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الخلاصة

لتكن R حلقة ابدالية ذا محايد وليكن M مقاساً أيسر على R. نُعرف ان مقاساً جزئياً فعلياً N في M يكون أولياً ضعيفاً اذا كان لكل $r \in$ R ، و $x \in$ M ، و $0 \neq r\,x \in$ N يؤدي الى $x \in$ N أو $r \in$ (N:M) . في الحقيقة ان هذا المفهوم هو تعميم لمفهوم مثالي أولي ضعيف، اذ ان مثالياً فعلياً P في R، يسمى أولياً ضعيفاً اذا كان لكل $a, b \in$ R ، و $0 \neq a\,b \in$ P يؤدي الى ان $b \in$ P أو $a \in$ P.

خواص مختلفة عن المقاسات الجزئية الأولية الضعيفة قد أعطيت.

On Weakly Prime Submodules

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Abstract

Let R be a commutative ring with unity and let M be a left R-module. We define a proper submodule N of M to be a weakly prime if whenever $r \in$ R, $x \in$ M, $0 \neq rx \in$ N implies $x \in$ N or $r \in$ (N:M). In fact this concept is a generalization of the concept weakly prime ideal, where a proper ideal P of R is called a weakly prime, if for all a, $b \in$ R, $0 \neq ab \in$ P implies $a \in$ P or $b \in$ P. Various properties of weakly prime submodules are considered.

1.Introduction

Throughout this paper, R be a commutative ring with identity and M be a unity R-module. A proper submodule N of M is said to be Prime if whenever $r \in$ R , $x \in$ M , $rx \in$ N implies either $x \in$ N or $r \in$ (N:M), where (N:M) = $\{r \in$ R: rM $\subseteq$ N$\}$, see (1).

Semiprime submodules was given by Dauns in (2), as a generalization of prime submodules, where a proper submodule N of M is semiprime if $r^k x \in$ N, for $r \in$ R, $x \in$ M, $k \in Z_+$ (set of positive integers) implies $rx \in$ M. Also Eman A.A. in (3) studied these notions.

In 1999, quasi-prime submodules was introduced and studied by Muntaha (see (4)), as another generalization of prime submodules, where a proper submodule N of M is quasi-prime if $r_1 r_2 m \in$ N, for r_1, $r_2 \in$ R, $m \in$ M implies $r_1 m \in$ N or $r_2 m \in$ N; equivalently, N is quasi-prime if the ideal (N:M) is prime for all $m \in$ M.

In 2004, M.Behoodi and H.Koohi in (5) gave the notion of weakly prime submodules, where a proper submodule N of M weakly prime if (N:K) is a prime ideal, for all submodules K of M. Also this notion was studied by A.Azizi in (6), 2006.

By Th.2.14 in (4), we obtain that the two concepts weakly prime submodules and quasi-prime submodules are equivalent.

In this paper, we give another generalization of prime submodules namely weakly prime submodules, however this concept is different from the concept of quasi-prime submodule (see Remarks 2.1.(5)).

In fact, D.D.Anderson and E.Smith in (7) gave the following: A proper ideal I of R is said to be a weakly prime if $0 \neq ab \in$ I, for a, $b \in$ R, then $a \in$ I or $b \in$ I. We define a proper submodule N of M is weakly prime if whenever $r \in$ R , $x \in$ M , $0 \neq rx \in$ N implies $x \in$ N or $r \in$ (N:M). Moreover S.E.Atani and F.Farzalipour in (8) introduced the notion of weakly primary submodules, where a proper submodule N of M is a weakly primary if whenever $r \in$ R , $x \in$ M , $0 \neq rx \in$ N implies $x \in$ N or $r^n \in$ (N:M) for some $n \in Z_+$. Also they gave that : a proper ideal of R is a weakly primary if it is a weakly prime submodule of the R-module R, (see (8)).

In this paper we study weakly prime submodules and give many basic properties related to this concept.

2.Basic Properties

As we mentioned in the introduction, we introduce the following:

Definition 2.0 : A proper submodule N of an R-module M is weakly prime if whenever $r \in R$, $x \in M$, $0 \neq rx \in N$ implies $x \in N$ or $r \in (N:M)$.

In this section, we will give basic properties of weakly prime submodules. Some of these are extension of the results about weakly prime ideals, which are given in (7).

Let us start with the following:

Remarks 2.1:

(1) It is clear that every prime submodule is weakly prime. However, since (0) the zero submodule of any module) is always weakly prime (by definition), a weakly prime submodule may not be prime; for example: the zero submodule of the Z-module Z_4 is weakly prime, but it is not prime.

Moreover it is easy to check that in the class of torsion free modules, the concepts of prime submodule and weakly prime submodule are equivalent.

(2) Every weakly prime ideal P of a ring R is a weakly prime submodule of the R-module R.

(3) Every weakly prime submodule is weakly primary, but the converse is false as the following example shows.

The submodule $N = (\bar{4})$ of the Z-module Z_8 is weakly primary but it is not weakly prime.

(4) It is easy to check that :if P is a weakly primary submodule of an R-module M and (P:M) is a semiprime ideal, then P is weakly prime.

(5) (*a*) Weakly prime submodule need not be quasi-prime as the following example shows: The zero submodule of the Z-module Z_{12} is weakly prime, but it is not quasi-prime since $(\bar{0}:\bar{3}) = 4Z$ which is not a prime ideal of Z.

(*b*) Quasi-prime submodule need not be weakly prime submodule, as the following example shows

If M is the Z-module $Z \oplus Z$, and $N = 2Z \oplus (0)$, then N is a quasi-prime submodule of M (see (4), Rem.2.1.2(1)). But N is not weakly prime submodule, since $(0,0) \neq 2\,(3,0) \in N$, $(3,0) \notin N$ and $2 \notin (N:M) = (0)$.

(6) If P is a proper submodule of an R-module M. Then P is a weakly prime R-submodule of M iff P is a weakly prime R / I-submodule of M, where I is an ideal of R with $I \subseteq$ *ann* M.

The following result gives characterizations of weakly prime submodules.

Theorem 2.2 : Let M be an R-module. The following asserations are equivalent:

1. P is a weakly prime submodule of M.
2. $(P:x) = (P:M) \cup (0:x)$ for any $x \in M$, $x \notin P$.
3. $(P:x) = (P:M)$ or $(P:x) = (0:x)$ for any $x \in M$, $x \notin P$.
4. If $(0) \neq (a)N \subseteq P$, then either $N \subseteq P$ or $(a) \subseteq (P:M)$, where $a \in R$, N is a submodule of M.

Proof. (1) $\Rightarrow$ (2) Let $r \in (P:x)$ and $x \notin P$. Then $rx \in P$. Suppose $rx \neq 0$. Hence $r \in (P:M)$ because P is weakly prime and $x \notin P$. If $rx = 0$, then $r \in (0:x)$. Thus $(P:x) \subseteq (P:M) \cup (0:x)$. Now if $r \in (P:M) \cup (0:x)$, then either $r \in (P:M)$ or $r \in (0:x)$. Hence, when $r \in (0:x)$, $rx = 0 \in P$ and so $r \in (P:x)$. If $r \in (P:M)$ then $rM \subseteq P$, and this implies $rx \in P$. Hence $r \in (P:x)$. Thus $(P:M) \cup (0:x) \subseteq (P:x)$ and therefore $(P:M) \cup (0:x) = (P:x)$.

(2) $\Rightarrow$ (3) It is well-known that the union of two ideals A, B of R is an ideal if $A \subseteq B$ or $B \subseteq A$. By condition, the ideals (P:M) is the union of the ideals (P:M), $(0:x)$, so either $(P:M) \subseteq (0:x)$ or $(0:x) \subseteq (P:M)$. Thus either $(P:x) = (0:x)$ or $(P:x) = (P:M)$.

(3) $\Rightarrow$ (4) If $0 \neq (a)\, N \subseteq P$. Suppose that $N \not\subseteq P$ and $(a) \not\subseteq (P:M)$. $N \not\subseteq P$ implies that there exists $x \in N$ and $x \notin P$, hence $ax \in aN \subseteq P$; that is $a \in (P:x)$. By condition (3), either $(P:x) = (P:M)$ or $(P:x) = (0:x)$. If $(P:x) = (P:M)$, we get $a \in (P:M)$ which is a contradiction. Thus $(P:x) = (0:x)$ and so $ax = 0$.
On the other hand, $0 \neq (a)\, N \subseteq P$ implies that there exists $y \in N$ such that $0 \neq ay \in P$ and so $a \in (P:y)$. Moreover we can see that $y \in P$, for if we assume that $y \notin P$, then by condition 3, either $(P:y) = (P:M)$ or $(P:y) = (0:y)$. If $(P:y) = (P:M)$, then $a \in (P:M)$ which is a contradiction. If $(P:y) = (0,y)$, we get $ay = 0$ which is a contradiction. Moreover $0 \neq ay = ay + ax = a(y + x) \in P$; that is $a \in (P:y + x)$. But $y + x \notin P$ because $x \notin P$, $y \in P$, hence by condition 3 , either $(P: y + x) = (P:M)$ or $(P: y + x) = (0: y + x)$. If $(P: y + x) = (P:M)$ then $a \in (P:M)$ which is a contradiction. If $(P: y + x) = (0: y + x)$, then $a(y + x) = 0$ and hence $ay + ax = ay + 0 = 0$ which is a contradiction. Therefore our assumption is false and so either $N \subseteq P$ or $(a) \subseteq (P:M)$.
(4) $\Rightarrow$ (1) Let $r \in R$, $x \in M$, such that $0 \neq r x \in P$. Then $0 \neq (r)\,(x) \subseteq P$. By condition (4), $(x) \subseteq P$ or $(r) \subseteq (P:M)$ and hence either $x \in P$ or $r \in (P:M)$; that is, P is weakly prime.

Remark 2.3

It is known that if P is a prime submodule of an R-module M, then (P:M) is a prime ideal of R. However the "weak" analogs of this statement is not true in general, for example:
The zero submodule of the Z-module Z_4, is weakly prime, but $(0 \underset{Z}{:} Z_4) = 4\,Z$ is not a weakly prime ideal of Z.

We give the following:

Proposition 2.4: If P is a weakly prime submodule of a faithful R-module M, then $(P \underset{R}{:} M)$ is a weakly prime ideal of R.

Proof. Let $a, b \in R$. If $0 \neq a\, b \in (P:M)$; then $a\, b\, M \subseteq P$. Since M is faithful, $a\, b\, M \neq (0)$, hence $0 \neq (a)\,(b\,M) \subseteq P$ and so by Th.2.2 either $(a) \subseteq (P:M)$ or $b\,M \subseteq P$; that is, either $a \in (P:M)$ or $b \in (P:M)$. Thus (P:M) is a weakly prime ideal of R.

The converse of prop.2.4 is not true as the following example shows:

Let M be the Z-module $Z \oplus Z$, let $P = (0) \oplus 2Z$. Then $(P \underset{Z}{:} M) = (0)$ which is a weakly prime ideal in Z, however P is not a weakly prime submodule of M because $(0,0) \neq 2(0,1) \in P$, but $(0,1) \notin P$ and $2 \notin (P \underset{Z}{:} M) = (0)$.

Also, we have the following:-

Proposition 2.5 : Let P be a weakly prime submodule of an R-module M. Then $(P \underset{\bar{R}}{:} M)$ is a weakly prime ideal of $\bar{R}$, where $\bar{R} = R/ann\mathrm{M}$.

Proof. By remark 2.1 (6), P is a weakly prime $\bar{R}$-submodule of M. But M is a faithful $\bar{R}$-module, so by prop.2.6, $(P \underset{\bar{R}}{:} M)$ is a weakly prime ideal of $\bar{R}$.

Recall that an R-module M is called multiplication module if for each submodule N of M, N = I M for some ideal I of R, equivalently N = (N:M)M (see (9)).

In the class of finitely generated faithful multiplication modules, we have the following:

Theorem 2.6: Let M be a faithful finitely generated multiplication R-module, let N be a proper submodule of M. Then the following statements are equivalent

1. N is a weakly prime submodule of M.
2. $(N \underset{R}{:} M)$ is a weakly prime ideal of R.
3. N = I M for some weakly prime ideal I of R.

Proof. (1) $\Rightarrow$ (2) It holds by prop. 2.4

$(2) \Rightarrow (1)$ Let $r \in R$, $x \in M$, such that $0 \neq r\,x \in N$. (x) is a submodule of M, hence $(x) = J\,M$ for some ideal J of R. Thus $0 \neq r\,J\,M \subseteq N = (N{:}M)\,M$. But M is a faithful finitely generated multiplication R-module, so by (10, Th.3.1) $r\,J \subseteq (N \underset{R}{:} M)$. Moreover $r\,J \neq (0)$ and since $(N \underset{R}{:} M)$ is weakly prime ideal, either $r \in (N \underset{R}{:} M)$ or $J \subseteq (N \underset{R}{:} M)$ (see Th.3 in (7)). Hence either $r \in (N \underset{R}{:} M)$ or $(x) = J\,M \subseteq (N \underset{R}{:} M)\,M = N$, that is $r \in (N \underset{R}{:} M)$ or $x \in N$. Thus N is weakly prime.

$(2) \Rightarrow (3)$ Since $(N \underset{R}{:} M)$ is weakly prime and $N = (N \underset{R}{:} M)\,M$, so condition (3) hold.

$(3) \Rightarrow (2)$ By (3), $N = I\,M$ for some weakly prime ideal I of R. But M is a multiplication module, so $N = (N \underset{R}{:} M)\,M$. Hence $I\,M = (N{:}M)\,M$ and so by (10, Th.3.1) $I = (N \underset{R}{:} M)$.

Proposition 2.7 : Let P be a weakly prime submodule of an R-module M. If P is not prime, then $(P{:}M)\,P = (0)$.

Proof. Suppose $(P{:}M)\,P \neq 0$. We will show that P is prime. Let $r\,x \in P$. If $r\,x \neq 0$, then either $x \in P$ or $r \in (P{:}M)$, since P is weakly prime. Now assume $r\,x = 0$. First suppose $rP \neq (0)$, so there exists $y \in P$ such that $0 \neq r\,y \in P$. Hence $0 \neq r\,y = r\,(x + y) \in P$ which implies that either $x + y \in P$ or $r \in (P{:}M)$. Hence either $x \in P$ or $r \in (P{:}M)$. Now we can assume that $r\,P = 0$ and $(P{:}M)\,x = 0$. Since $(P{:}M)\,P \neq (0)$, there exists $s \in (P{:}M)$, $y \in P$ such that $0 \neq s\,y \in P$. Thus

$$\begin{aligned}(r+s)\,(x+y) &= r\,x + s\,x + r\,y + s\,y\\ &= 0 + 0 + 0 + s\,y\\ &= s\,y\end{aligned}$$

That is $0 \neq (r+s)\,(x+y) \in P$. Then P is weakly prime gives $x + y \in P$ or $r + s \in (P{:}M)$. Hence $x \in P$ or $r \in (P{:}M)$.

Now we have the following:

Proposition 2.8: Let M and M' be R-modules and let $f\colon M \to M'$ be an R-epimorphism. If N is a weakly prime submodule of M such that $\ker f \subseteq N$, then $f\,(N)$ is a weakly prime submodule of M'.

Proof. Let $r \in R$, $y \in M'$, such that $0 \neq r\,y \in f(N)$. Then there exists $x \in N$ such that $0 \neq r\,y = f(x)$, and since f is an epimorphism, $y = f(x_1)$ for some $x_1 \in M$. Thus $f(r\,x_1 - x) = 0$ and so $r\,x_1 - x \in \ker f \subseteq N$. It follows that $0 \neq r\,x_1 \in N$ and since N is weakly prime either $x_1 \in N$ or $r \in (N \underset{R}{:} M)$. Thus $y = f(x_1) \in f(N)$ or $r \in (f(N){:}\,M')$; that is $f(N)$ is weakly prime.

As a particular case of prop.(2.8), we have the following: if N, W are submodules of an R-module M such that $N \supseteq W$ and N is weakly prime, then N / W is a weakly prime R-submodule of M / N.

The following result discussos the localization of weakly prime submodules.

Proposition 2.9 : Let P be a weakly prime R-submodule and S be a multiplicative subset of R with $(P \underset{R}{:} M) \cap S = \varnothing$. Then P_S is weakly prime R_S-submodule of M_S.

Proof. Let $\frac{a}{b} \in R_S$ and $\frac{x}{c} \in M_S$ such that $0_s \neq \frac{a}{b}\frac{x}{c} \in P_S$. Hence $0_s \neq \frac{a\,x}{b\,c} \in P_S$ and so there exists $y \in P$ and $d \in S$ such that $\frac{a\,x}{b\,c} = \frac{y}{d}$, and this implies that there exists $t \in S$ such that $t\,a\,d\,x = t\,b\,c\,y$. On the other hand $\frac{a\,x}{b\,c} \neq \frac{0}{1} = (0_s)$ which implies that $f\,a\,x \neq 0$ for all $f \in S$. Hence $0 \neq t\,a\,d\,x \in P$. But P is a weakly prime R-submodule of M, so either $t\,d\,x \in P$ or $a \in$

(P:M) and hence either $\frac{tdx}{tdc} \in P_S$ or $\frac{a}{b} \in (P{:}M)_S$. Because $(P \underset{R}{:} M)_S \subseteq (P_S \underset{R_S}{:} M_S)$, we have either $\frac{x}{c} \in P_S$ or $\frac{a}{b} \in (P_S \underset{R_S}{:} M_S)$.

As a generalization of Cohen theorem, the following was given in ((3),Prop.4.15,ch.1).

Let M be a finitely generated R-module, then M is Noetherian iff every prime submodule is finitely generated.

Since every prime submodule is weakly prime (by Rem.2.1.(11)), we have the following:

Proposition2.10 : Let M be a finitely generated R-module. Then M is Noetherian if every weakly prime submodule is finitely generated.

Noteice that the condition M is finitely generated that cann't be dropped from Prop.2.10, as the following example shows:

The Z-module Z_{p^∞} is not finitely generated, also it is not Noetherian. However if G is a nonzero submodule, then $G = < \frac{1}{p^i} + z >$ for some $i \in Z_+$, and $0 \neq P(\frac{1}{p^{i+1}} + z) \in G$. But $P \notin (G{:}\ Z_{p^\infty}) = 0$ and $\frac{1}{p^{i+1}} + z \notin G$; that is G is not weakly prime. Thus (0) is the only weakly prime submodule of Z_{p^∞} and it is obviously finitely generated.

Theorem 2.11: Let M_1, M_2 be R-modules and let N be a proper submodule of M_1. Then $W = N \oplus M_2$ is a weakly prime submodule of $M = M_1 \oplus M_2 \Leftrightarrow$ N is a weakly prime submodule of M_1 and for $r \in R, x \in M_1$ with $rx = 0$, but $x \notin N$, $r \notin (N{:}M_1)$ implies $r \in ann\, M_2$.

Proof. ($\Rightarrow$) Let $r \in R$, $x \in M_1$ such that $0 \neq rx \in N$. Then $(0,0) \neq r(x,0) \in W$, but W is weakly prime, so either $(x,0) \in W$ or $r \in (W \underset{R}{:} M)$. Thus either $x \in N$ or $r \in (N \underset{R}{:} M_1)$, so that N is weakly prime. Now, if $r \in R$, $x \in M_1$ such that $rx = 0$, $x \notin N$ and $r \notin (N{:}M_1)$. Assume that $r \notin ann\, M_2$, so there exists $m \in M_2$ such that $r m_2 \neq 0$. Thus $r(x,m_2) = (rx, r m_2) = (0, r m_2) \neq (0,0)$ and hence $(0,0) \neq r(x,m_2) \in N \oplus M_2 = W$. Since W is weakly prime, so either $(x,m_2) \in N \oplus M_2$ or $r \in (N \oplus M_2 \underset{R}{:} M_1 \oplus M_2)$. Thus either $x \in N$ or $r \in (N \underset{R}{:} M_1)$ which is a contradiction with hypothesis.

($\Leftarrow$) Let $r \in R$, $(x,y) \in M$. Assume $(0,0) \neq r(x,y) \in N \oplus M_2$, so if $rx \neq 0$, then either $x \in N$ or $r \in (N{:}M_1)$, since N is weakly prime. Thus either $(x,y) \in N \oplus M_2$ or $r \in (N \oplus M_2{:}M_1 \oplus M_2)$. If $rx = 0$. Suppose $x \notin N$ and $r \notin (N_1{:}M_1)$, then by hypothesis $r \in ann\, M_2$ and so $r(x,y) = (0,0)$ which is a contradiction. Thus either $x \in N$ or $r \in (N_1 \underset{R}{:} M_1)$ and hence either $(x,y) \in N \oplus M_2$ or $r \in (N_1 \oplus M_2 \underset{R}{:} M_1 \oplus M_2)$.

It is known that if Q is a primary submodule then $P = \sqrt{(Q{:}M)}$ is a prime ideal, see (11, prop. 2.11, p.41). Sometimes Q is called P-primary, see(11, p.42).

Now we have the following result:

Corollary 2.12: Let Q_α be P-primary submodules of an R-module M_1 with $\cap Q_\alpha = (0)$. If N is a weakly prime submodule of M_1 and M_2 is an R-module such that $P \subseteq ann_R M_2$, then $N \oplus M_2$ is a weakly prime submodule in $M_1 \oplus M_2$.

Proof. Let $r \in R$, $x \in M_1$ with $rx = 0$. If $x \notin N_1$ (so $x \neq 0$) and $r \notin (N{:}M_1)$. We will prove that $r \in ann\, M_2$ and hence the result is obtained by previous theorem. Suppose that $r \notin ann$ M_2. Hence $r \notin P$.

On the other hand, $rx = 0 = \cap Q_\alpha$, but $\cap Q_\alpha$ is a P-primary submodule by (11,prop.1.1, p.15), so either $x \in \cap Q_\alpha = 0$ or $r \in P$, which is a contradiction. Thus $r \in ann\, M_2$.

Remark 2.13: Let M_1, M_2 be R-modules. If (0) is a prime submodule of M_1, then $(0) \oplus M_2$ is a weakly prime submodule of $M = M_1 \oplus M_2$.

Proof. Let $r \in R$, $(x,y) \in M$. If $(0,0) \neq r\,(x,y) \in (0) \oplus M_2$, then $r\,x = 0$ and $r\,y \in M_2$. Since (0) is prime in M_1, either $x = 0$ or $r \in (0{:}M_1)$. Hence either $(x,y) = (0,y) \in (0) \oplus M_2$ or $r \in ((0) + M_2{:}M_1 \oplus M_2)$; that is $(0) \oplus M_2$ is weakly prime in M.

Thus we can give the following example:

$N = (0) \oplus Z_4$ is a weakly prime submodule of the Z-module $Z \oplus Z_4$.

Next we have the following:

Proposition 2.14: Let M_1, M_2 be R-modules. If $N = U \oplus W$ be a weakly prime submodule in $M = M_1 \oplus M_2$, then U, W are weakly prime submodules in M_1, M_2 respectively.

Proof. The proof is a straight forword, so it is omitted.

Remark 2.15: The converse of proposition 2.14 is not true in general as the following example shows.

Example: (0) is a weakly prime submodule of the Z-module Z, (2Z) is a prime submodule of the Z-module Z so it is weakly prime. But $N = (0) \oplus 2Z$ is not weakly prime in the Z-module $Z \oplus Z$.

For the next results we will assume that $R = R_1 \times R_2$ where each R_i is a commutative ring with identity, M_i be an R_i-module, where $i = 1,2$. and $M = M_1 \times M_2$ be the R-module with action $(r_1,r_2)\,(m_1,m_2) = (r_1\,m_1, r_2\,m_2)$ where $r_i \in R_i$, $m_i \in M_i$, $i = 1,2$.

Proposition 2.16 : If P is a proper R_1-submodule of M_1, then the following statements are equivalent

1. P is a prime R_1-submodule of M_1.
2. $P \times M_2$ is a prime R-submodule of $M = M_1 \times M_2$.
3. $P \times M_2$ is a weakly prime R-submodule of $M = M_1 \times M_2$.

Proof. (1) $\Rightarrow$ (2) Let $(r_1,r_2) \in R$, $(x,y) \in M$ such that $(r_1,r_2)\,(x,y) \in P \times M_2$. Then $r_1\,x \in P$ and since P is prime, either $x \in P$ or $r_1 \in (P \underset{R_1}{:} M_1)$. If $x \in P$, then $(x,y) \in P \times M_2$. If $r_1 \in (P \underset{R_1}{:} M_1)$, then $(r_1,r_2) \in (P \times M_2 \underset{R}{:} M)$. Thus $P \times M_2$ is a prime R-submodule of M.

(2) $\Rightarrow$ (3) It holds by remark 2.1 (1).

(3) $\Rightarrow$ (1) Let $r \in R_1$, $x \in M_1$ such that $r\,x \in P$. Then for each $y \in M_2$, $y \neq 0$, $(0,0) \neq (r,1)\,(x,y) \in P \times M_2$. But $P \times M_2$ is a weakly prime R-submodule of M, so either $(x,y) \in P \times M_2$ or $(r,1) \in (P \times M_2 \underset{R}{:} M)$. Thus either $x \in P$ or $r \in (P \underset{R_1}{:} M_1)$; that is P is a prime R_1-submodule of M_1.

Similarly we have

Proposition 2.17: If P is a proper R_2-submodule of M_2, then the following statements are equivalent

1. P is a prime R_2-submodule of M_2.
2. $M_1 \times P$ is a prime R-submodule of $M = M_1 \times M_2$.
3. $M_1 \times P$ is a weakly prime R-submodule of $M = M_1 \times M_2$.

Proposition 2.18: Let M_1, M_2 be R_1, R_2-modules respectively. If $P = P_1 \times P_2$ is a weakly prime R-submodule of $M = M_1 \times M_2$, then either $P = 0$ or P is a prime submodule of M.

Proof. Assume $P \neq 0$, so either $P_1 \neq 0$ or $P_2 \neq 0$. Suppose that $P_2 \neq 0$, hence there exists $y \in P_2$, $y \neq 0$. Let $r \in (P_1 \underset{R_1}{:} M_1)$ and let $x \in M_1$, then $(0,0) \neq (r,1)\,(x,y) = (r\,x,y) \in P_1 \times P_2 = P$. Since P is weakly prime in M, either $(x,y) \in P$ or $(r,1) \in (P_1 \times P_2{:}\ M_1 \times M_2)$. Hence if $(x,y) \in P$, then $x \in P_1$ and so $M_1 = P_1$ which implies $P = M_1 \times P_2$. If $(r,1) \in (P_1 \times P_2{:}\ M_1 \times M_2)$, then $M_2 = P_2$ which implies $P = P_1 \times M_2$. Hence by propositions 2.16, 2.17, P is a prime R-submodule of M.

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