutative rings with identity and M_1, M_2 be R_1 and R_2 – module respectively, put $R = R_1\times R_2$. Then $M = M_1 \times M_2$ is an R – module and each submodule of M is of the form $N = N_1 \times N_2$ for some N_1 of M_1 and N_2 of M_2 .Furthermore $N = N_1 \times N_2$ is a prime submodule of M if and only if $N = N_1\times M_2$ or $N = M_1 \times N_2$ for some prime submodules N_1 of M_1 and N_2 of M_2 .

Theorem (2.29):

Let $R = R_1\times R_2$ that each R_i is a commutative rings with identity . Let M_i be R_i – module and $M = M_1 \times M_2$ with $(r_1, r_2)(m_1,m_2) = (r_1m_1,r_2m_2)$, be an R – module ,where $r_i \in R_i$, $m_i \in M_i$, and let $\psi_i : \delta(M_i) \to \delta(M_i) \cup \{\phi\}$ be functions , $\psi = \psi_1 \times \psi_2$. Then we have :

1) $N_1 \times N_2$ is a ψ- prime submodule , where N_i is a ψ_i – prime submodule of M_i , with $N_i \subseteq \psi_i(N_i)$.
2) $N_1\times M_2$ is a ψ- prime submodule of M , where N_1 is a prime submodule of M_1.
3) $N_1\times M_2$ is a ψ- prime submodule of M, whereN_1 is a ψ_1- prime submodule of M_1 and $\psi_2(M_2) = M_2$.

4) $M_1 \times N_2$ is a ψ- prime submodule of M, where N_2 is a prime submodule of M_2 .
5) $M_1 \times N_2$ is a ψ- prime submodule of M, where N_2 is a ψ_2- prime submodule of M_2 and $\psi_1(M_1) = M_1$.

Proof:

(1) Suppose N_i is a ψ_i – prime submodule of M_i , with $N_i \subseteq \psi_i(N_i)$ and let $(r_1, r_2)(m_1,m_2) = (r_1m_1,r_2m_2) \in N_1 \times N_2$, then $r_1m_1 \in N_1$ and $r_2m_2 \in N_2$ and since N_i is a ψ_i – prime submodule of M_i , so either $r_1 \in [N_1 + \psi_1(N_1): M_1]$ and . $r_2 \in [N_2 + \psi_2(N_2) : M_2]$ or $m_1 \in N_1 + \psi_1(N_1)$ and $m_2 \in N_2 + \psi_2(N_2)$. Hence either $(r_1, r_2) \in [N_1 + \psi_1(N_1): M_1] \times [N_2 + \psi_2(N_2) : M_2] = [\psi_1(N_1): M_1] \times [\psi_2(N_2) : M_2] = [\psi_1(N_1) \times \psi_2(N_2) : M_1 \times M_2] = [N_1 \times N_2 + \psi(N_1 \times N_2) : M_1 \times M_2]$ or $(m_1,m_2) = [N_1 + \psi_1(N_1)] \times [N_2 + \psi_2(N_2)] = [\psi_1(N_1) \times \psi_2(N_2) \subseteq [N_1 \times N_2] + [\psi(N_1 \times N_2)]$. Therefore $N_1 \times N_2$ is a ψ- prime submodule .

(2) If N_1 is a prime submodule of M_1, then $N_1 \times M_2$ is a prime submodule of M by [6,Th.2.14,p.1446] and hence $N_1 \times M_2$ is a ψ- prime submodule of M by (2.2,(1)).

3) Let N_1 is a ψ_1- prime submodule of M_1 and $\psi_2(M_2) = M_2$ .Let $(r_1, r_2) \in R$ and $(m_1,m_2) \in M$ be such that $(r_1, r_2)(m_1,m_2) = (r_1m_1,r_2m_2) \in N_1 \times M_2$. Then $r_1m_1 \in N_1$ and $r_2m_2 \in M_2$ and since N_1 is a ψ_1 – prime submodule of M_1, so either $r_1 \in [N_1 + \psi_1(N_1): M_1]$ or $m_1 \in N_1 + \psi_1(N_1)$ .So either $(r_1, r_2) \in [N_1 + \psi_1(N_1): M_1] \times [M_2 : M_2]$ or $(m_1,m_2) \in [N_1 + \psi_1(N_1)] \times M_2$. Hence either $(r_1, r_2) \in [(N_1 + \psi_1(N_1)) \times M_2 : M_1 \times M_2] = [N_1 \times M_2 + \psi(N_1 \times M_2) : M_1 \times M_2]$ or $(m_1,m_2) \in [N_1 \times M_2 + \psi_1(N_1) \times \psi_2(N_2)] = N_1 \times M_2 + \psi(N_1 \times M_2)$.Therefore $N_1 \times M_2$ is a ψ- prime submodule of M .

Parts (4) ,(5) are proved similar to (2) ,(3) respectively.

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المقاسات الجزئية الأولية من النمط -- ψ

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استلم في:9/اذار/2016،قبل في:5/حزيران/2016

الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد، وليكن M مقاساً معرفاً على الحلقة R. لتكن $\delta(M)$ مجموعة كل المقاسات الجزئية من M ولتكن $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ دالة. في هذا البحث، نقول ان المقاس الجزئي P من M هو مقاس جزئي أولي من النمط -- ψ اذا كان لكل $r \in R$, $m \in M$ اذا ان $rx \in P$ ، فانه يؤدي الى اما $m \in P + \psi(P)$ أو $r \in [P + \psi(P): M]$ 0 لقد درسنا واعطينا بعض خواص و مميزات هذا النوع من المقاسات الجزئية وبرهنا تحت شروط معينة ان المقاسات الجزئية الاولية وهذا النوع من المقاسات الجزئية تكون متكافئة 0

الكلمات المفتاحية : المقاسات الجزئية الاولية ، المقاسات الجزئية الضعيفة ، المقاسات الجزئية الاولية من النمط - ϕ 0