Fuzzy Semimaximal ideals

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Abstract

Let R be a commutative ring with identity. A proper ideal I of R is called semimaximal if I is a finite intersection of maximal ideals of R. In this paper we fuzzify this concept to fuzzy ideals of R, where a fuzzy ideal A of R is called semimaximal if A is a finite intersection of fuzzy maximal ideals. Various basic properties are given. Moreover some examples are given to illustrate this concept.

Introduction

Let R be a commutative ring with unity. It is well-known that a proper ideal M of a ring R is called maximal if for every ideal B of R, $M \subset B \subseteq R$ implies B=R.

D.S.Malik and J.N.Mordeson in (1) introduced and studied the concept of fuzzy maximal ideal of R, where a fuzzy ideal A of R is maximal if

1. A is not constant,
2. for any fuzzy ideal B of R if $A \subseteq B$, then either $A_* = B_*$ or $B = \lambda_R$

In fact D.S.Malik and J.N.Mordeson in (1) explained that a fuzzy maximal ideal A on R can not be defined as a fuzzy ideal $A \neq \lambda_R$ such that for each fuzzy ideal B of R, if $A \subset B \subseteq \lambda_R$ implies $B = \lambda_R$.

Goodreal in (2) introduced the concept of semimaximal ideals, where an ideal I of R is called semimaximal if it is a finite intersection of maximal ideals.

Also, this concept was studied by Hatem in (3).

In this paper, we fuzzify this concept to fuzzy ideals of R, where a fuzzy ideal A of R is called semimaximal if A is a finite intersection of fuzzy maximal ideals of R.

Moreover, we generalize many properties of maximal and semimaximal ideals in to fuzzy semimaximal ideals of a ring.

This paper consists of four sections. In S.1, we recall many definitions and properties which are needed in our work. In S.2, Various basic properties about fuzzy semimaximal ideals are discussed. In S.3, the image and inverse image of fuzzy semimaximal ideals are studied. In S.4, we study the behavior of fuzzy semimaximal ideals in a ring R, where $R = R_1 \oplus R_2$ (direct sum of two rings R_1, R_2).

S.1 Preliminaries

This section contains some definitions and properties of fuzzy subset, fuzzy ideals and fuzzy rings, which we used in the next section. First we give some basic definitions and properties of fuzzy subsets.

Let R be a commutative ring with unity, A fuzzy subset of R is a function from R into [0,1], (4). A fuzzy subset A is called a fuzzy constant if A(x) = t, $\forall\, x \in R$, $t \in [0,1]$, (4). For each $t \in [0,1]$, the set $A_t = \{x \in R, A(x) \geq t\}$ is called a level subset of A, (4). A_* denoted the set $\{x \in R, A(x) = A(0)\}$, (1). If $x \in R$ and $t \in [0,1]$, we let x_t denote the fuzzy subset of A define by $x_t(y) = 0$ if $x \neq y$ and $x_t(y) = t$ if $x = y$, x_t is called a fuzzy singleton, (5).

If A and B are fuzzy subsets of R, then:

1. $A \subseteq B$ if $A(x) \leq B(x)$, for all $x \in R$, (6),
2. $A = B$ if $A(x) = B(x)$, for all $x \in R$, (6).

We define $A \cap B$ by, $(A \cap B)(x) = \min\{A(x), B(x)\}$, $\forall\ x \in R$, and let $\{A_\alpha : \alpha \in \wedge\}$ be a collection of fuzzy subsets of R. Define the fuzzy subset of R (intersection) by $\bigcap_{\alpha \in \wedge} A_\alpha(x) = \inf\{A_\alpha : \alpha \in \wedge\}$, for all $x \in R$, (6).

Let A and B be fuzzy subsets of R, then for all $t \in [0,1]$, $(A \cap B)_t = A_t \cap B_t$,(3).

Let f be a mapping from a set M into a set N. Let A be a fuzzy subset of M and B be a fuzzy subset of N. The image of A denoted by f (A) is the fuzzy subset of N defined by:

$$f(A) = \begin{cases} \sup\{A(z) \mid z \in f^{-1}(y) \neq \phi, \text{if } f^{-1}(y) \neq \phi, \text{for all } y \in N\}, \\ 0 \qquad \text{otherwise} \end{cases}$$

where $f^{-1}(y) = \{x \in M, f(x) = y\}$.

And the inverse image of B, denoted by $f^{-1}(B)$ is the fuzzy subset of M, denoted by $f^{-1}(B)(x) = B(f(x))$, for all $x \in M$, (4).

Let f be a function from a set M into a set N. A fuzzy subset A of M is called f – invariant if $A(x) = A(y)$, whenever $f(x) = f(y)$ where $x, y \in M$, (7).

If f a function from a set M into a set N, A_1 and A_2 are fuzzy subsets of M and B_1, B_2 are fuzzy subsets of N, then

1. $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$, whenever A_1, A_2 f –invariant, (8)
2. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$, (8).

Moreover the following definitions and properties are needed later

Definition 1.1: (9)

A fuzzy subset K of R is called a fuzzy ideal of R if for each $x, y \in R$, then:

1. $K(x - y) \geq \min\{K(x), K(y)\}$,
2. $K(x\, y) \geq \max\{K(x), K(y)\}$.

Definition 1.2: (10)

Let X be a fuzzy subset of a ring R, then X is called fuzzy ring of R if for each $x, y \in R$, then

1. $X \neq 0$,
2. $X(x - y) \geq \min\{X(x), X(y)\}$,
3. $X(x\, y) \geq \max\{X(x), X(y)\}$.

Proposition 1.3: (1)

Let $\{A_\alpha, \alpha \in \wedge\}$ be a family of fuzzy ideals of R, then $\bigcap_{\alpha \in \wedge} A_\alpha$ is a fuzzy ideal of R.

Definition 1.4: (11)

Let X be a fuzzy ring of a ring R, let A be a fuzzy subset of X such that $A \subseteq X$. Then A is called a fuzzy ideal of a fuzzy ring X if for each $x, y \in R$

1. $A(x - y) \geq \min\{A(x), A(y)\}$,
2. $A(x\, y) \geq \min\{\max\{A(x), A(y), X(x\, y)\}$.

Note 1.5:

It is clear that any fuzzy ideal of a ring R is a fuzzy ideal of a fuzzy ring X of R such that $X(a) = 1$, $\forall\ a \in R$.

Proposition 1.6:

Let f be a homomorphism from a ring R_1 into a ring R_2, then the following are true:

1. $f(A)$ is a fuzzy ideal of R_2, for each fuzzy ideal A of R_1, (6).
2. $f^{-1}(B)$ is a fuzzy ideal of R_1, for each fuzzy ideal B of R_2, (7)

Proposition 1.7: (1)

Let A, B be fuzzy ideals of a ring R such that A(0) = 1 = B(0). Then $(A\cap B)_* = A_* \cap B_*$.

Proposition 1.8: (1)

Let $\{A_\alpha : \alpha\in\wedge\}$ be a family of fuzzy ideals of R such that $A_\alpha(0) = 1$, for all $\alpha\in\wedge$. Then $\bigcap_{\alpha\in\wedge}(A_\alpha)_* = \left(\bigcap_{\alpha\in\wedge} A_\alpha\right)_*$.

S.2 Basic Properties of Fuzzy Semimaximal Ideals

First, we give the following lemma which summarized the basic properties of fuzzy maximal ideals.

Lemma 2.1: (1)

1. Let A be a fuzzy maximal ideal of R, then A(0) = 1 (see Th.3.3).

2. Let A be a fuzzy maximal ideal of R, then $|\text{Im}(A)| = 2$; That is A is a two valued, where Im(A) denotes image of A and $|\text{Im}(A)|$ denotes the cardinality of Im(A) (see Th.3.4).

3. If A is a fuzzy maximal ideal of R, then A_* is a maximal ideal of R (see Th.3.5).

4. If A is a fuzzy ideal of R and A_* is a maximal ideal of R, then A is two valued (see Th. 3.6).

5. If A is a fuzzy ideal of R and A_* is a maximal ideal of R such that A(0) = 1. Then A_* is a fuzzy maximal ideal of R (see Th. 3.7).

6. If $I \neq R$ be an ideal of R. Then I is a maximal ideal of R if and only if λ_I is a fuzzy maximal ideal of R (see Cor. 3.8).

Thus we introduce the following:

Definition 2.2:

Let A be a fuzzy ideal of R, A is called a fuzzy semimaximal ideal if A is a finite intersection of fuzzy maximal ideals of R.

Remarks 2.3:

1. It is clear that every fuzzy maximal ideal is fuzzy semimaximal ideal. However the converse is not true as the following example shows:

Let $A : Z \longrightarrow [0,1]$ defined by:

$$A(x) = \begin{cases} 1 & x \in 6Z, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

It is clear that A is fuzzy ideal of Z and A is not fuzzy maximal ideal since $A_* = 6Z$ is not maximal ideal (see Lemma 2.1(3)).

However $A = A_1 \cap A_2$, where A_1 and A_2 are fuzzy ideals defined by:

$A_1 : Z \longrightarrow [0,1]$ $\qquad$ $A_2 : Z \longrightarrow [0,1]$

$$A_1(x) = \begin{cases} 1 & x \in 2Z, \\ 0 & \text{otherwise} \end{cases} \qquad A_2(x) = \begin{cases} 1 & x \in 3Z, \\ 0 & \text{otherwise} \end{cases}$$

A_1 and A_2 are fuzzy maximal ideals of Z since $(A_1)_* = 2Z$ and $(A_1)_* = 3Z$ are maximal ideals (see Lemma 2.1(5)).

2. If A is a fuzzy semimaximal ideal of R, then A(0) = 1.

Proof. Since A is a fuzzy semimaximal ideal of R, $A = A_1 \cap A_2 \cap \ldots \cap A_n$, where A_i is a fuzzy maximal ideal of R, for all i = 1, 2, …, n. Since $A_i(0) = 1$ by lemma 2.1(1), then

$$A(0) = \bigcap_{i=1}^{n} A_i(0)$$

$= \min\{ A_i(0), i = 1, 2, \ldots, n\} = 1$

3. If A and B are fuzzy semimaximal ideals of R, then $A \cap B$ is a fuzzy semimaximal ideal of R.

Proof. Since A and B are fuzzy semimaximal ideal of R, then A = $\bigcap_{i=1}^{n} A_i$, B = $\bigcap_{i=1}^{m} B_i$, where A_i is a fuzzy maximal ideal, for all i = 1, 2, …, n and B_i is a fuzzy maximal ideal, for all i = 1, 2, …, m. Thus

$A \cap B = A_1 \cap A_2 \cap \ldots \cap A_n \cap B_1 \cap B_2 \cap \ldots \cap B_m$; That is $A \cap B$ is a finite intersection of fuzzy maximal ideals of R.

4. If { A_i, i = 1, 2, …, n} be a family of fuzzy semimaximal ideals of R, then $\bigcap_{i=1}^{n} A_i$ is a fuzzy semimaximal ideal of R.

Proof. It is easy, so it is omitted.

Compare the following result with lemma 2.1(3)

Proposition 2.4:

If A is a fuzzy semimaximal ideal of R, then A_* is semimaximal ideal of R.

Proof. Since A is a fuzzy semimaximal ideal, so A = $\bigcap_{i=1}^{n} A_i$, where A_i is a fuzzy maximal ideal for all i = 1, 2, …, n. Since $A_i(0) = 1$ (by Lemma2.1(1)), so that

$A_* = \left(\bigcap_{i=1}^{n} A_i\right)_* = \bigcap_{i=1}^{n} (A_i)_*$ (by prop.1.8)

But $(A_i)_*$ is maximal ideal, $\forall$ i = 1, 2, …, n by lemma 2.1(3).

Hence $A_* = \bigcap_{i=1}^{n} (A_i)_*$.

Thus A_* is a maximal ideal.

The converse of this proposition is not true in general. However an example which will explain this depend on theorem 2.10. So we shall give it later (see Remark 2.11). Before giving our next result, we need to recall the following:

Definition 2.5: (1)

Let A be a fuzzy ideal of R, then A is called fuzzy prime if either A = λ_R or

1. A is not constant and

2. For any fuzzy ideals B and C of R, if $B.C \subseteq A$, then either $B \subseteq A$ or $C \subseteq A$.

Definition 2.6: (1)

Let A be a fuzzy ideal of R. The fuzzy radical of A denoted by $\sqrt{A}$ defined by $\sqrt{A}$ = $\cap$ { P:P $\in$ £(A)}, where £(A) denotes the set of all fuzzy prime ideals of R which contains A.

Proposition 2.7:

If A is a fuzzy semimaximal ideal of R, then $\sqrt{A}$ = A

Proof. Since A is a fuzzy semimaximal ideal, then $A = A_1 \cap A_2 \cap \ldots \cap A_n$, where $A_1, A_2, \ldots, A_n$ are fuzzy maximal ideals of R. But for each i = 1, 2, …, n A_i is a fuzzy prime ideal, hence $\sqrt{A_i} = A_i$ by (Theorem 5.13,(1)). Thus $\sqrt{A} = \bigcap_{i=1}^{n} A_i$ and so $\sqrt{A}$ = A

Remark 2.8:

If A is a fuzzy semimaximal ideal, then it is not necessary that A is a fuzzy prime ideal. We can give the following example:

Example: Let A :Z $\longrightarrow$ [0,1] defined by

$$\mathrm{A}(x) = \begin{cases} 1 & x = 0, \\ 0 & \text{otherwise} \end{cases}$$

By (Theorem 2.4(12)). A is a fuzzy prime ideal of Z, but $A_* = (0)$ is not a semimaximal ideal in Z. Thus A is not fuzzy semimaximal ideal (by prop.2.4).

Compare the following with (Lemma 2.1(6)).

Proposition 2.9:

Let I be an ideal of R, then I is a semimaximal ideal of R if and only if λ_I is a fuzzy semimaximal ideal of R, where

$$\lambda_{\mathrm{I}}(x) = \begin{cases} 1 & x \in \mathrm{I}, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since I is a semimaximal ideal, then $\mathrm{I} = \bigcap_{i=1}^{n} \mathrm{I}_i$, I_i is a maximal ideal, $\forall$ i = 1, 2, …, n.

It is clear that $\lambda_{\mathrm{I}} = \lambda_{\mathrm{I}_1} \cap \lambda_{\mathrm{I}_2} \ldots \cap \lambda_{\mathrm{I}_n}$. But for each i = 1, 2, …, n, $\left(\lambda_{\mathrm{I}_i}\right)_* = \mathrm{I}_i$

so $\lambda_{\mathrm{I}_1}, \lambda_{\mathrm{I}_2} \ldots, \lambda_{\mathrm{I}_n}$ are fuzzy maximal ideals by (Lemma 2.1(6)).Thus $\lambda_{\mathrm{I}} = \bigcap_{i=1}^{n} \lambda_{\mathrm{I}_i}$ is a fuzzy semimaximal ideal.

Conversely; If λ_I is a fuzzy semimaximal ideal of R, then by (Lemma 2.1(3)), $(\lambda_I)_*$ is semimaximal, and since $(\lambda_I)_* = I$. So the result is obtained.

Compare the following with (Lemma.2.1(2)).

Theorem 2.10:

Let A be a fuzzy semimaximal ideal, then $|\mathrm{Im}\ A| = 2$.

Proof. $1 \in \mathrm{Im}\ A$ since A(0) = 1 (by Rem.2.3(2)).

We claim that for any $0 \le t < 1$, $A_t = R$. Since A is a fuzzy semimaximal ideal, then $A = A_1 \cap A_2 \cap \ldots \cap A_n$, where $A_1, A_2, \ldots, A_n$ are fuzzy maximal ideals of R. Since $0 \le t < 1$, then by the same proof of theorem 3.4 (1), we have

$(A_1)_t = (A_2)_t = \ldots = (A_n)_t = R$

But $A_t = (A_1)_t \cap (A_2)_t \cap \ldots \cap (A_n)_t$, so $A_t = R$, for all t, $0 \le t < 1$.

Suppose there exist $t_1, t_2 \in [0,1]$, $t_1, t_2 \in \mathrm{Im}\ A$. Then $\mathrm{A}_{t_1} = \mathrm{A}_{t_2}$ = R which implies $t_1 = t_2$. Thus $|\mathrm{Im}\ A|$ has two valued namely 1, t.

Remark 2.11:

By using theorem 2.10, we can give an example which explains that the converse of proposition 2.4 is not true in general.

Example: Let $A : Z \longrightarrow [0,1]$ defined by

$$\mathrm{A}(x) = \begin{cases} 1 & x \in 6\mathrm{Z}, \\ \frac{1}{2} & x \in 2\mathrm{Z} - 6\mathrm{Z}, \\ 0 & \text{otherwise} \end{cases}$$

A is not a fuzzy semimaximal ideal, since $|\mathrm{Im}\ A| = 3$. However A(0) = 1, $A_* = 6Z$ is a semimaximal ideal of Z.

Remark 2.12:

If A a fuzzy semimaximal ideal of R and B is a fuzzy ideal of R such that $B \neq \lambda_R$ and $A \subseteq B$. Then it is not necessary that B is a fuzzy semimaximal ideal.

Example: Let $A : Z \longrightarrow [0,1]$ defined by

$$A(x) = \begin{cases} 1 & x \in 6Z, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

A is a fuzzy semimaximal ideal (see Remark 2.3(1)).
Let $B : Z \longrightarrow [0,1]$ defined by

$$B(x) = \begin{cases} 1 & x \in 6Z, \\ \frac{3}{4} & x \in 2Z - 6Z, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

It is clear that $A \subseteq B$. However $|\mathrm{Im}\, A| = 3$, which implies that B is not a fuzzy semimaximal ideal, by theorem 2.10.

Remark 2.13:

If A a fuzzy semimaximal ideal and $t \in [0,1)$, then A_t does'nt need to be a semimaximal ideal of R. As can be seen by the following example:

Example: Let $A : Z \longrightarrow [0,1]$ defined by

$$A(x) = \begin{cases} 1 & x \in 2Z, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

A is a fuzzy semimaximal ideal of Z and $A_* = 2Z$ which is amaximal ideal and $A(0) = 1$, this implies that A is a fuzzy maximal ideal, so it is semimaximal. But $A_{1/2} = \{x : A(x) \geq \frac{1}{2}\} = Z$ which is not a semimaximal ideal.

Recall that, the fuzzy Jacobson radical of a ring R denoted by F-J(R) is the intersection of all fuzzy maximal ideals of R (1).

F-J(R) does'nt need to be a fuzzy semimaximal ideal of R.

Example: Let $\{A_i, i = 1, 2, \ldots, n\}$ be the collection of all fuzzy maximal ideals of Z, where

$$A_i(x) = \begin{cases} 1 & x \in pZ, \\ \frac{1}{i} & x \notin pZ. \end{cases}$$, p is a prime number.

$$\text{F - J(R)} = \bigcap_{i=1}^{\infty} \mathrm{A}_i = \begin{cases} 1 & x \in \bigcap\limits_{p \text{ is a prime no.}} pZ, \\ \inf\{\frac{1}{\mathrm{i}}, i \in \mathrm{Z}_+\} & x \notin \bigcap\limits_{p \text{ is a prime no.}} pZ. \end{cases}$$

$$\text{F - J(R)} = \bigcap_{i=1}^{\infty} \mathrm{A}_i = \begin{cases} 1 & x \in \bigcap\limits_{p \text{ is a prime no.}} pZ, \\ 0 & x \notin \bigcap\limits_{p \text{ is a prime no.}} pZ. \end{cases}$$

$= 0_1$ which is not a fuzzy semimaximal ideal of Z.

Now, let F-J′(R) denotes the intersection of all fuzzy semimaximal ideals of R. Then F-J′(R) is called a fuzzy semijacobson radical of R.

Remark 2.14:

F-J(R) = F-J′(R).

Proof. It is clear that F-J′(R) $\subseteq$ F-J(R).

Let $x_t \in$ F-J(R). Then x_t belongs to any fuzzy maximal ideal. Since any fuzzy semimaximal ideal A of R is a finite intersection of fuzzy maximal ideals, so $x_t \in$ A. It follows that $x_t \in$ F-J′(R).

Thus F-J(R) = F-J′(R).

S.3 Image and Inverse Image of Fuzzy Semimaximal Ideals

In this section, we consider the homomorphic image and inverse image of fuzzy semimaximal ideals.

Theorem 3.1:

Let R_1, R_2 be two rings, let f : $R_1 \longrightarrow R_2$ be an epimorphisim and every fuzzy ideal of R_1 is f-invariant. Then if A is a fuzzy semimaximal ideal of R_1, then f (A) is a fuzzy semimaximal of R_2.

Proof. A is a fuzzy semimaximal ideal of R_1, then A = $A_1 \cap A_2 \cap \ldots \cap A_n$, where $A_1, A_2, \ldots, A_n$ are fuzzy maximal ideals of R_1.

Also, since every fuzzy ideal of R_1 is f-invariant.

So f (A) = f ($A_1 \cap A_2 \cap \ldots \cap A_n$) = f ($A_1$) $\cap$ f (A_2) … $\cap$ f (A_n).

On the other hand, f (A_i) is a fuzzy maximal ideal of R_2, $\forall$ i = 1, 2, …, n by (Th. 3.2 (1)) in(13) and note 1.5.

Hence f (A) is a finite intersection of fuzzy maximal ideals.

Thus f (A) is a fuzzy semimaximal ideal of R_2.

Theorem 3.2:

Let R_1, R_2 be two rings, let f : $R_1 \longrightarrow R_2$ be an epimorphisim. If B is a fuzzy semimaximal ideal of R_2, then f^{-1}(B) is a fuzzy semimaximal ideal of R_1.

Proof. Since B is a fuzzy semimaximal ideal of R_2, B = $B_1 \cap B_2 \cap \ldots \cap B_n$, where B_i is a fuzzy maximal ideals of R_2, for all i = 1, 2, …, n.

But $f^{-1}(B) = f^{-1}(B_1 \cap B_2 \cap \ldots \cap B_n) = f^{-1}(B_1) \cap f^{-1}(B_2) \ldots \cap f^{-1}(B_n)$

But for each i = 1, 2, …, n, $f^{-1}(B_i)$ is a fuzzy maximal ideal of R_1 by (Th. 3.2)(2) in (13) and note 1.5.

Hence f^{-1}(B) is a finite intersection of fuzzy maximal ideals.

Thus f^{-1}(B) is a fuzzy semimaximal ideal of R_1.

S.4 Direct Sum of Fuzzy Semimaximal Ideals

In this section, we turn out attention to study fuzzy semimaximal ideals and direct sum. First we give the following lemmas which are useful in our work.

Lemma 4.1:

Let R_1, R_2 be two rings, let A, B be fuzzy ideals of R_1, R_2 respectively. Then $A\oplus B$ is a fuzzy ideal of $R_1\oplus R_2$, where

$(A \oplus B)(a,b) = \min\{A(a),B(b)\}$, for all $(a,b) \in R_1 \oplus R_2$.

Proof. By using note 1.5 and (Th.2.4.1.8)(14) the result follows directly.

Lemma 4.2:

Let R_1, R_2 be two rings, let A be a fuzzy ideals of $R_1 \oplus R_2$, then there exist fuzzy ideals B_1 and B_2 of R_1, R_2 respectively such that $A = B_1 \oplus B_2$.

Proof. By using note 1.5 and (Th.2.4.1.9)(14) the result is obtained.

Lemma 4.3:

If A and B are fuzzy ideals of rings R_1, R_2 respectively then $(A \oplus B)_* = A_* \oplus B_*$.

Proof. Let $(x,y) \in (A \oplus B)_*$, then $(A \oplus B)(x,y) = 1$ and so $\min\{A(x),B(y)\} = 1$. This implies that $A(x) = 1$, $B(y) = 1$. Hence $x \in A_*$ and $y \in B_*$.

Thus $(x,y) \in A_* \oplus B_*$, so $(A \oplus B)_* \subseteq A_* \oplus B_*$.

Conversely; Let (x,y) $A_* \oplus B_*$. Then $x \in A_*$ and $y \in B_*$. Hence $A(x) = 1$, $B(y)=1$. Thus $\min\{A(x),B(y)\} = 1$ and so $(A\oplus B)(x,y) = 1$; That is $(x,y)\in(A\oplus B)_*$.

Thus $(A \oplus B)_* \subseteq A_* \oplus B_*$ and hence $(A \oplus B)_*=A_* \oplus B_*$.

It is known that (see (15)p.53):

If R_1, R_2 be rings, $R = R_1 \oplus R_2$ and A is an ideal of R then A is a maximal ideal of R iff $A = A_1 \oplus R_2$ or $A = R_1 \oplus A_2$, where A_1 is a maximal ideal of R_1, A_2 is a maximal ideal of R_2.

We generalize this result, to the following:

Lemma 4.4:

Let R_1, R_2 be two rings, $R = R_1 \oplus R_2$ and A is a fuzzy ideal of R then A is a fuzzy maximal ideal of R if and only if either $A = B \oplus \lambda_{R_2}$, where B is a fuzzy maximal ideal of R_1 or $A = \lambda_{R_1} \oplus C$, where C is a fuzzy maximal ideal of R_2.

Proof. If A is a fuzzy maximal ideal of R. Since A is a fuzzy ideal of R, so by lemma 4.2, $A = B \oplus C$ for some fuzzy ideals B and C of R_1, R_2 respectively.

Hence $A_* = (B \oplus C)_* = B_* \oplus C_*$ by lemma 4.3. Then by (Lemma 2.1.(3)), $B_* \oplus C_*$ is a maximal ideal. So either $B_* \oplus C_* = R_1 \oplus C_*$ or $B_* \oplus C_* = B_* \oplus R_2$.

hat is either $B_* = R_1$ or $C_* = R_2$.

If $B_* = R_1$, then $B = \lambda_{R_1}$. If $C_* = R_2$, then $C = \lambda_{R_2}$. Hence either $B \oplus C = \lambda_{R_1} \oplus C$ or $B \oplus C = B \oplus \lambda_{R_2}$.

Conversely; If $A = B \oplus \lambda_{R_2}$ and B is a fuzzy maximal ideal of R_1. To prove A is a fuzzy maximal ideal of R.

By (Lemma 4.3), $A_* = B_* \oplus (\lambda_{R_2})_*$ and so $A_* = B_* \oplus R_2$. Since B is a fuzzy maximal ideal of R_1, then by (Lemma 2.1(3)), B_* is a maximal ideal of R_1. Hence $B_* \oplus R_2 = A_*$ is a maximal ideal of R.

On the other hand, $A(0,0) = \min\{B(0), \lambda_{R_2}(0)\}$. But $B(0) = 1$ by (Lemma 2.1(1)), so $A(0,0) = \min\{1,1\} = 1$. Then by (Lemma.2.1(5)), A is a fuzzy maximal ideal of R.

Similarly, if $A = \lambda_{R_1} \oplus C$, C is a fuzzy maximal ideal of R_2, then A is a fuzzy maximal ideal of R.

Now, we can give the main results, first we have the following:

Theorem 4.5:

Let R_1, R_2 be two rings, let $R = R_1 \oplus R_2$ and A, B be fuzzy ideals of R_1, R_2 respectively. Then

(1) A is a fuzzy semimaximal ideal of R if and only if $A \oplus \lambda_{R_2}$ is a fuzzy semimaximal ideal of R.

(2) B is a fuzzy semimaximal ideal of R_2 if and only if $\lambda_{R_1} \oplus B$ is a fuzzy semimaximal ideal of R.

Proof

(1). Since A is a fuzzy semimaximal ideal of R_1, $A = \bigcap_{i=1}^{n} A_i$, where A_i is a fuzzy maximal ideal of R_1, for all i = 1, 2, …, n.

Hence, $A \oplus \lambda R_2 = \bigcap_{i=1}^{n} A_i \oplus \lambda_{R_2}$

$$= \bigcap_{i=1}^{n} A_i \oplus \bigcap_{n-times} (\lambda_{R_2}), \text{ since } \lambda_{R_2} = \underbrace{\lambda_{R_2} \cap \ldots \cap \lambda_{R_2}}_{n-times}$$

$$= \bigcap_{i=1}^{n} (A_i \oplus \lambda_{R_2}) \qquad \text{, by Lemma 2.4 (16)}$$

But by lemma 4.4, $A_i \oplus \lambda_{R_2}$ is a fuzzy maximal ideal of R, for all i = 1, 2, …, n. Thus $A \oplus \lambda_{R_2}$ is a fuzzy semimaximal ideal.

Conversely; If $A \oplus \lambda_{R_2}$ is a fuzzy semimaximal ideal of R, then

$A \oplus \lambda_{R_2} = \bigcap_{i=1}^{n} D_i$, where D_i is a fuzzy maximal ideal of R for all i = 1, 2, …, n.

By lemma 4.2, for each i = 1, 2, …, n, $D_i = B_i \oplus C_i$, where B_i is a fuzzy ideal of R_1, C_i is a fuzzy ideal of R_2.

Hence $A \oplus \lambda_{R_2} = \bigcap_{i=1}^{n} (B_i \oplus C_i)$

$$= \bigcap_{i=1}^{n} B_i \oplus \bigcap_{i=1}^{n} C_i \quad \text{, by Lemma2.4 (16)}$$

It follows that $A = \bigcap_{i=1}^{n} B_i$, $\bigcap_{i=1}^{n} C_i = \lambda_{R_2}$.

But $\bigcap_{i=1}^{n} C_i = \lambda_{R_2}$ implies that $C_i = \lambda_{R_2}$, $\forall$ i = 1, 2, …, n.

Hence $B_i \oplus C_i = B_i \oplus \lambda_{R_2}$; That is $D_i = B_i \oplus \lambda_{R_2}$.

Then by lemma 4.4, B_i is a fuzzy maximal ideal of R_1 and so $A = \bigcap_{i=1}^{n} B_i$ is a fuzzy semimaximal ideal of R_1.

(2). The proof is similarly.

Next, we can give the following:

Theorem 4.6:

Let R_1, R_2 be two rings, let $R = R_1 \oplus R_2$ and let A be a fuzzy ideal of R. If A is a fuzzy semimaximal of R, then either:

(1) There exists fuzzy semimaximal ideals B, C of R_1, R_2 respectively such that $A = B \oplus C$, or

(2) There exists a fuzzy semimaximal ideal B of R_1 such that $A = B \oplus \lambda_{R_2}$, or

(3) There exists a fuzzy semimaximal ideal C of R_2 such that $A = \lambda_{R_1} \oplus C$.

Proof. If A is a fuzzy semimaximal ideal of R, then $A = \bigcap_{i=1}^{n} A_i$, where A_i is a fuzzy maximal ideal of R. By lemma 4.4, for each i = 1, 2, …, n, either $A_i = B_i \oplus \lambda_{R_2}$ or $A_i = \lambda_{R_2} \oplus C_i$, where B_i, C_i are fuzzy maximal ideals of R_1, R_2 respectively.

If $A_i = B_i \oplus \lambda_{R_2}$, for all i = 1, 2, …, n, then $A = \bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} B_i \oplus \lambda_{R_2}$, putting $\bigcap_{i=1}^{n} B_i$ = C, we get $A = B \oplus \lambda_{R_2}$ and B is a fuzzy semimaximal ideal of R_1.

If $A_i = \lambda_{R_1} \oplus C_i$, for all i = 1, 2, …, n, then $A = \lambda_{R_1} \oplus C$, where $C = \bigcap_{i=1}^{n} C_i$ and C is a fuzzy semimaximal ideal of R_2.

Now if $A_i = B_i \oplus \lambda_{R_2}$, for some i = 1, 2, …, n. Then without loss of generality, we can assume that $A_i = B_i \oplus \lambda_{R_2}$, for some i = 1, 2,…, k, k < n and $A_i=\lambda_{R_1} \oplus C_i$, for all i = k+1, …, n.

Hence

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{k} A_i \cap \left(\bigcap_{i=k+1}^{n} A_i \right)$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{k} (B_i \oplus \lambda_{R_2}) \bigcap_{i=k+1}^{n} (\lambda_{R_1} \oplus C_i)$$

$$= \left(\bigcap_{i=1}^{k} B_i \oplus \lambda_{R_2} \right) \cap \left(\lambda_{R_1} \oplus \bigcap_{i=k+1}^{n} C_i \right) \text{, by (Lemma 2.4(16))}$$

$$= \bigcap_{i=1}^{k} B_i \oplus \bigcap_{i=k+1}^{n} C_i$$

Letting $B = \bigcap_{i=1}^{k} B_i$, $C = \bigcap_{i=k+1}^{n} C_i$, then B, C are fuzzy semimaximal of R_1, R_2 respectively. Thus $A = B \oplus C$.

Remark 4.7:

The converse of theorem 4.6 is not necessary true in general, in fact when B, C are fuzzy semimaximal ideals of $R_1 \oplus R_2$, then $B \oplus C = A$ need not a fuzzy semimaximal ideal of $R_1 \oplus R_2$.

We can give the following example:

Example: Let $B, C : Z \longrightarrow [0,1]$ defined by

$$B(x) = \begin{cases} 1 & x \in 2Z, \\ \frac{1}{3} & \text{otherwise} \end{cases} \qquad C(x) = \begin{cases} 1 & x \in 3Z, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

B, C are fuzzy semimaximal ideals of Z since B, C are fuzzy maximal ideals of Z.
On the other hand, $B \oplus C$: $Z \oplus Z \longrightarrow [0,1]$ and

$$(B \oplus C)(a+b) = \begin{cases} 1 & (a,b) \in 2Z \oplus 3Z, \\ \frac{1}{2} & (a,b) \in 2Z \oplus (Z-3Z), \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

$B \oplus C$ is not a fuzzy semimaximal ideal, since $|\text{Im}\,(B \oplus C)| = 3$.

References

1. Malik, D.S. and Mordeson, J.N., (1991), "Fuzzy Maximal, Radical and Primary Ideals of a Ring", Information Sciences, 53: 237-250.

2. Goodreal, K.R., (1979), "Ring Theorey-Non Singular Ring and Modules", Marcei-Dekker, New York and Basel.

3. K.Y.Hatem, (2007), "Semimaximal Submodules", Ph.D.Thesis, University of Baghdad.

4. Zaheb, L.A. (1965), "Fuzzy Sets, Information and Control", 8:338-353.

5. Zahedi, M.M. (1992), "On L-Fuzzy Residual Quotient Modules and P.Primary Submodules", Fuzzy Sets and Systems,51:333-344.

6. Zahedi, M.M, (1991), "A Characterization of L-Fuzzy Prime Ideals", Fuzzy Sets and Systems, Vol.44, pp.147-160.

7. Kumar, R. (1991), "Fuzzy Semiprimary Ideals of Ring", Fuzzy Sets and Systems, 42:263-272.

8. Zhao Jiandi, Shik. Yue M., (1993), "Fuzzy Modules Over Fuzzy Rings", The J. of Fuzzy Math. 3: 531-540.

9. Kumar, R., (1992), "Fuzzy Cosets and Some Fuzzy Radicals", Fuzzy Sets and Systems, Vol.46, pp.261-265.

10. Liu, W.J, (1982), "Fuzzy Invariant Subgroups and Fuzzy Ideals", Fuzzy Sets and Systems, 8:133-139.

11. Mordeson, J.N. (1996), "Fuzzy Intersection Equations and Primary Representations", Fuzzy Sets and Systems, 83: 93-98.

12. Swamy, K.L.N. and Swamy, V.M., (1988), "Fuzzy Prime Ideals of Rings, J.Math. Anal.Appl, 134:94-103.

13. Hadi, I.M.A., (2001), "On Fuzzy Ideals of Fuzzy Rings", Math. and Physics J. 16(4):

14. Abu-Dareb, A.T.H., (2000), "On Qusi-Frobenius Fuzzy Rings", M.Sc. Thesis, University of Baghdad.

15. Larsen, M.D. and Carthy, P.J.Mc, (1971), "Multiplicative Theory of Ideals", Academic Press, New York.

16. Hadi, I.M.A. and Abu-Dareb, A.T.H., (2004), "P-F Fuzzy Rings and Normal Fuzzy Ring", Ibn-Al-Haitham J. of Pure and Applies Sciences, 17(1):

المثاليات الضبابية شبه الأعظمية

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الخلاصة

لتكن R حلقة ابدالية ذا محايد.I مثالي فعلي في R يسمى شبه أعظمي إذا كان I تقاطع عدد منته من مثاليات عظمى. في هذا البحث قمنا بتنصيب هذا المفهوم إلى المثاليات الضبابية على R، اذ يكون المثالي الضبابي A على R مثاليا شبه أعظمي إذا كان تقاطع عدد منته من المثاليات الضبابية العظمى. خواص أساسية مختلفة قد أعطيت فضلا عن هذا قد أعطيت بعض الأمثلة لتوضيح هذا المفهوم.