2-Regular Modules II

Nuhad S.AL-Mothafar
Dept. of Mathematic/ College of Science/ University of Baghdad
Ghaleb A. Humod
Dept. of Mathematic/ College of Education for Pure Science (Ibn Al-Haitham)/ University of Baghdad
Received in:28/March/2015,Accepted in:7/June/2015

Abstract

An R-module M is called a 2-regular module if every submodule N of M is 2-pure submodule, where a submodule N of M is 2-pure in M if for every ideal I of R, $I^2M\cap N = I^2N$, [1].

This paper is a continuation of [1]. We give some conditions to characterize this class of modules, also many relationships with other related concepts are introduced.

Key Words: 2-pure submodules, 2-regular modules, pure submodule, regular modules.

0- Introduction

Throughout this paper, R is a commutative ring with identity and all R-modules are unitary. A submodule N of an R-module M is called 2-pure submodule if for every ideal I of R, $I^2M \cap N = I^2N$. If every submodule of M is 2-pure, then M is said to be 2-regular module. This work consists of two sections. In the first section we give some properties of 2-regular rings. Next we present a characterization of 2-regular modules. In the second section we illustrate some relationships between the concept 2-regular modules and other modules such as semiprime divisible, projective and multiplication modules.

1- 2-Regular Modules

In this section, we first define 2-regular rings and study some of its properties. Next we consider some conditions to characterize 2-regular modules.

Definition (1.1): [1]

An ideal I of a ring R is called **2-pure ideal** of R if for each ideal J of R, $J^2 \cap I = J^2I$.
If every ideal of a ring R is 2-pure ideal, then we say R is 2-regular ring.

Remarks and Examples (1.2):

(1) It is clear every (von Neumman) regular ring is 2-regular ring, but the converse is not true, for example: the ring Z_4 is 2-regular ring, since every ideal of Z_4 is 2-pure. But Z_4 is not regular since the ideal $\{\overline{0},\overline{2}\}$ is not pure because $\{\overline{0},\overline{2}\} \cap \{\overline{0},\overline{2}\} = \{\overline{0},\overline{2}\}$, on the other hand $\{\overline{0},\overline{2}\} \cdot \{\overline{0},\overline{2}\} = \{\overline{0}\}$ implies $\{\overline{0},\overline{2}\} \cap \{\overline{0},\overline{2}\} \neq \{\overline{0},\overline{2}\} \cdot \{\overline{0},\overline{2}\}$.

(2) It is clear that $\{0\}$ and R are always 2-pure ideals of any ring R.

(3) Every field is 2-regular ring.

(4) Let R be an integral domain. If R is 2-regular ring, then R is a field.

Proof:

Let I be an ideal of R. Since R is 2-regular ring then $J^2 \cap I = J^2I$ for every ideal J of R. If we take J = I implies $I^2 = I^3$. Thus for each element $0 \neq a \in R$, $< a >^2 = < a >^3$, hence $a^2 \in < a >^3$. Let $a^2 = r\, a^3$ for some $r \in R$, then $a^2(1 - ra) = 0$ but R is domain and $a \neq 0$ implies $1 - ra = 0$, thus $1 = ra$. Therefore a is an invertible element of R. Thus R is a field

(5) If R is a 2-regular ring then every prime ideal of R is a maximal ideal.

Proof:

Let I be a prime ideal of R. Since R is a 2-regular ring then $\frac{R}{I}$ is 2-regular by [1,Cor.3.2]. But $\frac{R}{I}$ is a domain since I is a prime ideal. Thus $\frac{R}{I}$ is a field by the above remark. Therefore I is a maximal ideal.

(6) Every 2-regular ring is nearly regular, where a ring R is called nearly regular if $\frac{R}{J(R)}$ is regular ring, see [2], where J(R) = the intersection of all maximal ideals of R.

Proof:

Let R be a 2-regular ring. Then $\frac{R}{J(R)}$ is 2-regular by corollary (1.2.3). So by above remark (5), every prime ideal of $\frac{R}{J(R)}$ is a maximal ideal and since $J\left(\frac{R}{J(R)}\right)=0$, therefore by [3], $\frac{R}{J(R)}$ is regular.

Proposition (1.3):

Let M be 2-regular R-module then for every element x of Mand every element $r \in R$, $r^2x = r^2tr^2x$ for some $t \in R$.

Proof:

Let x be an element of M and r be an element of R. Since $r^2x \in r^2M$ and $r^2x \in < r^2x >$ implies $r^2x \in r^2M \cap < r^2x >$. But M is 2-regular, then $r^2M \cap < r^2x > = r^2 < r^2x >$. Thus, $r^2x \in r^2 < r^2x >$ implies $r^2x = r^2t\, r^2x$ for some $t \in R$.

Proposition (1.4):

Let M be a module over principal ideal ring R. If for every element x of M and every element $r \in R$, $r^2x = r^2tr^2x$ for some $t \in R$ implies M is a 2-regular module.

Proof:

Let N be a submodule of M and I is an ideal of R. First, to prove $r^2M \cap N = r^2N$ for every element $r \in R$. Let $x \in r^2M \cap N$ implies $x \in r^2M$, $x \in N$. Thus $x = r^2m$ for some $m \in M$. Then $x = r^2tr^2m$ for some $t \in R$ by hypothesis. Hence $x \in r^2N$. But R is a principal ideal ring. Therefore $I^2M \cap N = I^2N$.

Proposition (1.5):

Let M be a cyclic R-module. If for every element x of M and every element r of R, $r^2x = r^2tr^2x$ for some $t \in R$, implies M is a 2-regular module.

Proof:

Let $M = Rm$ be a cyclic module for some $m \in M$. Let N be a submodule of M and I is an ideal of R. Let $y \in I^2M \cap N$ then $y \in I^2M$ and $y \in N$. Thus $y = r^2m = r^2tr^2m \in r^2N$ for some $t \in R$ and $r \in I$. Therefore $y \in I^2N$ implies M is 2-regular.

The proof of the following result is similar to that of propositions (1.3) and (1.4).

Corollary (1.6):

Let R be a 2-regular ring then for every element $a \in R$, $a^2 = a^2t\, a^2$ for some $t \in R$, and the converse is true if R is a principal ideal ring.

Proposition (1.7):

Let R be a principal ideal ring and M be an R-module. The following statements are equivalent:

(1) M is 2-regular module.

(2) $\frac{R}{\operatorname{ann}_R(x)}$ is 2-regular for every element x of M.

(3) For every element x of M and every element r of R, $r^2x = r^2tr^2x$ for some $t \in R$.

Proof:

(1) ⇒ (3) It follows by Proposition (1.3).

(3) ⇒ (1) By Proposition (1.4).

(1) ⇒ (2) Let $r + \underset{R}{\operatorname{ann}}(x) \in \dfrac{R}{\underset{R}{\operatorname{ann}}(x)}$ where $x \in M$ and $r \in R$.

Since M is 2-regular, then $r^2x = r^2tr^2x$ for some $t \in R$. Thus $r^2 - r^2tr^2 \in \underset{R}{\operatorname{ann}}(x)$ implies $\dfrac{R}{\underset{R}{\operatorname{ann}}(x)}$ is 2-regular.

(2) ⇒ (1) Let $x \in M$ and $r \in R$. Since $\dfrac{R}{\underset{R}{\operatorname{ann}}(x)}$ is 2-regular, then $r^2 + \underset{R}{\operatorname{ann}}(x) = (r^2 + \underset{R}{\operatorname{ann}}(x)$ $(t + \underset{R}{\operatorname{ann}}(x))(r^2 + \underset{R}{\operatorname{ann}}(x))$ for some $t \in R$. Thus $r^2x = r^2tr^2x$ implies M is 2-regular.

We have the following results:

Corollary (1.8):

Let R be a principal ideal ring. Then R is 2-regular if and only if all R-modules are 2-regular.

Proof:

(⇒) Let R be 2-regular ring and M is an R-module. Then $\dfrac{R}{\underset{R}{\operatorname{ann}}(x)}$ is 2-regular for every element $x \in M$ by [1,Cor.(3.3)]. Therefore M is 2-regular by proposition (1.7).

(⇐) Assume all R-modules are 2-regular. Thus R is 2-regular R-module. By Proposition (1.7), $\dfrac{R}{\underset{R}{\operatorname{ann}}(x)}$ is 2-regular for some every element $x \in R$, so if take $x = 1 \in R$ implies $\dfrac{R}{\underset{R}{\operatorname{ann}}(x)} = \dfrac{R}{<0>} \cong R$, therefore R is 2-regular.

Corollary (1.9):

Let R be a principal ideal ring. Then R is a 2-regular if and only if R is 2-regular R-module.

Proof: By the same argument of Corollary (1.8).

Corollary (1. 10):

Let R be a principal ideal ring. If $\dfrac{R}{\underset{R}{\operatorname{ann}}(M)}$ is 2-regular then M is 2-regular R-module.

Proof:

Let x be a non-zero element of M. Since $\underset{R}{\operatorname{ann}}(M) \subseteq \underset{R}{\operatorname{ann}}(x)$, there exists an epimorphism f: $\dfrac{R}{\underset{R}{\operatorname{ann}}(M)} \rightarrow \dfrac{R}{\underset{R}{\operatorname{ann}}(x)}$ defined by $f(r + \underset{R}{\operatorname{ann}}(M)) = r + \underset{R}{\operatorname{ann}}(x)$. Therefore $\dfrac{R}{\underset{R}{\operatorname{ann}}(x)}$ is 2-regular by [1,Cor.(3.3)]. Then M is 2-regular by Proposition (1.7).

2- Regular Modules and Other Related Modules

In this section, we study the relationships between 2-regular modules and other modules such as semiprime, divisible, projective and multiplication modules.

Recall that a proper submodule N of an R-module M is called a semiprime submodule if for every $r \in R$, $x \in M$, $k \in Z^+$ such that $r^k x \in N$ implies $rx \in N$ implies $rx \in N$, see [4]. Equivalently, a proper submodule N of M is semiprime if for every $r \in R$, $x \in M$ such that $r^2x \in N$ implies $rx \in N$, see [5].

An R-module M is called semiprime if <0> is a semiprime submodule of M.

The proof of the following result follows by [5].

Proposition (2.1):

Let R be a principal ideal ring and M is an R-module. If every proper submodule of M is semiprime then M is a 2-regular module. The converse is not true, for example: The module Z_4 as Z-module is 2-regular but <0> is not semiprime.

The following proposition gives a partial converse of proposition (2.1).

Proposition (2.2):

Let M be 2-regular and semiprime R-module then every proper submodule of M is semiprime.

Proof:

Let N be a proper submodule of M and $r^2x \in N$ where $r \in R$, $x \in M$ implies $r^2x \in r^2M \cap N = r^2N$ since M is 2-regular. Then $r^2x = r^2n$ for some $n \in N$, thus $r^2(x - n) \in$ <0>. But <0> is semiprime, hence $rx = rn \in N$. Therefore N is semiprime submodule of M.

Before we give a consequence of Proposition (2.2), we need the following lemma:

Lemma (2.3):

Let M be 2-regular and semiprime R-module then J(R)M = <0>.

Proof:

Let $r \in J(R)$ and $x \in M$ then $r^2x=r^2tr^2x$ for some $t \in R$ since M is 2-regular, $r^2x(1 - r^2t)=0$ implies $1 - r^2t$ is invertible in R. Then $r^2x = 0$, but M is semiprime thus $rx = 0$. Therefore J(R)M = <0>.

Recall that an R-module M is called semisimple if every submodule of M is a summand. The sum of all simple submodules of a module M is called the socle of M is denoted by Soc(M), moreover if Soc(M) = 0 , then M has no simple submodule and if Soc(M) = M then M is semisimple module, see [6].

A commutative ring is a local ring in case it has a unique maximal ideal, see [7].

Corollary (2.4):

Let R be a local ring and M is 2-regular and semiprime R-module then M is a semisimple and hence is regular.

Proof:

Since R is a local ring, then $\frac{R}{J(R)}$ is a simple ring and hence is semisimple. By [6], $Soc(M) = \underset{M}{ann}(J(R)) = \{m \in M;\ mJ(R) = 0\}$. But $J(R)M = <0>$ by lemma (2.3), thus $Soc(M) = M$. Therefore M is semisimple.

Now, we have the following:

Proposition (2.5):

Let N be a semiprime submodule of an R-module M and K is a 2-pure submodule of M containing N, then $\frac{K}{N}$ is semiprime submodule in $\frac{M}{N}$.

Proof:

Let $r^2(x + N) \in \frac{K}{N}$ for some $r \in R$ and $x + N \in \frac{M}{N}$.

Then $r^2x \in K$, imples $r^2x \in r^2M \cap K = r^2K$ since K is 2-pure in M. Let $r^2x = r^2m$ for some $m \in K$. Thus $r^2(x - m) = 0 \in N$ implies $r(x - m) \in N$ since N is semiprime submodule in M, hence $r(x + N) = rm + N \in \frac{K}{N}$. Therefore $\frac{K}{N}$ is semiprime submodule in $\frac{M}{N}$.

Corollary (2.6):

Let N be a semiprime submodule of an R-module M and K is a 2-pure in M with $N \subseteq K$ then K is semiprime submodule in M.

Proof:

Let $r^2x \in K$ for some $r \in R$ and $x \in M$. Thus $r^2(x + N) \in \frac{K}{N}$, but $\frac{K}{N}$ is semiprime in $\frac{M}{K}$ by Proposition (2.5) therefore $r(x + N) \in \frac{K}{N}$. Hence $rx \in K$, that is K is semiprime in M.

Let R be an integral domain, an R-module M is said to be divisible if and only if $rM = M$ for every non-zero element r of R, see [8].

An R-module M is said to be a prime module if $\underset{R}{ann}(M) = \underset{R}{ann}(N)$ for every non-zero submodule N of M, see [9].

Proposition (2.7):

Let M be a module over a principal ideal domain R and N is a divisible R-submodule of M then N is a 2-pure submodule in M.

Proof: Since N is divisible then for each $r \in R$, $r^2N=N$. Therefore $N \cap r^2M = r^2N$.

Remark (2.8):

The converse of proposition (2.7) is not true, for example: the submodule $\{\bar{0},\bar{2}\}$ of the module Z_4 as Z-module where $\{\bar{0},\bar{2}\}$ is 2-pure in Z_4, but is not divisible since there exists $2 \in Z$ and $2\cdot\{\bar{0},\bar{2}\} = \{\bar{0}\}$. That is $2\cdot\{\bar{0},\bar{2}\} \neq \{\bar{0},\bar{2}\}$.

The following proposition gives a condition under which the converse of proposition (2.7) is true.

Proposition (2.9):

Let M be divisible module over a principal ideal domain R and N is a 2-pure in M then N is divisible.

Proof:

Assume N is 2-pure in M, let $m \in N$ and $r \in R$. Since M is divisible implies $m = r^2x$ for some $x \in M$. But $m = r^2x \in r^2M \cap N = r^2N \subseteq rN$. Therefore $N = rN$.

As an immediate consequence we have the following:

Corollary (2.10):

Let R be a principal ideal domain and every proper submodule of an R-module M is divisible then M is 2-regular. The converse is true if M is divisible.

Proof:

Follows by Propositions (2.7) and (2.9).

Corollary (2.11):

Let R be a principal ideal domain and M is 2-regular and divisible R-module then M is prime module.

Proof:

By above corollary (2.10), every submodule N of M is divisible. Thus rN = N for every $r \in R$. Therefore $\underset{R}{\operatorname{ann}}(N) = \underset{R}{\operatorname{ann}}(M) = \langle 0 \rangle$. Hence M is prime module.

Corollary (2.12):

Let R be a principal ideal domain and M is 2-regular injective R-module then M is prime module.

Proof: Clear

We give the following theorem.

Theorem (2.13):

Let R be any ring. The following statements are equivalent:

(1) $\underset{\Lambda}{\oplus} R$ is 2-regular R-module for any index set Λ.

(2) Every projective R-module is 2-regulaar module.

Proof:

(1) ⇒ (2) Let M be projective R-module then there exists a free R-module F and an R-epimorphism $f : F \longrightarrow M$, and $F \cong \underset{\Lambda}{\oplus} R$ where Λ is an index set. We have the following short exact sequence

$$0 \longrightarrow \ker f \xrightarrow{\ i\ } \underset{\Lambda}{\oplus} R \xrightarrow{\ f\ } M \longrightarrow 0$$

Where i is the inclusion mapping.
Since M is projective, the sequence is split implies that $\underset{\Lambda}{\oplus} R \cong \ker f \oplus M$. But $\underset{\Lambda}{\oplus} R$ is 2-regular R-module. Therefore by [1,Cor.(3.4)] M is 2-regular module.

(2) ⇒ (1) Assume that every projective R-module is 2-regular module. Since R is projective R-module, then $\underset{\Lambda}{\oplus} R$ is projective because the direct sum of projective modules is projective. Therefore $\underset{\Lambda}{\oplus} R$ is 2-regular R-module for any index set Λ.

Recall that an R-module M is called multiplication module if for every submodule N of M there exists an ideal I of R such that N = IM, see [10]

We have the following:

Proposition (2.14):

If M is a finitely generated faithful multiplication R-module. The following statements are equivalent:

(1) R is 2-regular ring.
(2) M is 2-regular R-module.

Proof:

(1) ⇒ (2) Let N be a submodule of M and I is an ideal of R. Since

$I^2M \cap N = I^2M \cap JM$ for some ideal J of R
$= (I^2 \cap J)M$ since M is faithful multiplication, see [10]
$= (I^2J)M$ since R is 2-regular
$= I^2(JM)$
$= I^2N$

Therefore M is 2-regular.

(2) ⇒ (1) Let I and J be ideals of R. Since

$(I^2 \cap J)M = I^2M \cap JM$ because M is faithful multiplication
$= I^2(JM)$ since M is 2-regular
$= (I^2J)M$

Thus $I^2 \cap J = I^2J$ since M is finitely generated faithful multiplication, see [10]. Therefore R is 2-regular ring.

Recall that an R-module M is said to be I-multiplication module if each submodule N of M of the form JM for some idempotent ideal J of R, see [11].
It is clear that every I-miltiplication module is multiplication but not the converse.
Clearly the two concepts multiplication and I-multiplication modules are equivalent over regular rings. However we have the following:

Proposition (2.15):

If M is I-multiplication and 2-regular R-module then M is regular module.

Proof:

Let N be a submodule of M and I is an ideal of R. Since

$IM \cap N = IM \cap JM$
$= IM \cap J^2M$ for some idempotent $J = J^2$
$= J^2(IM)$ since M is 2-regular
$= (I^2J)M$ since R is 2-regular
$= I(J^2M)$
$= I(JM)$
$= IN$
Therefore M is regular module.

Proposition (2.16):

If M is I-multiplication and 2-regular R-module then every submodule N of M is I-multiplication as R-module.

Proof:

Let N be a submodule of M and K is any submodule in N, then K is a submodule of M and $K = IM = I^2M$ for some idempotent ideal I of R. Since
$K = N \cap K$
$= N \cap I^2M$
$= I^2N$ because M is 2-regular
$= IN$
Thus N is I-multiplication R-module.

References

1. Nuhad, S.AL-Mothafar and Ghaleb, A.Humod, 2-Regular Modules, to appear.
2. Naoum, A.G. and Nuhad, S.AL-Mothafar, (1994),Nearly Regular Ring and Nearly Regular Modules, Mutah. Journal For Research and Studies, 9, 6.
3. Aziz, B.B., (1975), Regular rings and Bear rings, M.Sc.Thesis, University of Baghdad.
4. .Duns, (1988), Modules and One-Sided Ideal in Ring Theory and Algebra, 19, 755-779.
5. Athab, E.A., (1996), Prime Submodules and Semiprime Submodules, M.Sc. Thesis, University of Baghdad, Iraq.
6. Kasch, F., (1982), Modules and Rings, Academic Press, New York.
7. Anderson, F.W. and Fuller K.R., (1992), Rings and Categories of Modules, Springer Verlag, New York.
8. Sharp, D.W. and Vamos, P., (1972), Injective Modules, Combrdige University, Press.
9. Desale, G.and Nicholson, W.K., (1981), Endoprimitive Rings, J.Algebra, 70, 548-560.
10. EL-Bast, Z.A., and Smith, P.F., (1988), Multiplication Modules, Commun. Algebra, 16, 4, 755-779.
11. Abbas, M.S., (1990), On Fully Stable Modules, Ph.D. Thesis, University of Baghdad, Iraq.

المقاسات المنتظمة من النمط -2 II

نـهاد سالم عبد الكريم
قسم الرياضيات/ كلية العلوم/ جامعة بغداد

غالب أحمد حمود
قسم الرياضيات / كلية التربية للعلوم الصرفة(ابن الهيثم) / جامعة بغداد

استلم البحث في:28/نيسان/ 2015،قبل البحث في:7/حزيران/2015

الخلاصة

ليكن M مقاساً على R إذ R حلقة إبدالية ذات محايد. يقال ان المقاس M بأنه منتظم من النمط – 2 اذا كان كل مقاس جزئي في M هو مقاس جزئي نقي من النمط-2 إذ يقال عن المقاس الجزئي N بأنه نقي من النمط-2 في M اذا حقق $I^2M \cap N = I^2N$ لكل مثالي I في R، [1].

في هذا البحث نستمر بدراسة مفهوم الانتظام من النمط-2 [1]. في القسم الاول من هذا البحث أعطينا تمييزاً للمقاسات المنتظمة من النمط-2. في القسم الثاني درسنا العلاقة بين المقاسات المنتظمة من النمط-2 وانواع اخرى من المقاسات.

الكلمات المفتاحية : المقاسات الجزئية النقية من النمط – 2، المقاسات النتظمة من النمط – 2، المقاسات الجزئية النقية، المقاسات المنتظمة.