Zernike Polynomiales for Opticl Ssytem With Horizantal Rectangular Aperture

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Abstract

For small aberrations, the Strehl ratio of an imaging system depends on the aberration variance. Its aberration function is expanded in terms of Zernike polynomials, which are orthogonal over a circular aperture. Their advantage lies in the fact that they can be identified with classical aberrations balanced to yield minimum variance, and thus maximum Strehl ratio. In recent paper, we derived closed form of Zernike polynomials that are orthonormal over a horizontal rectangular pupil (parallel to the x-axies) with area equal π. Using the circle polynomials as the basis functions for their orthogonalization over such pupil, we derive closed-form polynomials that are orthonormal over rectangular pupil by using Gram-shmit method These polynomials are unique in that they are not only orthogonal across such pupils, but also represent balanced classical aberrations, just as the Zernike circle polynomials are unique in these respects but also represent balanced classical aberrations.

Introduction

For systems with noncircular pupils, the Zernike circle polynomials are neither orthogonal over such pupils nor do they represent balanced aberrations. Hence their special utility is lost. However, since they form a complete set, an aberration function over a noncircular wavefront can be expanded in terms of them. The expansion coefficients are no longer independent of each other, and their values change as the number of polynomials used in the expansion changes. The piston term does not represent the mean value of the aberration function, and the sum of the squares of the Zernike coefficients does not yield the aberration variance. An obvious example of a system with a noncircular pupil is a Cassegrain telescope, which has an annular pupil. For systems with annular pupils, we need polynomials that are orthogonal and represent balanced aberrations yielding minimum variance over an annulus. Such polynomials are the Zernike annular polynomials (1–4). The amount of defocus that optimally balances spherical aberration to yield minimum variance over an annular pupil and thus yield the plane of best focus for a small aberration is different from that for a circular pupil. Similarly, the amount of wavefront tilt that optimally balances coma to yield minimum variance and thus the brightest point of the image of a point object, again for a small aberration, is different for circular and annular pupils. The optimal combination or balancing is built into the form of the annular polynomials. Although in many imaging applications the amplitude across the pupil is uniform, such is not always the case, e.g., in a system with an apodized pupil. The polynomials that are orthogonal over an annular or a Gaussian pupil have been obtained by orthogonalizing the Zernike circle polynomials over the pupil of interest with the appropriate weighting (e.g., Gaussian for a Gaussian pupil), using the Gram–Schmidt orthogonalization process (5). The primary mirrors of large telescopes, such as the Keck, consist of hexagonal segments (6). The wavefront analysis of such segments requires polynomials that are orthogonal over a hexagon. The pupil for offaxis imaging by a system with an axial circular pupil is vignetted, but can be approximated by an ellipse (7). High-power laser beams have rectangular or square cross sections (8) and require polynomials that are orthogonal over a rectangle or a square, respectively. In recent papers, we gave closed-form expressions for the low-order polynomials that are orthonormal over a square pupil by using the recursive Gram–Schmidt orthogonalization process (9). In this paper, we extend such work to other noncircular pupils. We first give a brief introduction on the Zernike circle polynomials and the orthogonalization process. We then derive closed-form analytical expressions for polynomials that are orthogonal over horizontal rectangle. We use Zernike circle polynomials as the basis functions for the orthogonalization process, so that the relationship of the new polynomial to the circle polynomials is evident, since the former is a linear combination of the latter. Moreover, this relationship, in turn, helps to determine the Zernike coefficients in terms of the orthonormal coefficients. Also uniquely represent the orthogonal and balanced aberrations across such pupils. Although high-power rectangular laser beams have been around for a long time, there is nothing in the literature on rectangular polynomials representing balanced aberrations for such beams.

The rectangular polynomials thus obtained up to the fourth order are given in Table (1) in the same manner. Each rectangular polynomial also consists of either cosine or sine terms, but not both. The rectangular polynomials also consist of a larger number of circle polynomials.

Aberration theory

The main motivation for using Zernike polynomials is that they describe with high precision the shapes of four conventional Seidel aberrations. Because there is no limit on the number of terms that may be used, many higher order aberrations can be described by Zernike polynomials, among them coma, 3rd order spherical aberration, etc.. Actually it is enough to use the first 15 linearly independent Zernike polynomials, and it can be written for the unit circular aperture and in Cartesian coordinates as follows(9):

$$W(x,y)= Z_0+Z_1 x+Z_2 y+Z_3(2x^2+2y^2-1)+Z_4(x^2-y^2) +Z_5(2 xy) +Z_6(3x^3+3xy^2-2x) +Z_7(3x^2y+3y^3-2y) +Z_8(6 x^4+12 x^2 y^2+6 y^4-6x^2-6y^2+1) +Z_9(x^3-3xy^2) +Z_{10}(3 x^2y- y^3)+Z_{11}(4 x^3y-4 xy^3)+Z_{12}(8x^3y+8 xy^3-6xy) +Z_{13}(4 x^4-3 x^2-4 y^4+3 y^2)+Z_{14}(x^4-6 x^2y^2+ y^4) \quad [2.1]$$

To describe a wavefront across an aperture using a complete set of functions $Z_n(x,y)$, the criterion for the base selection imposes some constraints on the functions properties. One fundamental parameter is the *Varience* V which depends on the mean, <W(x,y)>, and mean square value, $<W^2(x,y)>$, of the wavefront, as follows(10):

$$V = \langle W^2(x,y)\rangle - \langle W(x,y)\rangle^2$$

$$= \frac{1}{A}\iint_A W^2(x,y)dA - \frac{1}{A^2}\left(\iint_A W(x,y)dA\right)^2 \qquad [2.2]$$

where A is the area of the aperture. The orthogonolization of Zernike polynomials used to describe the aberration function simplify equation [2.2] to be(11):

$$V = \sum_1^n Z_n^2(x,y)/C_n^2$$

$$= \frac{1}{C_1^2} Z_1^2(x,y) + \frac{1}{C_2^2} Z_2^2(x,y) + \frac{1}{C_3^2} Z_3^2(x,y) + \ldots\ldots \qquad [2.3]$$

Where C_n is the normalization constant for the concerned polynomial which satisfy the equation:

$$C_n^{\,2}\,\frac{\iint_A Z^2(x,y)dA}{A} = 1 \qquad [2.4]$$

Zernike Polynomials for Rectangular Aperture of AN Area Equal to π

The first 15^{th} The Zernike polynomials for the rectangular aperture of an area equal to π is found by taking Zernike polynomials of a circular aperture with an area equal to π (equation [2.1]) and make them orthogonal to each other at an area of a rectangular by using Gram Shmidt orthogonolization method as follows:[7,12]

$$f_i(x,y)=g_j(x,y)-\sum_{1}^{i}\frac{\iint_A f_i(x,y)g_j(x,y)dA}{\iint_A f_i^2(x,y)dA} \qquad [3.1]$$

where $f_i(x,y)$ is the ith Zernike polynomial for the rectangular aperture and $g_j(x,y)$ is the jth Zernike polynomial for the circular aperture, and i and j are integer number equal 1,2,3,......,15 , so the first 15th Zernike orthogonal polynomials equation for a horizontal rectangular aperture (figure 1) is computed and found to be:

$$W(x,y)= Z_0 + Z_1 x + Z_2 y + Z_3(x^2+y^2-0.53895) + Z_4(0.55125 x^2-1.44875 y^2+0.11413)$$
$$+ Z_5(2xy) +Z_6(x^3+xy^2-0.80562x) + Z_7(x^2y+y^3-0.70344y)$$
$$+Z_8(y^4+x^4+2x^2y^2-1.26838 x^2-1.1954 y^2+0.25541)$$
$$+Z_9(1.32127x^3-2.67873xy^2-0.24197x) +Z_{10}(0.90769x^2y-3.09231y^3+0.84192y)$$
$$+Z_{11}(4x^3y-4xy^3-0.91956xy)+Z_{12}(0.55124x^3y+1.44876xy^3-0.86695xy)$$
$$+Z_{13}(2.936x^4-5.064y^4-2.128x^2y^2-2.07901x^2+3.38682y^2-0.05936)$$
$$+Z_{14}(1.06396x^4+5.18396y^4-1.75208x^2y^2-0.55181x^2-2.15769y^2+0.07912) \qquad [3.2]$$

To convert the orthogonal set of Zernike polynomials for the horizontal rectangular aperture into a set of orthonormal polynomials, each polynomial must be multiplied by a constant called normalization constant C_n , and it found by using equation [2.4] as follows:

$$1 = C^2 \frac{\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \int_{-1}^{+1} (f(x,y))^2 \, dx \cdot dy}{\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \int_{-1}^{+1} dx \cdot dy} \qquad [3.3]$$

where *f(x,y)* is the Zernike polynomial . Table (3.1) states the normalization constants C for the first 15 Zernike polynomials for the rectangular aperture of an area equal to π.

Conclusions

We have considered orthogonal polynomials for analysis of wavefronts across noncircular pupils, such as, rectangular, and square. High-power laser beams often have a rectangular or a square cross section[8]. Zernike circle polynomials are in widespread use for wavefront analysis in optical design and testing because they are orthogonal over a unit circle and represent balanced aberrations of systems with circular pupils. When a wavefront is expanded in terms of them, the value of an expansion coefficient is independent of the number of terms used in the expansion. Accordingly, one or more terms can be added or subtracted without affecting the other coefficients. The piston term represents the mean value of the aberration function, and the other coefficients represent the standard deviation of the corresponding terms. The variance of the aberration is given simply by the sum of the squares of the aberration coefficients (with the exception of the piston coefficient). Since the circle polynomials form a complete set, any wavefront, regardless of the shape of the pupil (which defines the perimeter of the wavefront) can be expanded in terms of them. However, unless the pupil is circular, advantages of orthogonality and aberration balancing are lost. For example, the mean value of a circle polynomial across a noncircular pupil is not zero. The value of a Zernike coefficient changes as the number of polynomials used in the expansion of an aberration function changes. Hence, the circle polynomials are not appropriate for wavefront analysis of noncircular pupils. An obvious example of a noncircular pupil is a an annular pupil, which is common for a centered mirror telescope. The appropriate polynomials for annular pupils are the Zernike annular polynomials, which can be obtained from the circle polynomials by their orthogonalization over an annulus using the recursive Gram–Schmidt process. We can extend this procedure to rectangular, and square pupils and determined the polynomials that are orthonormal over and represent balanced aberrations for such pupils, thereby retaining the advantages of orthogonal polynomials. The orthonormal polynomials representing balanced aberrations of a one-dimensional slit pupil are obtained as a limiting case of a rectangular pupil as the Legendre polynomials.

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Table: (3.1) The first 15 Zernike polynomials and their corresponding normalization constants for horizontal rectangular aperture of an area equal to π.

mode	Polynomial	C
0	1	1
1	X	1.732917
2	Y	2.203263
3	$x^2+y^2-0.53895$	2.85133
4	$0.55125\,x^2-1.44875\,y^2+0.11413$	3.194383
5	$2xy$	1.910402
6	$x^3+xy^2-0.80562x$	5.423261
7	$x^2y+y^3-0.70344y$	6.085806
8	$y^4+x^4+2x^2y^2-1.26838\,x^2-1.1954\,y^2+0.25541$	7.254763
9	$1.32127x^3-2.67873xy^2-0.24197x)$	2.874798
10	$0.90769x^2y-3.09231y^3+0.84192y$	3.892495
11	$4x^3y-4xy^3-0.91956xy$	3.100868
12	$0.55124x^3y+1.44876xy^3-0.86695xy$	13.8675
13	$2.936x^4-5.064y^4-2.128x^2y^2-2.07901x^2+3.38682y^2-$	3.429972
	0.05936	
14	$1.06396x^4+5.18396y^4-1.75208x^2y^2-0.55181x^2-2.15769y^2+0.07912$	4.8795

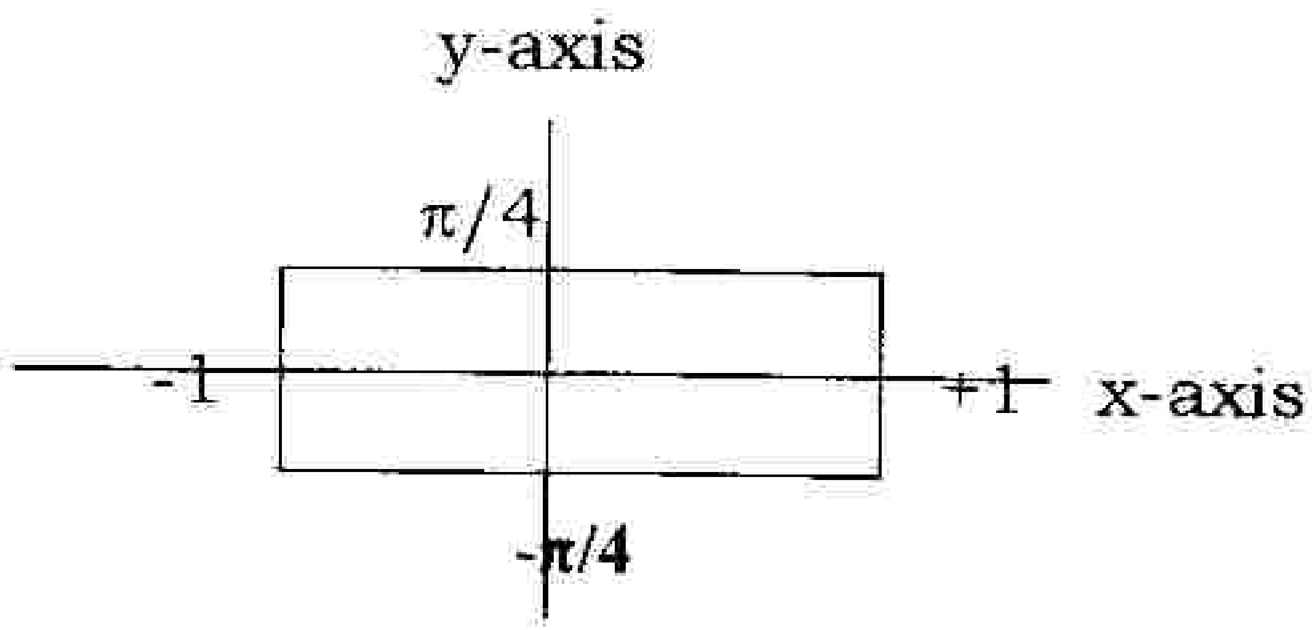

Fig.(1) horizontal rectangular aperture of an area equals π

متسلسلة زرنيخ لمنظومة بصرية ذات فتحه مستطيلة أفقيا"

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الخلاصة

تعتمد نسبة ستريل للمنظومات البصرية ذات الزيوغ القليلة على تغاير هذة الزيوغ وتكتب دالة الزيوغ بدلالة متسلسلة زرنيخ والتي تكون حدودها متعامدة للفتحة الدائرية وهذة المتسلسلة مفيدة كونها تمثل حالات توازن الزيوغ وبالتالي تعطي القيمة الصغرى لتباين الزيوغ وبالتالي نسبة ستريل عظمى. في هذا البحث تم اشتقاق معادلات متعددة حدود زرنيك للمنظومة البصرية ذات فتحة دخول مستطيلة الشكل موازية للمحور السيني والتي مساحتها تساوي π ، وبسبب فقدان متسلسلة الزيوغ للفتحة المستطيلة لصفة التعامد فقد تم استخدام طريقة كرام- شميت لأيجاد الثوابت العيارية لها والتي تجعل معادلة زرنيك للفتحة المستطيلة كما هو الحال للفتحة الدائرية ، معادلة عيارية متعامدة وكذلك تمثل حالة الزيوغ المتوازنة .