Algorithm to Solve Linear Volterra Fractional Integro-Differential Equation via Elzaki Transform

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Abstract

In this work, Elzaki transform (ET) introduced by Tarig Elzaki is applied to solve linear Volterra fractional integro-differential equations (LVFIDE). The fractional derivative is considered in the Riemman-Liouville sense. The procedure is based on the application of (ET) to (LVFIDE) and using properties of (ET) and its inverse. Finally, some examples are solved to show that this is computationally efficient and accurate.

Key word: integral, differential, Volterra, fractional, Elzaki.

Introduction

Many fields of science and engineering are described by integro-differential equation of fractional order such as, fluid mechanics, vis co-elasticity, diffusion processes, biology and so on [1-4]. Several methods to solve VFIDEs have been proposed such as, expansion methods and spline method [5], variation iteration method [6], Laplace decomposition method [7] and Legender pseudo spectral method [8].

In recent years, Elzaki transform (ET) was introduced by Tarig Elzaki (2010) has been used to find solutions of a wide class of differential, integral, partial differential and the integro-differential equations, see [9-12].

In this paper, ET and some of its fundamental properties are used to solve LVFIDEs with initial value problems:

$$D^{\alpha} y(t) = g(t) + \int_0^t k(x,\tau)\, y(\tau)\, d\tau \qquad (1)$$

Subject to the initial conditions

$$D^{\alpha-z} y(t) = c_z \quad , \quad z=1,2,\ldots,n \quad , \quad n\text{-}1< \alpha \le n \quad , \quad n \in N$$

where c_z is a specified constant, α is a parameter describing the order of the time fractional, g and k are known functions, y(t) is the unknown function and D^{α} is the fractional differential operator of order α.

The present paper has been organized as follows. In section 2, some definitions, theorems and properties of the fraction calculus and ET are presented. In section 3, the solution steps to solve LVFIDE by using ET are described. Finally, in section 4, some test examples are solved.

Preliminaries

In this section, we present some basic definitions, some theorems and useful properties of the fractional calculus and Elzaki transform, as well as we proved some theorems which are used in this work.

Definition (1): [1, 13]

The Riemann–Liouville fractional derivative of order α, is defined to be for $\alpha > 0$:

$$D^{\alpha} f(x)= D^{n} D^{\alpha-n} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\, dt$$

or for $\alpha < 0$: $D^{\alpha} f(x) = \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-t)^{-\alpha-1} f(t)\, dt$

where $n-1 < \alpha \le n$

Definition (2): [6, 13, 14]

The Riemann – Liouville fractional integral operator of order $\alpha \ge 0$, of a function $f \in C_{\mu}$, $\mu \ge -1$ is defined as:

$$D^{-\alpha} f(x) = J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

It has the following properties:

For $\alpha, \beta > 0$ and $\gamma \ge 0$

1) $J^0 f(x) = f(x)$ 2) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

3) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$ 4) $D^\alpha J^\alpha f(x) = f(x)\ ,\ \alpha > 0$

5) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$ 6) $D^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(-\alpha+\gamma+1)} x^{-\alpha+\gamma}$ $,\ \gamma = 0,1,\ldots$

7) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=1}^{n} \dfrac{C_k\, x^{\alpha-k}}{\Gamma(\alpha-k+1)}$, where $n-1 \le \alpha < n$ and $D^{\alpha-k} f(0) = C_k$

Definition (3): [11, 15]

Elzaki transform is defined by:

$$E[f(t)] = T(u) = u^2 \int_0^\infty f(ut)\ e^{-t}\ dt \quad , \quad u \in (k_1, k_2)$$

Where the function f (t) in te set A defined by:

$$A = \left\{ f(t) \,\middle|\, \exists\ M, k_1, k_2 > 0 \,\middle|\, f(t) < M\, e^{|t|/k_j} \ \text{if}\ \ t \in (-1)^j x\, [0,\infty) \right\}$$

Definition (4): [15, 12]

Defined for Re(s) > 0 , the Laplace transform (LT) is given by:

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)\ e^{-st}\ dt$$

Some properties of ET are given by following theorems:

Theorem (1): [9, 15]

Let $f(t) \in A = \left\{ f(t) \,\middle|\, \exists\ M, k_1, k_2 > 0 \,\middle|\, f(t) < M\, e^{|t|/k_j}\ \text{if}\ \ t \in (-1)^j x\, [0,\infty) \right\}$

with LT , F(s) then the ET, T(u) of f (t) is given by:

$$T(u) = u\,F\left(\frac{1}{u}\right) \quad \text{and} \quad F(s) = s\,T\left(\frac{1}{s}\right)$$

Theorem (2): [9]

Let T(u) denote the ET of the f(t), the ET of the definite integral of f(t), that is

$$h(t) = \int_0^t f(\tau)\ d\tau \quad \text{then} \quad E[h(t)] = u\,T(u)$$

Theorem (3): [9, 11, 15]

Let the ET, F(u) and G(u) of the functions f(t) and g(t) respectively, then the ET of the convolution of f and g is:

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\ d\tau \quad \text{is given by} \quad E[(f * g)(t)] = \frac{1}{u} F(u)\, G(u)$$

Theorem (4): [1, 5, 13]

The LT of the Riemann – Liouville is defined as:

$$L\{D^\alpha f(t)\} = F^\alpha(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \frac{d^{\alpha-k-1} f(0)}{dx^{\alpha-k-1}} \ , \text{for all } \alpha \text{ and } n-1 \le \alpha < n, n \in N$$

Theorem (5):

If F(s) and T(u) are LT and ET of f(t) respectively, then

$$E\{D^{\alpha} f(t)\} = T^{\alpha}(u) = u\, F^{\alpha}(\frac{1}{u}) \quad \text{where} \quad F^{\alpha}(s) = L\{D^{\alpha} f(t)\}$$

Proof:-

From definition(3) we have:

$$E[D^{\alpha} f(t)] = T^{\alpha}(u) = u^2 \int_0^{\infty} [D^{\alpha} f(ut)]\, e^{-t}\, dt$$

Now, let $w = ut$ then we get:

$$T^{\alpha}(u) = u \int_0^{\infty} [D^{\alpha} f(w)]\, e^{-w/u}\, dw = u\, F^{\alpha}(\frac{1}{u})$$

Theorem (6):

The ET of the Riemann – Liouville fractional derivative is defined:

$$E\{D^{\alpha} f(t)\} = T^{\alpha}(u) = \frac{T(u)}{u^{\alpha}} - \sum_{k=0}^{n-1} u^{1-k} \frac{d^{\alpha-k-1} f(0)}{dx^{\alpha-k-1}}$$

Proof:-

At first from the above theorem we have:

$$E[D^{\alpha} f(t)] = T^{\alpha}(u) = u\, F^{\alpha}(\frac{1}{u})$$

And using theorem (4) with $s = \frac{1}{u}$ we obtain:

$$T^{\alpha}(u) = u\left[(\frac{1}{u})^{\alpha} F(\frac{1}{u}) - \sum_{k=0}^{n-1} (\frac{1}{u})^{k} \frac{d^{\alpha-k-1} f(0)}{dx^{\alpha-k-1}} \right]$$

Then by theorem (1), we get:

$$T^{\alpha}(u) = \frac{T(u)}{u^{\alpha}} - \sum_{k=0}^{n-1} u^{1-k} \frac{d^{\alpha-k-1} f(0)}{dx^{\alpha-k-1}}$$

Theorem (7):

Let T(u) be ET of the f(x), then ET for f(x) = $e^{x}\, \mathrm{erf}(\sqrt{x})$ is given by:

$$T(u) = \frac{u^2 \sqrt{u}}{1-u}$$

Proof:-

The proof of this equation is easy, by using table of LT, [16] and theorem (1) as follows: $L\left\{e^{x}\, \mathrm{erf}(\sqrt{x})\right\} = \frac{1}{\sqrt{s}\,(s-1)}$

and $$T(u) = u\, F(\frac{1}{u}) = u \frac{1}{\sqrt{\frac{1}{u}}\,(\frac{1}{u}-1)} = \frac{u^2\sqrt{u}}{1-u}$$

Applications to (LVFIDE)

Recall equation (1), the LVFIDE with initial value problems:

$$D^{\alpha} y(t) = g(t) + \int_0^t k(x,\tau)\, y(\tau)\, d\tau$$

Subject to the initial conditions

$$D^{\alpha-z} y(t) = c_z \quad , \quad z=1,2,\dots,n \quad , \quad n-1<\alpha\le n \quad , \quad n\in N$$

The first step, we apply ET on both sides we have:

$$E\left[D^{\alpha} y(t)\right] = E\left[g(t)\right] + E\left[\int_0^t k(x,\tau)\, y(\tau)\, d\tau\right]$$

Can easily be transformed into its ET using theorem (6), equivalent to:

$$\frac{T(u)}{u^{\alpha}} - C = G(u) + E\left[\int_0^t k(x,\tau)\, y(\tau)\, d\tau\right]$$

where $C = \sum_{k=0}^{n-1} u^{1-k} \frac{d^{\alpha-k-1} y(0)}{dx^{\alpha-k-1}}$, T(u) and G(u) are ET for y(t) and g(t) respectively.

The general Elzaki solution is:

$$T(u) = u^{\alpha} C + u^{\alpha} G(u) + u^{\alpha} E\left[\int_0^t k(x,\tau)\, y(\tau)\, d\tau\right] \qquad (2)$$

The second step is to find ET of integral in above equation which depends on the type function $k(x,\tau)$ (kernel of the integral equation) as follows:

A. If the kernel is constant that is, $k(x,\tau) = R$, then by using theorem (2) we have:

$$E\left[\int_0^t R\, y(\tau)\, d\tau\right] = R\, u\, T(u) \qquad (3)$$

where T(u) is the ET of y(t).

B. If the kernel is deference that is, $k(x,\tau) = k(x-\tau)$, by theorem (3) we get:

$$E\left[\int_0^t k(x-\tau)\, y(\tau)\, d\tau\right] = \frac{1}{u} K(u) * T(u) \qquad (4)$$

where K(u) and T(u) are ET of the functions k and y respectively.

C. If the kernel is any function, we represent the solution as an infinite series, that is:

$$y(t) = \sum_{i=0}^{\infty} y_i(t)$$

The ET of the integral is become:

$$E\left[\int_0^t k(x,\tau)\, y(\tau)\, d\tau\right] = E\left[\int_0^t k(x,\tau) \sum_{i=0}^{\infty} y_i(\tau)\, d\tau\right] \qquad (5)$$

The third step, we substitute either eq.(3) or eq.(4) or eq.(5) into eq.(2), we will get: either,

$$T(u) = u^{\alpha} C + u^{\alpha} G(u) + R\, u^{\alpha+1}\, T(u)$$

Then the general Elzaki solution:

$$T(u) = \frac{u^{\alpha}(C + G(u))}{1 - R\, u^{\alpha+1}} \qquad (6)$$

or

$$T(u) = u^{\alpha}(C + G(u)) + u^{\alpha-1}\, K(u) * T(u)$$

Then

$$T(u) = \frac{u^{\alpha}(C + G(u))}{1 - u^{\alpha-1}\, K(u)} \qquad (7)$$

or

$$E\left[\sum_{i=0}^{\infty} y_i(t)\right] = u^{\alpha}(C + G(u)) + u^{\alpha}\, E\left[\int_0^t k(x,\tau) \sum_{i=0}^{\infty} y_i(\tau)\, d\tau\right]$$

Matching both sides of this equation yields the following iterative relations:

$$E\left[y_0(t)\right] = u^{\alpha}(C + G(u)) \qquad (8)$$

$$E\left[y_1(t)\right] = u^{\alpha}\, E\left[\int_0^t k(x,\tau)\, y_0(\tau)\, d\tau\right]$$

$$E\left[y_2(t)\right] = u^{\alpha}\, E\left[\int_0^t k(x,\tau)\, y_1(\tau)\, d\tau\right]$$

In general,

$$E\left[y_{i+1}(t)\right] = u^{\alpha}\, E\left[\int_0^t k(x,\tau)\, y_i(\tau)\, d\tau\right], \quad i = 0, 1, \ldots \qquad (9)$$

The fourth step, we apply inverse ET of either eq.(6) or eq.(7) or (eq.(8) and eq.(9)) give the general solution for (LVFIDE):

either $$y(t) = T^{-1}\left[\frac{u^{\alpha}(C + G(u))}{1 - R\, u^{\alpha+1}}\right] \qquad (10)$$

or $$y(t) = T^{-1}\left[\frac{u^{\alpha}(C + G(u))}{1 - u^{\alpha-1}\, K(u)}\right] \qquad (11)$$

or $$y_0(t) = E^{-1}\left[u^{\alpha}(C + G(u))\right] = P(t) \qquad (12)$$

with $$y_{i+1}(t) = E^{-1}\left[u^{\alpha} E\left[\int_0^t k(x,\tau)\, y_i(\tau)\, d\tau\right]\right], \quad i = 0, 1, \ldots \tag{13}$$

The function $P(t)$ defined in eq.(12) can be decomposed into two parts:

$$P(t) = P_1(t) + P_2(t)$$

So, the modified recursion relation is obtained

$$\left.\begin{aligned} y_0(t) &= P_1(t) \\ y_1(t) &= P_2(t) + E^{-1}\left[u^{\alpha} E\left[\int_0^t k(x,\tau)\, y_0(\tau)\, d\tau\right]\right] \\ y_{i+1}(t) &= E^{-1}\left[u^{\alpha} E\left[\int_0^t k(x,\tau)\, y_i(\tau)\, d\tau\right]\right], \quad i = 1, 2, \ldots \end{aligned}\right\} \tag{14}$$

The solution depends on the choice of $P_1(t)$ and $P_2(t)$. We will show how to suitably choose $P_1(t)$ and $P_2(t)$ as well as testing the method by some examples in the next section.

Test Examples

In this section, we present some test examples to show the effectiveness of the ET method for solving LVFIDE's.

Example (1):

Consider the following linear LVFIDE [5]

$$D^{0.5} y(x) = \frac{1}{\sqrt{\pi x}} + e^{x}\, \mathrm{erf}(\sqrt{x}) - e^{x} + 1 + \int_0^x y(t)\, dt$$

with $D^{-0.5} y(0) = c_1 = 0$

since $\alpha = 0.5$ and $n-1 \le \alpha < n$, then $n = 1$

Applying (ET) of the above equation and since $k(x, t) = 1$, then by using eq.(10) we have the solution:

$$y(x) = T^{-1}\left[\frac{u^{0.5}(C + G(u))}{1 - u^{1.5}}\right] \tag{15}$$

where $C = u\, D^{-0.5} y(0) = 0$

and $G(u) = E\left[\frac{1}{\sqrt{\pi x}} + e^{x}\, \mathrm{erf}(\sqrt{x}) - e^{x} + 1\right]$

from appendix given in [11] (ET of some functions) and theorem (7) we have:

$$G(u) = u\sqrt{u} + \frac{u^2\sqrt{u}}{1-u} - \frac{u^2}{1-u} + u^2 = \frac{u\sqrt{u} - u^3}{1-u}$$

Substituting in eq.(15), we obtain:

$$y(x) = T^{-1}\left[\frac{u^2 - u^3\sqrt{u}}{(1-u)(1-u\sqrt{u})}\right] = T^{-1}\left[\frac{u^2}{1-u}\right]$$

Again from appendix given in [11], we get the exact solution: $y(x) = e^{x}$.

Example (2):

Let us consider the following LVFIDE with difference kernel

$$D^{6/5} y(x) = g(x) + \int_0^x e^{x-t}\, y(t)\, dt \qquad (16)$$

where $g(x) = \frac{6}{\Gamma(14/5)} x^{9/5} + x^3 + 3x^2 + 6x + 6 - 6e^x$

with initial conditions: $D^{1/5} y(0) = c_1 = 0$ and $D^{-4/5} y(0) = c_2 = 0$

Take the (ET) of eq.(16) and since k(x, t) is difference kernel, so by eq.(11) yields:

$$y(x) = T^{-1}\left[\frac{u^{6/5}(C + G(u))}{1 - u^{1/5}\, K(u)}\right] \qquad (17)$$

where $C = u\,c_1 + c_2 = 0$.

$$G(u) = E[g(x)] = 6\left[u^{19/5} + u^5 + u^4 + u^3 + u^2 - \frac{u^2}{1-u}\right]$$

$$= 6\left[\frac{u^{19/5} - u^{24/5} - u^6}{1-u}\right]$$

and $K(u) = E\left[e^x\right] = \frac{u^2}{1-u}$

Now, eq.(17) can be written in the form:

$$y(x) = 6\,T^{-1}\left[\frac{u^5 - u^6 - u^{36/5}}{(1-u)\,(1 - \frac{u^{11/5}}{1-u})}\right] = T^{-1}\left[6u^5\right]$$

Upon inverting, we get the exact solution: $y(x) = x^3$.

Example (3):

Consider the following LVFIDE

$$D^{0.5} y(t) = \frac{-56}{15} t^2 \sqrt{t} + \int_0^t (t+\tau)^2 \; y(\tau) \; d\tau$$

with initial conditions: $c_1 = \sqrt{\pi}$.

Applying the ET to both sides of above equation as well as using eq.(12) and eq.(13) give:

$$y_0(t) = E^{-1}\left[u^{0.5}(C + G(u))\right]$$

$$y_{i+1}(t) = E^{-1}\left[u^{0.5} \; E\left[\int_0^t (t+\tau)^2 \; y_i(\tau) \; d\tau\right]\right], \; i = 0, 1, \ldots$$

where $C = u\, D^{-0.5} y(0) = u\, c_1 = u\sqrt{\pi}$

and $G(u) = E\left[\frac{-56}{15} t^2 \sqrt{t}\right] = \frac{-56\,\Gamma(7/2)}{15} u^{9/2}$

Then

$$y_0(t) = E^{-1}\left[\sqrt{\pi}\, u^{3/2} - \frac{56\,\Gamma(7/2)}{15} u^5\right]$$

$$= \frac{1}{\sqrt{t}} - \frac{28\,\Gamma(7/2)}{45} t^3$$

$$= P(t) = P_1(t) + P_2(t)$$

So we have the following relations:

$$y_0(t) = \frac{1}{\sqrt{t}}$$

$$y_1(t) = -\frac{28\,\Gamma(7/2)}{45} t^3 + E^{-1}\left[u^{0.5} \; E\left[\int_0^t (t+\tau)^2 \; y_0(\tau) \; d\tau\right]\right] \quad (18)$$

and

$$y_{i+1}(t) = E^{-1}\left[u^{0.5} \; E\left[\int_0^t (t+\tau)^2 \; y_i(\tau) \; d\tau\right]\right], \; i = 1, 2, \ldots$$

Now, we find the ET of integral in eq.(18) we have:

$$E\left[\int_0^t (t+\tau)^2 \frac{1}{\sqrt{\tau}} \; d\tau\right] = E\left[\frac{56}{15} t^2 \sqrt{t}\right] = \frac{56\,\Gamma(7/2)}{15} u^{9/2} \quad \text{Then}$$

$$y_1(t) = -\frac{28\,\Gamma(7/2)}{45} t^3 + E^{-1}\left[\frac{56\,\Gamma(7/2)}{15} u^5\right]$$

That is

$y_1(t) = 0$ and $y_{i+1}(t) = 0$ $\quad i=1,2,\ldots$

Therefore, the solution is obtained to be and its exact solution:

$$y(t) = \sum_{i=0}^{\infty} y_i(t) = \frac{1}{\sqrt{t}} \; .$$

Conclusions

The properties of the Elzaki transform are used to solve and to get the general solution of linear Volterra fractional integro-differentioal equations. The fractional derivative is considered in the Riemman-Liouville sense. The results of illustrate examples as well as the simplicity of the algorithm and obtained exact solution show efficiency and accuracy of the method also show that it is a special case of the analysis methods. Finally, the proposed approach is very powerful to find analytical solution of linear problems in fractional calculus field.

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