أنواع جديدة من المجموعات القالبية في المستوي الاسقاطي PG(2,q)

علي طالب محمد ، ميساء جليل محمد

قسم الرياضيات - كلية التربية- ابن الهيثم - جامعة بغداد

الخلاصة

لقد تم في هذا البحث الحصول على انواع جديدة من المجموعات القالبية في المستوي الاسقاطي حول حقل كالوا PG(2,q). هذان النوعان هما المجموعة القالبية الكاملة والمجموعة القالبية العظمى. كذلك تم الحصول على بعض النتائج حول المجموعات.

New Kinds of Blocking sets in a Projective Plane PG(2,q)

A.T. Mahammad, M. J. Mahammad
Department of Mathematics, College of Education Ibn-Al-Haitham , University of Baghdad

Abstract

In this work, new kinds of blocking sets in a projective plane over Galois field PG(2,q) can be obtained. These kinds are called the complete blocking set and maximum blocking set. Some results can be obtained about them.

Introduction

Let PG(2,q) denotes the 2-dimensional projective plane over Galois field GF(q). A blocking set in PG(2,q) is a set of points that nonempty intersection with every line in PG(2,q), (1).

This paper is divided into two sections, section one consists of standard basic, theorems and definitions of projective plane, blocking set, minimal, trivial, t-fold, maximal, complete, and committee blocking sets. Section two consists of the relation between (k,n)-arc and blocking sets.

Section One

Definition "Projective Plane" (1)

A projective plane PG(2,q) over Galois field GF(q) is a two-dimensional projective space, which consists of points and lines with relation between them, in PG(2,q) there are $q^2 + q + 1$ points, and $q^2 + q + 1$ lines, every line contains $1 + q$ points and every point is on $1 + q$ lines, any point in PG(2,q) has the form of a triple (a_1,a_2,a_3) where $a_1, a_2, a_3 \in GF(q)$; such that $(a_1,a_2,a_3) \neq (0,0,0)$. Two points (a_1,a_2,a_3) and (b_1,b_2,b_3) represent the same point if there exists $\lambda \in GF(q)\backslash\{0\}$, such that $(b_1,b_2,b_3) = \lambda (a_1,a_2,a_3)$. Similarly any line in PG(2,q) has the form of a triple $[a_1,a_2,a_3]$, where $a_1,a_2,a_3 \in GF(q)$; such that $[a_1,a_2,a_3] \neq [0,0,0]$. Two lines $[a_1,a_2,a_3]$ and $[b_1,b_2,b_3]$ represent the same line if there exists $\lambda \in GF(q)\backslash\{0\}$, such that $[b_1,b_2,b_3] = \lambda [a_1,a_2,a_3]$.

There exists one point of the form (1,0,0). There exists q points of the form (x,1,0). There exists q^2 points of the form (x,y,1). A points $p(x_1,x_2,x_3)$ is incident with the line $L[a_1,a_2,a_3]$ if and only if $a_1x_1 + a_2x_2 + a_3x_3 = 0$, i.e.

a point represented by (x_1,x_2,x_3) and the line represented by $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, then

$$(x_1,x_2,x_3)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \Leftrightarrow a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

Any projective plane PG(2,q) satisfies the following axioms:

1. Any two distinct lines intersected in a unique point.
2. Any two distinct points are contained in a unique line.
3. There exists at least four points such that no three of them are collinear.

Definition "Blocking Set"(1)

A blocking set B of PG(2,q) is a set of points intersecting every line of PG(2,q) in at least one point, so B is blocking set if and only if PG(2,q)\B is blocking set.

Definition "Minimal Blocking Set" (1)

A Blocking set B is called minimal in PG(2,q) when no proper subset of it is still a blocking set such that, $\forall$ p $\in$ B, B\{p} is not a blocking set.

Example

In PG(2,3), there exists 13 points 1,2,…,13 and 13 lines L_1, L_2,…, L_n, table (1), such that every point is on four lines and every line contains four points, let B={1,2,4,5,6,7}. B is a minimal blocking set of PG(2,3).

Definition "Trivial Blocking Set"(2)

A blocking set in PG(2,q) is called trivial when contains a line.

Definition "Committe Blocking Set"(2)

A blocking set B of PG(2,q) is committee if and only if B contains a minimum value of humber of points.

Now the definition of complete blocking set can be given

Definition "Complete Blocking Set"

A Blocking set B of PG(2,q) is called complete blocking set if for every p in PG(2,q)\B, B$\cup${p} is not blocking set or B$\cup${p} is trivial blocking set. i.e.

B is complete blocking set if $\forall$ p $\notin$ B, $\exists$ L $\in$ PG(2,q) and p $\in$ L such that L $\cap$ (B$\cup${p}) = L.

B is incomplete blocking set of PG(2,q) if there exists at least one point p $\in$ PG(2,q) such that B$\cup${p} is still blocking set.

Example

In PG(2,q), there exists 13 points and 13 lines showed in the table (1), such that in the projective plane PG(2,3) in the above example.

Let B_1={1,2,3,4,5,6}.

B_1 is a blocking set of PG(2,3), since it intersects every line in at least one point.

PG(2,3)\B_1 = {7,8,9,10,11,12,13} is also blocking set.

B_1 is incomplete blocking set, since 7 $\in$ PG(2,3)\B_1 and $B_1\cup${7} is still blocking set.

Let B_2={1,2,3,4,5,6,7}.

B_2 is a blocking set of PG(2,3)

Since $L_1 \subset B_2 \cup \{8\}$, $L_2 \subset B_2 \cup \{9\}$, $L_3 \subset B_2 \cup \{10\}$, $L_4 \subset B_2 \cup \{11\}$, $L_5 \subset B_2 \cup \{12\}$, $L_6 \subset B_2 \cup \{13\}$. Then $B_2 \cup \{8\}$, $B_2 \cup \{9\}$, $B_2 \cup \{10\}$, $B_2 \cup \{11\}$, $B_2 \cup \{12\}$ and $B_2 \cup \{13\}$ are not blocking sets and hence B_2 is a complete blocking set.

Definition "New" "Maximum Blocking Set"

A blocking set B of PG(2,q) is maximum if and only if PG(2,q)\B is committee blocking set.

Example

In PG(2,5) there exists 31 points {1,2,…,31} and 31 lines {L_1,L_2,…,L_{31}}, in table (2), such that every point is on six lines and every line contains six points. Let B_1=(1,2,3,4,5,7,10,17,21,27}, is a committee blocking set, and Let B_2 = PG(2,5)\B_1={6,8,9,11,12,13,14,15,16,18, 19,20,22,23,24,25,26,28,29,30,31} is a blocking set and it is maximum blocking in PG(2,5). Since it does not exist a blocking set B in PG(2,5) such that $B_2 \subset$ B.

Definition "A (k,n)-arc" (2)

A (k,n)-arc in PG(2,q) is a set S of k points with property that every line contains at most n points of S, a (k,n)-arc S is called complete arc if it is not contained in a (k+1,n)-arc.

Definition "t-Fold Blocking Set" (2)

A t-fold blocking set B in PG(2,q) is a set of points in PG(2,q) that intersects every line in PG(2,q) in at least t points.

Theorem (1)

A blocking set B is minimal if and only if for every point $p \in B$, there is some line L such that $B \cap L = p$.

Proof:

Let B satisfies the condition that p is a point such that $B \cap L = \{P\}$ for some line L, then $B \backslash \{p\}$ does not intersect L and so is not a blocking set and that is a contradiction. Thus B is minimal.

Conversely suppose that B is minimal, so $B \backslash \{p\}$ is not a blocking set i.e. for if the condition is not satisfied there is some point p in B such that all lines through p contain another point of B so $B/\{p\}$ is a blocking set and B is not minimal point $p \in B$ there exists some line $L \in PG(2,q)$ such that $p = L \cap B$.

Theorem (1)

If B is a blocking set in PG(2,q), and $|B| = b$, then

(i) No line of PG(2,q) contains more that $b - q$ points of B,

(ii) If a line contains n points of B, then $b \geq n + q$,

(iii) There is some line containing at least three points of B,

(iv) $b \geq q + 3$.

Definition "Subgeometry"

Let PG(2,q), q is square be a projective plane, then PG(2, $\sqrt{q}$) is a subgeometry of PG(2,q) and contains $q + \sqrt{q} + 1$ points and lines, every line contain $\sqrt{q} + 1$ point and every points is on $\sqrt{q} + 1$ lines.

Theorem (1), (3)

In PG(2,q), q is square, then

(i) If $|B| = q + \sqrt{q} + 1$, B is blocking set, then B is subgeometry of PG(2,$\sqrt{q}$),

(ii) If $|B| = q^2 - \sqrt{q}$, B is blocking set, then B is complement of a subgeometry PG(2, $\sqrt{q}$).

Theorem (1)

In PG(2,q), q is square and let $|B| = b$, B is blocking set. Then $q + \sqrt{q} + 1 \leq b \leq q^2 - \sqrt{q}$.

Proof

Suppose $|b| = q + \sqrt{q} + 1 - n$, $n > 0$. By theorem (1.14) no line in PG(2,q) contains more than $\sqrt{q} + 1 - n$ points of B.

Let S be any set of n points such that $S \cap B = \phi$ and $S' = S \cup B$ is not subgeomety of PG(2,q), then S′ is a blocking set since no line contains more than $(\sqrt{q} + 1 - n) + n = \sqrt{q} + 1$ of its point. So by theorem (1.16) S′ is a subgeomety, contradicting the choice of S.

If $|B| > q^2 - \sqrt{q}$, then $|\pi \backslash B| < q + \sqrt{q} + 1$, where $\pi = PG(2,q)$.

Theorem

Let B be a blocking set in PG(2,q), then B is a minimal blocking set if and only if B* is complete blocking set. ($B^* = PG(2,q)\backslash B$)

Proof:

Suppose that B is minimal blocking set, then B* is blocking set. (by definition 1.2).
Now, we must prove B* is complete suppose that B* is not complete blocking set. Then there exists $p \notin B^*$ such that $B^* \cup \{p\}$ is blocking set.
Hence $PG(2,q) \setminus (B^* \cup \{P\})$ is blocking set (by definition 1.2), but $PG(2,q) \setminus (B^* \cup \{P\}) = B \setminus \{p\}$ that contradiction, since B is minimal, then B* is complete blocking set.
Conversely, suppose that B* is a complete blocking set, to prove that B is a minimal blocking set. Suppose that B is not minimal, then there exists $P \in B$ such that $B \setminus \{p\}$ is blocking set, then $PG(2,q) \setminus (B \setminus \{p\})$ is a blocking set (by definition 1.2) but $PG(2,q) \setminus (B \setminus \{p\}) = B^* \cup \{p\}$, and that is a contradiction, since B* is a complete blocking set.
Then B is a minimal blocking set.

Section Two

Blocking Sets and (k,n)-arcs (4)

A (k,n) –arc in PG(2,q) is a set S of k points with property that every line contains at most n points of S, most authers add the condition, that there should be some line meeting S in exactly n points, there is an obvious relation between (k,n)-arcs and blocking sets, the complement of (k,n)-arc is a (q – n)-fold blocking set of size q.

Theorem (5)

In PG(2,q). A (k,n)-arc K is complete if and only if $B=PG(2,q) \setminus K$ is $(q + 1 - n)$-fold minimal blocking set.

Proof:

suppose that K is a complete (k,n)-arc, it is clear that B is (q + 1 – n)-fold blocking set and $|B| = q^2 + q + 1 - k$, to prove that B is minimal blocking set, suppose B is not minimal, then there exists $p \in B$ such that $B \setminus \{p\}$ is blocking set, but B is (q + 1 – n)-fold blocking set and $|B| = q^2 + q + 1 - k$, then $K \cup \{p\}$ is a (k+1,n)-arc, and $k \subseteq$ (k+1,n)-arc (contradiction), since K is complete, then B is minimal.
Conversely suppose that B is (q + 1 – n)-fold minimal blocking set and $|B| = q^2 + q + 1 - k$, it is clear that K is (k,n)-arc, to prove K is complete suppose that K is not complete arc, then there exists $p \notin K$ such that $K \cup \{p\}$ is a (k+1,n)-arc but $PG(2q) \setminus K \cup \{p\} = B \setminus \{p\}$ is blocking set and that is contradiction, since B is minimal blocking, then K is complete arc.

Open Problem (5)

In Simeon Ball (6), he gives an open problem which says that, there exists a t-fold blocking set of PG(2,q) of size less than (t + 1) p". And he found the answer is no if q = 3, 5 and 7. And we check this problem if q = 4, 5, 8.

1. In PG(2,4), there is no 2-fold blocking set of size less than 12 points, and the all 2-fold blocking set we found exactly 12 points, for example, B={1,2,4,5,6,7,9,10,12,13,17,19}, and the all 2-fold blocking set of size less than 12 points that we found it is trivial blocking set.
2. In PG(2,5), there is no 3-fold blocking set of size less than 20 points, and the only 3-fold blocking set that we found tis exactly size is 20 points, for example B={1,2,3,5,6,7,8,9,11,12,14,16,20,24,27,28,29,30,31}, in PG(2,5), the all 3-fold blocking set of size is less than 20 points that we found which is trivial blocking set.
3. In PG(2,8), there is no 3-fold blocking set of size less than 33 points, for example B={1,2,3,5,7,10,11,12,13,15,16,17,19,20,27,29,33,34,35,37,45,46,47, 50,56,58,60,61,67,69,70,72,73}, and the all 3-fold blocking set of size 33 points that we found which is trivial blocking set.

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Table (1)The Points and Lines of PG(2,3)

i	P_i	L_i			
1	(1,0,0)	1	2	4	10
2	(0,1,0)	2	3	5	11
3	(1,1,0)	3	4	6	12
4	(2,1,0)	4	5	7	13
5	(0,0,1)	5	6	8	1
6	(1,0 ,1)	6	7	9	2
7	(2,0,1)	7	8	10	3
8	(0,1,1)	8	9	11	4
9	(1,1,1)	9	10	12	5
10	(2,1,1)	10	11	13	6
11	(0,2,1)	11	12	1	7
12	(1,2,1)	12	13	2	8
13	(2,2,1)	13	1	3	9

Table (2)The Points and Lines of PG(2,5)

i	P_i	L_i
1	1 0 0	2 7 12 17 22 27
2	0 1 0	1 7 8 9 10 11
3	1 1 0	6 7 16 20 24 28
4	2 1 0	4 7 14 21 23 30
5	3 1 0	5 7 15 18 26 29
6	4 1 0	3 7 13 19 25 31
7	0 0 1	1 2 3 4 5 6
8	1 0 1	2 11 16 21 26 31
9	2 0 1	2 9 14 19 24 29
10	3 0 1	2 10 15 20 25 30
11	4 0 1	2 8 13 18 23 28
12	0 1 1	1 27 28 29 30 31
13	1 1 1	6 11 15 19 23 27
14	2 1 1	4 9 16 18 25 27
15	3 1 1	5 10 13 21 24 27
16	4 1 1	3 8 14 20 26 27
17	0 2 1	1 17 18 19 20 21
18	1 2 1	5 11 14 17 25 28
19	2 2 1	6 9 13 17 26 30
20	3 2 1	3 10 16 17 23 29
21	4 2 1	4 8 15 17 24 31
22	0 3 1	1 22 23 24 25 26
23	1 3 1	4 11 13 20 22 29
24	2 3 1	3 9 15 21 22 28
25	3 3 1	6 10 14 18 22 31
26	4 3 1	5 8 16 19 22 30
27	0 4 1	1 12 13 14 15 16
28	1 4 1	3 11 12 18 24 30
29	2 4 1	5 9 12 20 23 31
30	3 4 1	4 10 12 19 26 28
31	4 4 1	6 8 12 21 25 29