On 2-Absorbing Submodules

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Abstract

Let R be a commutative ring with $1 \neq 0$ and M is a unitary R-module . In this paper , our aim is to continue studying 2-absorbing submodules which are introduced by A.Y. Darani and F. Soheilina . Many new properties and characterizations are given .

Key words: prime submodule , 2-absorbing submodule , quasi- prime submodule, multiplication module , P-primary submodule , pure submodule .

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Introduction

The concept of 2-absorbing ideal which is a generalization of prime ideal was introduced by Ayman Badawi , where "a nonzero proper ideal I of R is called a 2-absorbing ideal of R if whenever a , b , c ∈R , abc ∈ I then ab ∈ I or ac ∈I or bc ∈I" , [1] . This definition can obviously be made for any proper ideal . A. Y. Darani and F. Soheilnia in [2] introduced the concept of 2-absorbing submodule where "a proper submodule N of M is called 2-absorbing submodule of M if whenever a , b ∈ R , m∈ M and abm ∈N , then am ∈N or bm ∈N or ab ∈(N:M)"

Our concern in this paper is to give a comprehensive study of 2-absorbing submodule , where we give many new properties and characterizations .

1. 2-Absorbing Submodules – Basic Properties

"Definition 1.1:

Let M be an R-module. N a proper submodule of M is called 2-absorbing submodule if whenever a,b ∈ R , m ∈ M and abm ∈ N, then ab ∈ $(N :_R M)$ or am ∈ N or bm ∈ N."[2]

Note that an 2-absorbing ideal of a ring R is a 2-absorbing submodule of the R-module R .

Remarks and Examples.1.2 :

(1) "The intersection of each pair of distinct prime submodules of R-module M is 2-absorbing" [2]

(2) It is clear that every prime submodule is 2-absorbing. However the converse is not true in general, for example:

Consider Z_6 as Z-module , $(\bar{0})$ not prime submodule of Z_6 since $2 .\bar{3} = \bar{0} \in (\bar{0})$ but $\bar{3} \notin (\bar{0})$ and $2 \notin ((\bar{0}) :_Z Z_6)=6Z$ but $(\bar{0}) =(\bar{2}) \cap (\bar{3})$ is 2-absorbing submodule of Z_6 as Z-module by part (1)

(3) It is clear that every quasi-prime submodule is 2-absorbing, where" a proper submodule N of M is called quasi-prime submodule if whenever a, b ∈ R, m∈ M and a b m∈ N , then a m∈ N or b m∈ N"[3]

However a 2-absorbing submodule may not be quasi-prime , as the following example shown :

Consider the Z-module Z . The submodule N =4Z is a 2-absorbing submodule of Z since , if a,b,c ∈Z with abc ∈4Z=N , then at least two of a,b,c are even . Hence either ab ∈ N or ac∈ N or bc ∈ N

But 4Z is not quasi-prime , since 2.2.1 ∈4Z but 2.1∉4Z.

(4) Let N ,W be two submodules of an R-module M and W < N .if N is 2-absorbing submodule of M then it is not necessary that W is 2- absorbing submodule of M , for example in Z_{24} as Z-module . Take $N=(\bar{6})$, $W=(1\bar{2})$. Since $N = (\bar{2}) \cap (\bar{3})$ and both of them are prime submodules then N is 2-absorbing submodule by part (1) . But $2.2.\bar{3} \in W$, $2.\bar{3} \notin W$ and $2.2=4 \notin (W :_Z Z_{24}) =12Z$. Thus W is not 2-absorbing submodule in M .

(5) Let N ,W be two submodules of an R-module M and $N < W$. If N is 2-absorbing submodule of M , then N is a 2-absorbing submodule of W

Proof:

If W =M ,then nothing to prove

Let $a\,b\,x \in N$, where $a ,b \in R$, $x \in W$. Since $x \in W$ then $x \in M$. But N is 2-absorbing submodule of M , so either : $a\,x \in N$ or $b\,x \in N$ or $a\,b \in (N:M)$ and since $N < W$ implies $(N:M) \leq (N:W)$, then either $a\,x \in W$ or $b\,x \in W$ or $a\,b \in (N:W)$

Hence N is 2-absorbing in W ■

(6) The sum of 2-absorbing submodules is not necessary 2-absorbing . for example:
Let $N1 = 2Z$, $N2 = 3Z$, each of N_1 and N_2 is 2-absorbing submodule in the Z-module Z , but $N_1 + N_2 = Z$ which is not 2-absorbing .

(7) Let N and W be two submodules of an R-module M such that $N \cong W$
If N is 2-absorbing submodule , it is not necessary that W is 2-absorbing submodule as the following example explains this :

Consider the Z-module Z , the submodule 2Z is 2-absorbing submodule but $2Z \cong 30Z$ and $30Z$ is not 2-absorbing since $2.3.5 = 30 \in 30$ but $2.5 \notin 30Z$ and

$3.5 \notin 30Z$ and $2.3 = 6 \notin 30Z$

(8) The intersection of two 2-absorbing submodule need not be 2-absorbing submodule for example : 6Z and 5Z are 2-absorbing submodule in the Z-module Z , but $6Z \cap 5Z = 30Z$ which is not 2-absorbing

(9) Let N be a 2-absorbing submodule of M , then for each $A \subseteq M$, either $A \subseteq N$ or $A \cap N$ is a 2-absorbing submodule of A

Proof :

Suppose $A \nsubseteq N$. Then $A \cap N \lneqq A$.

Let $abx \in A \cap N$, and $x \in A$, $a,b \in R$. So $abx \in N$. Since N is 2-absorbing , either $ax \in N$ or $bx \in N$ or $ab \in (N:M)$, then $ax \in A \cap N$ or $bx \in A \cap N$ or $ab \in (A \cap N:A)$.

Proposition .1.3 :

Let $\varphi : M \longrightarrow M'$ be an R-epimorphism . If W is 2-absorbing submodule of M' , then $\varphi^{-1}(W)$ is 2-absorbing submodule of M.

Proof :

It is straight for word so it is omitted ■

Proposition 1.4 :

Let $f : M \longrightarrow M'$ be an epimorphism , $N < M$ such that $\ker f \subseteq N$, then N is 2-absorbing submodule of M if and only if f(N) is 2-absorbing submodule of M'

Proof:

($\Rightarrow$)Let $ab\acute{m} \in f(N)$, where $\acute{m} \in M'$ a,b $\in$ R . $\acute{m} = f(m)$ for some m $\in$ M , since f is onto .

Then abf(m) $\in$ f(N) , so abf(m)=f(n) for some n $\in$ N and hence f(abm)- f(n) = 0 . Thus we get that abm – n $\in$ kerf $\subseteq$ N which implies that abm $\in$ N . But N is 2-absorbing so either am $\in$ N or bm $\in$ N or ab $\in$ $(N :_R M)$.

If am $\in$ N , then f(am) $\in$ f(N) , that is a f(m) $\in$ f(N) so a$\acute{m}$ $\in$ f(N) .

Similarly, bm $\in$ N implies that b$\acute{m}$ $\in$ f(N) .

If ab $\in$ $(N :_R M)$, then abM $\subseteq$ N and so f(abM) $\subseteq$ f(N) which implies that $abM' \subseteq f(N)$ and we get that ab $\in$ $(f(N) :_R M')$.

($\Leftarrow$) Let abm $\in$ N then f(abm) $\in$ f(N) so abf(m) $\in$ f(N) . Since f(N) is 2-absorbing either af(m) $\in$ f(N) or bf(m) $\in$ f(N) or ab $\in$ $(f(N) :_R M')$.

1) If af(m) $\in$ f(N) then f(am) = f(n) for some n $\in$ f(N) , henc am-n $\in$ kerf $\subseteq$ N so am$\in$ N

2)If bf(m) $\in$ f(N) then Similarly that bm $\in$ N

3) If ab $\in$ $(f(N) :_R M')$ then $abM' \subseteq f(N)$ so abf(x) $\in$ f(N) for each x $\in$ M

so that f(abx)=f(n) for some n $\in$ N and hence abx $\in$ N for each x $\in$ M. Thus ab $\in$ $(N :_R M)$. ■

By using Proposition 1.4 we can get the following result which is given in [2] as a direct consequence

"**Corollary 1.5 :**

Let R be a ring , M an R- module and N , K submodules of M with K $\subseteq$N . Then N is a 2-absorbing submodule of M if and only if $\frac{N}{K}$ is a 2-absorbing submodule of $\frac{M}{K}$ "

A.Y. Darani and F. Soheilina in [2] introduced the following:

"**Proposition 1.6 :**

Let R be a commutative ring , M is a cyclic R-module and N is a submodule of M . Then N is a 2-absorbing submodule of M if and only if $(N :_R M)$ is a 2-absorbing ideal of R".

Sh. Payrovi and S. Babaei in [4] and S. moradi and A. Azizi in [5] introduced the following:

Theorem 1.7 :

"If N is a 2-absorbing submodule of M , then $(N :_R M)$ is a 2-absorbing ideal of R" .

Sh . Payrovi , Babaei in [4] proved the following " Let R be a Noetherian ring , M a finitely generated multiplication R-module and N a proper submodule of M such that $Ass_R(M/N)$ is a totally ordered set . If $(N :_R M)$ is a 2-absorbing ideal of R, then N is a 2-absorbing submodule of M " , where R is a commutative ring with non zero identity .

However we get the same conclusion under the class of multiplication modules . Before giving our result , recall that "An R-module is called a multiplication module if for every submodule N of M, there exists an ideal I of R such that IM = N . Equivalently , M is a multiplication module if for every submodule N of M ,N=(N $:_R$M)M" . [6]

Theorem 1.8 :

Let M be a multiplication R-module and N is a proper submodule of M , If $(N :_R M)$ is a 2-absorbing ideal of R , then N is 2-absorbing submodule of M

Proof:

Let abm ∈ N where a,b ∈ R , m ∈ M then ab(m)⊆ N .

But (m)= IM for some I ≤ R since M is a multiplication R-module , so abIM ⊆ N .

Hence abI ⊆ $(N :_R M)$,so we get that (a)(b)I ⊆ $(N :_R M)$. Since $(N :_R M)$ is a 2-absorbing ideal , therefore (a)I ⊆ $(N :_R M)$ or (b)I ⊆ $(N :_R M)$ or (a)(b) ⊆ $(N :_R M)$ by [1]

1) If (a)I ⊆ $(N :_R M)$, then (a)IM ⊆ N so (a)(m)⊆ N thus am ∈ N
2) If (b)I ⊆ $(N :_R M)$, then similarly bm ∈ N.
3) If (a)(b) ⊆ $(N :_R M)$, then ab ∈ $(N :_R M)$ ■

Corollary 1.9 :

Let M be a multiplication R-module and N is a proper submodule of M , then N is 2-absorbing submodule of M if and only if $(N :_R M)$ is 2-absorbing ideal.

Remark 1.10 :

The condition M is a multiplication R-module can't be dropped from Theorem 1.8 .

Consider the following example :

Let M be the Z-module Z_{P^∞} and N = $(\bar{0})$. N is not 2-absorbing since :

$p.p.\left(\frac{1}{p^2}+Z\right)=\bar{0}$ but $p.\left(\frac{1}{p^2}+Z\right)\neq\bar{0}$ and $P^2 \notin (\frac{1}{p^2}+Z :_R Z) = (\bar{0})$, Also notice that $(N: Z_{P^\infty})= (0)$, and (0) is a prime ideal in Z , so (0) is 2-absorbing ideal in Z .

Recall that " A proper submodule N of an R-module M is called a P-primary submodule of M if whenever a ∈ R and m ∈M and am ∈ N , then m ∈ N or a ∈ $\sqrt{N : M}$=P " [7]

Darani in [7] proved the following :

" Let N be a P-primary submodule of a cyclic R-module. Then N is 2-absorbing if and only if $(pM)^2 \subseteq N$ "

However we improve this Theorem as follows .

Proposition 1.11 :

Let N be a p-primary submodule of a multiplication R-module M . Then N is 2-absorbing if and only if $(pM)^2 \subseteq N$ (where $(pM)^2 = p^2M$) .

Proof :

(⟹) Since M is a multiplication R-module , M-radN=$\sqrt{N:M}$ M by[7,Th 2.10]

But N is P-primary , so P =$\sqrt{N:M}$ is a prime ideal . ; that is M-radN=PM .

It follows that $P^2 \subseteq$ (N:M) by [1] Th. 2.4 . Thus $P^2M \subseteq N$.

Hence $(PM)^2 \subseteq N$

(⟸) Let abm∈N where a , b ∈R , m∈M

Assume that am∉N and bm∉N . As N is primary with abm∈N and bm∉N , we get $a \in \sqrt{N:M}$=P

Also abm∈N and am∉N , we get $b \in \sqrt{N:M}$=P

Thus $ab \in P^2 \subseteq$ (N:M) and consequently N is 2-absorbing submodule of M ■

Recall that " A proper submodule N of an R-module M is called a 2-absorbing primary submodule of M if whenever a ; b ∈ R and m ∈M and abm ∈ N , then am ∈ M-rad(N) or bm ∈ M-rad(N) or ab ∈ $(N :_R M)$" [8]

It is clear that 2-absorbing submodule implies 2-absorbing primary , but the converse may be not hold , as the example shows

Let M be the Z-module Z , let N=8Z , so N is 2-absorbing primary ,but it is not 2-absorbing submodule .

Babaei in [4] proved the following :

"Let R be a Noetherian ring , I a 2-absorbing ideal of R , and M a faithful multiplication R-module such that $Ass_R(M/IM)$ is a non-empty totally ordered set . Then abm ∈ IM implies that am ∈IM or bm ∈ IM or ab ∈I whenever a; b ∈ R and m ∈ M " [4] , that IM is 2-absorbing . where R is a commutative ring with nonzero identity .

However we get the same conclusion of this theorem but with less conditions and also we give a simple proof .

Proposition 1.12 :

Suppose M is a finitely generated multiplication R-module . If I is 2-absorbing ideal of R such that annM ⊆I , then IM is 2-absorbing submodule of M.

Proof:

Let abm ∈ IM where a , b ∈ R , m ∈ M , hence ab(m) ∈ IM . Since M is multiplication , then (m)=JM for some ideal J of R . Thus abJM ⊆ IM and so abJ ⊆ I+annM = I . But I is a 2-absorbing ideal of R , so either ab ∈ I or aJ ⊆ I or bJ ⊆ I , it follows that ab ∈ $(IM :_R M)$ or aJM ⊆ IM or bJM ⊆ IM ; that is either ab ∈ $(IM :_R M)$ or a(m) ⊆ IM or b(m) ⊆ IM Thus ab ∈ $(IM :_R M)$ or am ∈ IM or bm ∈ IM and so IM is a 2-absorbing submodule of M ■

Corollary 1.13 :

Suppose M is a faithful finitely generated multiplication R-module . If I is a 2-absorbing ideal of R, then IM is a 2-absorbing submodule of M.

Proof:

It follows directly by Proposition 1.12 ■

Corollary 1.14 :
Let M be a faithful finitely generated multiplication R-module . Then every proper submodule of M is 2-absorbing if and only if every proper ideal of R is 2-absorbing .
Proof :

($\Leftarrow$) It follows directly by Corollary (1.13) .

($\Rightarrow$) Let I be a proper ideal of R . Then N= IM is a proper submodule of M

So it is 2-absorbing and hence by Theorem (1.7) , (N:M) is 2-absorbing ideal . But M is faithful finitely generated multiplication R-module , so (N:M)=I by ([6],Th.3.1) ■

" **Proposition 1.15 :** [4,Th 2.4]
Let M be an R-module , N a proper submodule of M , if N is 2-absorbing then $(N:_R(m))$ is 2-absorbing ideal for each $m \in M-N$ "
Proof:

First $(N:_R(m)) \neq R$ for any $m \notin N$

Let $abc \in (N:_R(m))$ then $ab(cm) \in N$, but N is 2-absorbing submodule then

$a(cm) \in N$ or $b(cm) \in N$ or $ab \in (N:_R M)$, so that $acm \in N$ or $bcm \in N$ or $abM \subseteq N$ that is $ac \in (N:_R(m))$ or $bc \in (N:_R(m))$ or $ab \in (N:_R(m))$ hence $(N:_R(m))$ is 2-absorbing ideal ■

Recall that "a submodule N of M is called a pure submodule of an R-module M if $IM \cap N = IN$ for any ideal I of R" [9]

Proposition 1.16 :

Let N be a proper pure submodule of an R-module M , If (0) is a 2-absorbing submodule of M , then N is 2-absorbing .

Proof:

Let $abm \in N$ where $a, b \in R$, $m \in M$

Put I =(ab) then $abm \in IM \cap N$, but $IM \cap N = IN$, so $abm = abn$ for some $n \in N$, then $ab(m-n) = 0$, but (o) is 2-absorbing then $a(m-n) = 0$ or $b(m-n) = 0$ or $ab \in annM \subseteq (N:M)$. So we get $am=an \in N$ or $bm=bn \in N$ or $ab \in (N:M)$,

Thus N is a 2-absorbing submodule ■

2. 2-absorbing Submodules Characterizations .

In this section we give some characterizations of 2-absorbing submodules . We start with the following proposition:

Proposition 2.1 :

Let N a proper submodule of an R-module M . Then N is 2-absorbing submodule of M if and only if $abK \subseteq N$ for some a , b $\in R$, $K \leq M$. implies $ab \in (N:M)$, or $aK \subseteq N$ or $bK \subseteq N$

Proof :

Suppose that $ab \notin (N:M)$ and $aK \nsubseteq N$ and $bK \nsubseteq N$. Then there exist m_1 , m_2 in K such that $am_1 \notin N$ and $bm_2 \notin N$.

Since $abm_1 \in N$ and $ab \notin (N:M)$, $am_1 \notin N$, we get $bm_1 \in N$.

Also since $abm_2 \in N$ and $ab \notin (N:M)$, $bm_2 \notin N$, we get $am_2 \in N$

Now , since $ab(m_1 + m_2) \in N$ and $ab \notin (N:M)$ we have $a(m_1 + m_2) \in N$ or $b(m_1 + m_2) \in N$

If $a(m_1 + m_2) \in N$; i.e. $am_1 + am_2 \in N$ and since $am_2 \in N$ we get $am_1 \in N$ which is contradiction !

If $b(m_1 + m_2) \in N$; i.e. $bm_1 + bm_2 \in N$ and since $bm_2 \in N$ we get $bm_1 \in N$ which is contradiction !

Then either $ab \in (N:M)$ or $aK \subseteq N$ and $bK \subseteq N$

the converse is clear ■

The following theorem give a useful characterization of 2-absorbing submodule .

Theorem 2.2 :

Let N a proper submodule of an R-module M , then the following statement are equivalent :

(1) N is a 2-absorbing submodule of M
(2) If $IJK \subseteq N$, for some ideal I and J of R and some submodule K of M then either $IK \subseteq N$ or $JK \subseteq N$ or $IJ \subseteq (N:M)$.

Proof : (1) $\Rightarrow$ (2)

Suppose N is a 2-absorbing submodule of M and $IJK \subseteq N$ for some ideals I and J of R and some submodule K of M and $IJ \nsubseteq (N:M)$.

To show that $IK \subseteq N$ or $JK \subseteq N$.

Suppose $IK \nsubseteq N$ and $JK \nsubseteq N$. then there exist $a_1 \in I$ and $a_2 \in J$ such that $a_1 K \nsubseteq N$ and $a_2 K \nsubseteq N$.

But $a_1 a_2 K \subseteq N$ and neither $a_1 K \nsubseteq N$ nor $a_2 K \nsubseteq N$ and N is 2-absorbing , so we have $a_1 a_2 \in (N:M)$ by Proposition (2.1)

Since $IJ \nsubseteq (N:M)$, then there exist $b_1 \in I$ and $b_2 \in J$ such that $b_1 b_2 \notin (N:M)$.

But $b_1 b_2 K \subseteq N$, so we have $b_1 K \subseteq N$ or $b_2 K \subseteq N$ by Proposition (2.1)

Now we have the following cases :

Case (1) $b_1 K \subseteq N$ and $b_2 K \nsubseteq N$

Since $a_1b_2K \subseteq N$ and $b_2K \nsubseteq N$ and $a_1 K \nsubseteq N$ so that $a_1 b_2 \in(N:M)$ by Proposition (2.1)

Since $b_1K \subseteq N$ and $a_1K \nsubseteq N$, we conclude $(a_1+b_1) K \nsubseteq N$. On the other hand , $(a_1+b_1) b_2 K \subseteq N$ and neither $(a_1+b_1) K \subseteq N$ nor $b_2K \subseteq N$, we get that $(a_1+b_1) b_2 \in(N:M)$ by Proposition (2.1)

But $((a_1+b_1) b_2 = a_1b_2+b_1b_2 \in(N:M)$ and $a_1b_2 \in(N:M)$, we get $b_1b_2 \in(N:M)$ which is a contradiction!

Case(2) If $b_2K \subseteq N$ and $b_1K \nsubseteq N$. By a similar argument of case (1), we reach to a contradiction!

Case(3) $b_1K \subseteq N$ and $b_2K \subseteq N$

Since $b_2K \subseteq N$ and $a_2K \nsubseteq N$, we conclude $(a_2+b_2) K \nsubseteq N$. But $a_1(a_2+b_2) K \subseteq N$ and neither $a_1K\subseteq N$ nor $(a_2+b_2)K \subseteq N$, hence $a_1(a_2+b_2)\in(N:M)$ by Proposition (2.1)

Since $a_1 a_2\in(N:M)$ and $a_1 a_2 + a_1 b_2 \in(N:M)$, we have $a_1 b_2 \in(N:M)$. Since $(a_1+b_1) a_2 K \subseteq N$ and neither $a_2 K \subseteq N$ nor $(a_1+b_1)K \subseteq N$, we conclude $(a_1+b_1) a_2\in(N:M)$ by Proposition (2.1)

But $(a_1+b_1) a_2 = a_1a_2+b_1 a_2$, so $a_1a_2+b_1 a_2 \in(N:M)$ and since $a_1 a_2\in(N:M)$, we get $b_1a_2 \in(N:M)$. Now , since $(a_1+b_1) (a_2+b_2)K \subseteq N$ and neither $(a_1+b_1)K \subseteq N$ nor $(a_2+b_2)K \subseteq N$,

We have $(a_1+b_1) (a_2+b_2) =a_1a_2+ a_1b_2+b_1a_2+b_1b_2 \in(N:M)$ by Proposition (2.1)

But $a_1a_2 , a_1b_2 , b_1a_2 \in(N:M)$, so $b_1b_2 \in(N:M)$ which is a contradiction! . Consequently $I_1K \subseteq N$ or $I_2K \subseteq N$

(2) $\Longrightarrow$ (1) It is clear ■

Recall that ". for any two submodules N , K of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1I_2M$."[2]

By using this definition of product of submodules , we give the following characterization of 2-absorbing submodules .

Theorem 2.3 :

Let N be a proper submodule of a multiplication R-module M , then

N is a 2-asorbing submodule of M if and only if $N_1 N_2 N_3 \subseteq N$ implies that $N_1 N_2 \subseteq N$ or $N_1 N_3 \subseteq N$ or $N_2 N_3 \subseteq N$, where N_1 , N_2 , N_3 are submodules of M.

Proof :

($\Longrightarrow$)Since M is multiplication , then $N_1= I_1M$, $N_2= I_2M$, $N_3= I_3M$ for some ideals I_1 , I_2 , I_3 of R . It follows that :

$N_1 N_2 N_3 = I_1 I_2 I_3M \subseteq N$. Hence $I_1 I_2 I_3 \subseteq (N:M)$. But N is 2-absorbing submodule of M implies that (N:M) is 2-absorbing ideal by [Theorem 1.7].

So by [1 ,Th 2.13] either $I_1 I_2 \subseteq (N:M)$ or $I_1 I_3 \subseteq (N:M)$ or $I_2 I_3 \subseteq (N:M)$. Then $I_1 I_2 M \subseteq N$ or $I_1 I_3M \subseteq N$ or $I_2 I_3M \subseteq N$. Thus $N_1 N_2 \subseteq N$ or $N_1 N_3 \subseteq N$ or $N_2 N_3 \subseteq N$

($\Leftarrow$) Let $I_1 I_2 K \subseteq N$ where I_1, I_2 are ideals of R and $K \leq M$. Since M is multiplication module, $K = JM$ for some J of R. Thus $I_1 I_2 J M \subseteq N$. Put $N_1 = I_1 M$ $N_2 = I_2 M$.

It follows that $N_1 N_2 K = I_1 I_2 J M \subseteq N$. So by hypotheses either $N_1 K \subseteq N$ or $N_2 K \subseteq N$ or $N_1 N_2 \subseteq N$ then $I_1 J M \subseteq N$ or $I_2 J M \subseteq N$ or $I_1 I_2 \subseteq (N:M)$; that is $I_1 K \subseteq N$ or $I_2 K \subseteq N$ or $I_1 I_2 \subseteq (N:M)$

Thus N is 2-absorbing submodule of M ■

Proposition 2.4 :

Let N be a proper submodule of an R-module M. The following statements are equivalent :

(1) N is a 2-absorrbing submodule of M
(2) $(N :_M I)$ is 2-absorbing, for each ideal I of R with $IM \not\subseteq N$
(3) $(N :_M (r))$ is 2-absorbing submodule for each $r \in R$ with $rM \not\subseteq N$

Proof:

(1) $\Rightarrow$(2) Let I be an ideal of R with $IM \not\subseteq M$, then $(N :_M I)$ is a proper submodule of M

Let $a\, b\, m \in (N :_M I)$, where $a, b \in R$, $m \in M$. Then $ab(Im) \subseteq N$.

But N is 2-absorbing submodule of M, so by Proposition(2.1), either $a(Im) \subseteq N$ or $b(Im) \subseteq N$ or $ab \in (N :_R M) \subseteq ((N :_M I): M)$. Hence either $am \in (N :_M I)$ or $bm \in (N :_M I)$ or $ab \in ((N :_M I): M$. Thus $(N :_M I)$ is a 2-absorbing submodule.

(2) $\Rightarrow$(3) It is clear.

(3) $\Rightarrow$(1) Take $r = 1$ then $(N :_M (1)) = N$, so N is 2-absorbing. ■

Now we have the following :

Theorem 2.5 :

Let N be a proper submodule of an R-module M. Consider the following statements:

1) N is a 2-absorbing submodule of M
2) For each $a,b \in R$, $m \in M$. If $abm \notin N$, then $(N :abm) = (N:am) \cup (N:bm)$
3) For each $a,b \in R$, $m \in M$ if $abm \notin N$ then $(N:abm) = (N :am)$ or $(N:abm) = (N:bm)$

Then (1) $\Rightarrow$(2) $\Rightarrow$(3) and if M is cyclic, then (3) $\Rightarrow$(1).

Proof : (1) $\Rightarrow$(2)

Let $c \in (N : abm)$ then $abcm \in N$. Since $abm \notin N$ and N is 2-absorbing, so by (1.2), either $acm \in N$ or $bcm \in N$. Then $c \in (N : am)$ or $c \in (N : bm)$

Thus $(N : abm) \subseteq (N : am) \cup (N : bm)$. To prove reverse inclusion:

Let $c \in (N : am) \cup (N : bm)$, then $c \in (N : am)$ or $c \in (N : bm)$ so that $acm \in N$ or $bcm \in N$

It follows that $abcm \in bN \subseteq N$ or $abcm \in aN \subseteq N$

Thus $abcm \in N$ and hence $c \in (N: abm)$.

Then (N:am) ∪(N:bm) ⊆ (N:abm) . So we get (N:abm)=(N:am)∪(N:bm).

(2) ⟹(3) Since (N : abm) is an ideal of R , and (N:abm)=(N:am)∪(N:bm).

So either (N : bm) ⊆ (N : am) or (N : am) ⊆ (N : bm) . Then (N : abm) = (N : am) or (N : abm) =(N : bm)

(3) ⟹(1) Now suppose that $M=(m_1)$ for some m_1 ∈M

Let abm ∈ N . But $m=rm_1$ for some r ∈ R then $abrm_1$ ∈ N and suppose that a b ∉ (N :M)= (N: (m_1)) , abm_1 ∉ N and r∈(N : abm_1) .

By (3) (N : abm_1) = (N : a m_1) or (N : ab m_1) = (N : b m_1) . Since r ∈(N : abm_1) , so either r ∈(N : a m_1) or r ∈(N : b m_1) . Then a rm_1 ∈ N or b rm_1 ∈ N , we get a m ∈ N or b m ∈ N . Hence N is 2-absorbing ■

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حول المقاسات الجزئية من النمط المستحوذه على 2

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الخلاصة

لتكن R حلقة ابدالية ذات محايد $1 \neq 0$ و M مقاس احادي على الحلقة R .
هدفنا في هذا البحث الاستمرار في دراسة المقاسات الجزئية من النمط المستحوذة على 2 والتي قدمت من قبل الباحثين دارياني وسهيليني .
العديد من الخواص والتمييزات لهذا المفهوم قد اعطيت

الكلمات المفتاحية : المقاسات الجزئية الاولية ، المقاسات الجزئية المستحوذة على**2** ، المقاسات الجزئية شبه الاولية ، المقاسات الجدائية ، المقاسات الجزئية الابتدائية ، المقاسات الجزئية النقية .