On e-Small Submodules

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Received in :3/June/2015,Accepted in :20/September/2015

Abstract

Let M be an R-module, where R is a commutative ring with unity. A submodule N of M is called e-small (denoted by $N \underset{e}{\ll} M$) if N + K = M, where $K \underset{e}{\leq} M$ implies K = M. We give many properties related with this type of submodules.

Keywords: Small submodule, δ-small submodule, e-small submodule, and e-coclosed submodule.

1- Introduction

Throughout this work, R is a commutative ring with unity and M is an R-module. A proper submodule N of M is called small ($N \ll M$), if $N + K = M$ where $K \leq M$ implies $K = M$, [1]. A submodule N of M is called δ-small if $N + K = M$ with $\frac{M}{K}$ is singular implies, $K = M$, [2]. A submodule N of M is called essential ($N \underset{e}{\leq} M$) if $N \cap W \neq (0)$ for any non zero submodule W of M, [3]. An R-module M is called singular (non singular) if $Z(M) = M$ ($Z(M) = (0)$), where $Z(M) = \{x \in M: \underset{R}{\text{ann}}(x) \leq R\}$. Zhou and Zhang in [4] introduce a new type of small submodule namely e-small submodule and give some basic properties of this kind of submodules.

In this paper, we continuo the work of Zhou [4] and give many other properties of e-small submodule and study the behavior of e-small submodules in certain class of module.

2- Preliminary

Definition (2.1): [4]

Let N be a submodule of a module M. N is said to be **e-small** in M (denoted by $N \underset{e}{\ll} M$), if $N + L = M$ with $L \underset{e}{\leq} M$ implies $L = M$.

Remark (2.2):

Obviously, every small (δ-small) submodule of an R-module M is e-small [4], but the converses are not true in general, for example:

In the Z-module Z_{12}, the submodule $N = <\bar{2}> \underset{e}{\ll} Z_{12}$ but $N \not\ll Z_{12}$, also $N \underset{\delta}{\not\ll} Z_{12}$.

Also, in the Z-module Z_6, $N = <\bar{3}> \underset{e}{\ll} Z_6$, but $N \underset{\delta}{\not\ll} Z_6$ and $N \not\ll Z_6$, [4].

Proposition (2.3): [4,Proposition 2.3]

Let N be a submodule of M. The following statements are equivalent:

(1) $N \underset{e}{\ll} M$

(2) If $X + N = M$, then $X \leq^{\oplus} M$ with $\frac{M}{X}$ a semisimple module. Where $X \leq^{\oplus} M$, means there exists $W \leq M$ such that $W \oplus X = M$.

Corollary (2.4): [4]

If M is a projective module, then every e-small submodule N of M is δ-small.

The next proposition explains how close the notion of e-small submodules to small submodules.

Proposition (2.5): [4,Proposition 2.5]

Let M be an R-module

(1) Assume N, K, L are submodules of M with $K \leq N$:

(a) If $N \underset{e}{\ll} M$, then $K \underset{e}{\ll} M$ and $\frac{N}{K} \underset{e}{\ll} \frac{M}{K}$.

(b) $N + L \underset{e}{\ll} M$ if and only if $N \underset{e}{\ll} M$ and $L \underset{e}{\ll} M$.

(2) If $K \underset{e}{\ll} M$ and $f : M \longrightarrow N$ is a homeomorphism, then $f(K) \underset{e}{\ll} N$. In particular, if $K \underset{e}{\ll} M \leq N$, then $K \underset{e}{\ll} N$.

(3) Assume that $K_1 \subseteq M_1 < M$, $K_2 \subseteq M_2 \leq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \underset{e}{\ll} M_1 \oplus M_2$ if and only if $K_1 \underset{e}{\ll} M_1$ and $K_2 \underset{e}{\ll} M_2$.

3- Main Results

Proposition (3.1):

Let M be an R-module, let $m \in M$. Then $Rm \underset{e}{\not\ll} M$ if and only if there exists an essential maximal submodule N with $m \notin N$.

Proof:

($\Leftarrow$) Suppose there exists an essential maximal submodule N such that $m \notin N$. Hence $M = Rm + N$ and so $Rm \underset{e}{\not\ll} M$.

($\Rightarrow$) Suppose $Rm \underset{e}{\not\ll} M$. Hence there exists an essential submodule W ($W \neq M$) such that $M = Rm + W$. Let $C = \{N \underset{e}{\leq} M: Rm + N = M\}$. Then $B \neq \phi$. By Zorn's lemma, there exists a maximal element N in C such that $Rm + N = M$. We claim that N is a maximal submodule. Suppose N is not maximal, so there exists a submodule K of M with $N \subset K \subset M$. But $N \underset{e}{\leq} M$, so $K \underset{e}{\leq} M$ and $M = Rm + N \subseteq Rm + K$, thus $Rm + K = M$ and hence $K \in C$. But this contradicts the maximality of N. Therefore N is a maximal submodule, and $N \underset{e}{\leq} M$ with $m \notin N$.

Proposition (3.2):

Let M be an R-module, let $K \leq N \leq M$ be submodules of M. If $K \underset{e}{\ll} M$ and $N \leq^{\oplus} M$, then $K \underset{e}{\ll} N$.

Proof:

Since $N \leq^{\oplus} M$, then $M = N \oplus W$ for some $W \leq M$. To prove $K \underset{e}{\ll} N$. Assume $N = K + U$ for some $U \underset{e}{\leq} N$. Then $M = (K + U) \oplus W = K + (U \oplus W)$. We claim that $(U \oplus W) \underset{e}{\leq} M$. To see this : let $m \in M$ and $m \neq 0$. As $M = N \oplus W$, $m = n + w$ for some $n \in N$, $w \in W$. If $n \neq 0$, then there exists $r \in R\backslash\{0\}$ such that $0 \neq rn \in U$, hence $rm = rn + rw \neq 0$ (because if $rm = 0$, then $rn = -rw \in N \cap W = (0)$ and hence $rn = 0$ which is a contradiction). Thus $0 \neq rm \in (U \oplus W)$.

Now if $n = 0$, then $m = w$ and so $0 \neq 1\cdot m = 1\cdot w \in W \subseteq (U \oplus W)$. Therefore $(U \oplus W) \underset{e}{\leq} M$. Since $K \underset{e}{\ll} M$, we get $U \oplus W = M$. Now, assume $x \in N \leq M$, so that $x = u_1 + w_1$ for some $u_1 \in U$, $w_1 \in W$. It follows that $x - u_1 = w \in (N \cap W) = (0)$, hence $x = u_1 \in U$. Thus $N = U$ and $K \underset{e}{\ll} N$.

Recall that a submodule N of an R-module M is called coclosed whenever $K \le N$, $\frac{N}{K} \ll \frac{M}{K}$ implies $N = K$ [5], [6].

Hasan in [7], gave the following definition:

Definition (3.3):
Let N be a submodule of an R-module M. N is called **e-coclosed** if whenever $K \le N$, $\frac{N}{K} \ll_e \frac{M}{K}$, then $N = K$.

Remarks (3.4):
(1) It is known that every direct summand is coclosed. However a direct summand may not be e-coclosed for example:

Let M be the Z-module Z_6, let $N = <\bar{2}> \le^{\oplus} M$. $\frac{N}{<\bar{0}>} \ll_e \frac{M}{(\bar{0})}$, but $N \ne <\bar{0}>$.

(2) It is clear that every e-coclosed submodule is coclosed, but the converse is true by the same example in (1), N is coclosed and it is not e-coclosed.

Proposition (3.5): [7,Lemma 4.2.8]
Let A be a submodule of an R-module M. If A is e-coclosed, then for each $X \le A$, $X \ll_e M$ implies $X \ll_e A$.

Proof:
To prove $X \ll_e A$. Assume $A = X + Y$ for some $Y \le_e A$. We claim that $\frac{A}{Y} \ll_e \frac{M}{Y}$. To see this, let $\frac{M}{Y} = \frac{A}{Y} + \frac{C}{Y}$ for some $\frac{C}{Y} \le_e \frac{M}{Y}$. Then $M = A + C$, so $M = X + Y + C$ implies $M = X + C$. Since $\frac{C}{Y} \le_e \frac{M}{Y}$, we have $C \le_e M$. Hence $C = M$ since $X \ll_e M$. This implies $\frac{M}{Y} = \frac{C}{Y}$ and $\frac{A}{Y} \ll_e \frac{M}{Y}$. But A is e-coclosed in M, so that $Y = A$. Thus $X \ll_e A$.

Proposition (3.6):
Let M be a non singular R-module. A proper submodule N of M is e-small if and only if it is δ-small.

Proof:
(⇐) it is clear by Remark (2.2).

(⇒) Let $N < M$. Assume $N + K = M$ with $\frac{M}{K}$ is singular. Since M is nonsingular, then by [8, Proposition 1.21,p.32], $K \le_e M$. But $N \ll_e M$, so $K = M$. Thus $N \ll_\delta M$.

Proposition (3.7):
Let M be an indecomposable R-module. A proper submodule N of M is small if and only if it is e-small.

Proof:
(⇒) It is clear by Remark (2.2).

($\Leftarrow$) Let N < M. Assume N + K = M with K ≤ M. Since $N \underset{e}{\ll} M$, then by Proposition (2.3), $K \leq^{\oplus} M$ and $\frac{M}{X}$ is semisimple. But M is indecomposable and K ≠ (0), so K = M. Thus $N \ll M$.

Now we get the following corollaries.

Corollary (3.8):

Let M be a an indecomposable R-module and let N < M. The following statements are equivalent:

(1) $N \ll M$.

(2) $N \underset{\delta}{\ll} M$.

(3) $N \underset{e}{\ll} M$.

Since every uniform module is indecomposable we have the following result which follows directly by Corollary (3.8).

Corollary (3.9):

Let M be uniform R-module and let N < M. Then the following statements are equivalent:

(1) $N \ll M$.

(2) $N \underset{\delta}{\ll} M$.

(3) $N \underset{e}{\ll} M$.

Recall that for an R-module M, if M has maximal submodule. Then $\underset{e}{\text{Rad}}\ M = \cap\{N \underset{e}{<} M \mid N \text{ is maximal in } M\}$, and if M has no maximal submodule, $\underset{e}{\text{Rad}}\ M = M$, [4].

The following is a characterization of $\underset{e}{\text{Rad}}\, M$.

Theorem (3.10): [4,Theorem 2.10]

Let M be an R-module. Then $\underset{e}{\text{Rad}}(M) = \sum\{N \leq M \mid N \underset{e}{\ll} M\}$.

Corollary (3.11): [4,Corollary 2.11]

Let M and N be R-modules.

(1) If $f : M \longrightarrow N$ is an R-homeomorphism, then $f\,(\underset{e}{\text{Rad}}(M)) \subseteq \underset{e}{\text{Rad}}(N)$.

(2) If every proper essential submodule of M is contained in a maximal submodule of M, then $\underset{e}{\text{Rad}}(M)$ is the largest e-small submodule of M.

Recall that an R-module M is called multiplication if for each $N \leq M$, there exists an ideal I of R such that $N = IM$. Equivalently M is multiplication if for each $N \leq M$, $N = (N:M)M$, where $(N \underset{R}{:} M) = \{r \in R: rM \subseteq N\}$, [9].

Corollary (3.12):

Let M be a finitely generated or multiplication R-module. Then $\underset{e}{\text{Rad}}(M)$ is the largest e-small submodule of M.

Proof:

Since M is finitely generated or multiplication, then every proper submodule is contained in maximal submodule. Hence the result is followed by Corollary (3.11).

Proposition (3.13):

Let M be an module, let $m \in M$ then $Rm \underset{e}{\ll} M$ if and only if $m \in \underset{e}{\text{Rad}}(M)$.

Proof:

Suppose $Rm \underset{e}{\ll} M$, then $Rm \leq \underset{e}{\text{Rad}}\, M$, hence $m \in \underset{e}{\text{Rad}}(M)$. Conversely, let $m \in \underset{e}{\text{Rad}}(M)$. Assume $\underset{e}{\text{Rad}}(M) \neq M$. Suppose $Rm \underset{e}{\not\ll} M$, then by Proposition(3.1), there exists an essential maximal submodule N in M and $m \notin N$. Hence $m \notin \underset{e}{\text{Rad}}(M)$ which is a contradiction. Thus $Rm \underset{e}{\ll} M$.

If $\underset{e}{\text{Rad}}(M) = M$, then M has no essential maximal submodule. Hence for each $m \in M$, $Rm \underset{e}{\ll} M$ (by Proposition (3.1)).

Proposition (3.14):

An arbitrary sum of e-small submodules of a module M is an e-small submodule of M if and only if $\underset{e}{\text{Rad}}(M) \underset{e}{\ll} M$.

Proof:

($\Rightarrow$) Since $\underset{e}{\text{Rad}}(M) =$ the sum of all e-small submodules (by Theorem (3.10)), $\underset{e}{\text{Rad}}(M) \underset{e}{\ll} M$.

($\Leftarrow$) Suppose $\underset{e}{\text{Rad}}(M) \underset{e}{\ll} M$. Let $\{K_\alpha\}_{\alpha\in\Lambda}$ be a family of e-small submodules of M. $\sum_{\alpha\in\Lambda} K_\alpha \leq \underset{e}{\text{Rad}}(M) \underset{e}{\ll} M$. Therefore $\sum_{\alpha\in\Lambda} K_\alpha \underset{e}{\ll} M$ by Proposition (2.5 (a)).

Proposition (3.15):

Let M be an R-module. Then $\underset{e}{\text{Rad}}(M) = M$ if and only if all finitely generated submodules are e-small submodules of M.

Proof:

($\Rightarrow$) Suppose $\underset{e}{\text{Rad}}(M) = M$ and Let N be a finitely generated submodule of M. Hence $N = Rx_1 + \ldots + Rx_n$ where $x_1, \ldots, x_n \in M = \underset{e}{\text{Rad}}(M)$, then by Proposition (3.13), $Rx_i \underset{e}{\ll} M$ and by Proposition (2.5(1.b)) $N \underset{e}{\ll} M$.

($\Leftarrow$) Let m $\in$ M. Then <m> = Rm is finitely generated, so by hypothesis, $Rm \underset{e}{\ll} M$ and hence <m> $\subseteq \underset{e}{\text{Rad}}(M)$. Thus $m \in \underset{e}{\text{Rad}}(M)$.

Next (3.16):

It is known that for a module M, if $\text{Rad}(M) \ll M$ then M/Rad M has no nonzero small submodulele. However this statement can not be generalized for $\underset{e}{\text{Rad}}(M)$,as the following example shows.

Example (3.17):

Consider the Z-module Z24 $\underset{e}{\text{Rad}}(Z_{24}) = <\bar{2}> \underset{e}{\ll} Z_{24}$. But $\dfrac{Z_{24}}{<\bar{2}>} \simeq Z_2$ and $Z_2 \underset{e}{\ll} Z_2$.

Proposition (3.18):

Let M be a faithful finitely generated multiplication R-module, let N < M. Then the following statements are equivalent $N \underset{e}{\ll} M$ if and only if $(N{:}M) \underset{e}{\ll} R$.

Proof:

($\Rightarrow$) Assume (N:M) + K = R with $K \underset{e}{\leq} R$. Then (N:M)M + KM = M, thus N + KM = M. But $K \underset{e}{\leq} R$, so by [9,theorem 2.13], $KM \underset{e}{\leq} M$ and since $N \underset{e}{\ll} M$, we get KM = M. Therefore K = R by [9,theorem 3.1].

($\Leftarrow$) Assume N + K = M with $K \underset{e}{\leq} M$. Since M is multiplication N = (N:M)M, K = (K:M)M, and $(K \underset{R}{:} M) \underset{e}{\leq} R$ by [1,theorem 2.13]. Thus (N:M)M + K:M)M = M and since M is a finitely generated faithful multiplication R-module, then (N:M) + (K:M) = R. As $(N{:}M) \underset{e}{\ll} R$ and $(K : M) \underset{e}{\leq} R$, we have (K:M) = R. It follows that K = M, and $N \underset{e}{\ll} M$.

Corollary (3.19):

Let M be a faithful finitely generated multiplication R-module, let N < M. The following statements are equivalent:

(1) $N \underset{e}{\ll} M$.

(2) $(N{:}M) \underset{e}{\ll} R$.

(3) N = IM for some $I \underset{e}{\ll} R$.

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حول المقاسات الجزئية الصغيرة من النمط -e

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استلم البحث في:3/حزيران/2015، قبل البحث في:20/أيلول/2015

الخلاصة

ليكن M مقاساً على R، إذ R حلقة ابدالية ذات محايد. المقاس N في M يسمى مقاساً جزئياً من النمط – e (يرمز له بالرمز $N \underset{e}{\ll} M$) اذا كان $N + K = M$، إذ $K \underset{e}{\leq} M$ تؤدي الى $K = M$. اعطينا العديد من الخواص المتعلقة لهذا النمط من المقاسات الجزئية.

الكلمات المفتاحية: مقاس جزئي صغير، مقاس جزئي صغير من النمط - δ، مقاس جزئي صغير من النمط – e، و مقاس جزئي ضد مغلق من النمط - e.