Modules with Chain Conditions on S-Closed Submodules

Rana Noori Majeed Mohammed
Dept. of Mathematic / College of Education for pure science-(Ibn Al-Haitham)/ University of Baghdad
rana.n.m@ihcoedu.uobaghdad.edu.iq

Received in :31 /May /2017, Accepted in : 27 /August/ 2017

Abstract

Let L be a commutative ring with identity and let W be a unitary left L- module. A submodule D of an L- module W is called s- closed submodule denoted by $D \leq_{sc} W$, if D has no proper s- essential extension in W, that is , whenever $D \leq W$ such that $D \leq_{se} H \leq W$, then D = H. In this paper, we study modules which satisfies the ascending chain conditions (ACC) and descending chain conditions (DCC) on this kind of submodules.

Keywords: s-essential submodules, s-closed submodules , ascending and descending chain conditions.

https://doi.org/10.30526/30.3.1618

Introduction

Throughout this paper, L represents a commutative ring with unity and W be a left unitary L- module. It′s well known that "a submodule D of W is called small denoted by D<< W if and only if D + U = W implies U=W for each U submodule of W (U≤W)" [2], and "a submodule D of an L - module W is called an essential submodule of W and denoted by $D\leq_e W$ if every non-zero submodule of W has non-zero intersection with D" [3], while "a submodule D of an L- module W is said to be a closed submodule of W if D has no proper essential extension inside W, that is if $D\leq_e H\leq W$ then D=H" [3]. As a generalization of essential submodules , in [4] "Zhou and Zhang" introduced the concept of s- essential submodule , where "a submodule D of an L-module W is said to be an s -essential submodule of W denoted by $D\leq_{se} W$ if D∩H=0 with H is a small submodule of W implies H= 0. "Mehdi Sadiq and Faten" in [1] introduced and studied the notion of s- closed submodules, "a submodule D of an L- module W is called s-closed submodule denoted by $D\leq_{sc} W$, if D has no proper s- essential extension in W, that is , whenever D ≤ W such that $D\leq_{se} H\leq W$, then D= H.

This paper consists of two sections. In section one, we give some other properties and examples of s-essential submodules and s- closed submodules. In section two, we study chain conditions (that is ascending and descending chain conditions) on s-closed submodules.

1. S-Essential Submodules and S- Closed Submodules

Definition 1 . 1 : [4]

A submodule D of an L-module W is said to be an s- essential submodule of W denoted by $D \leq_{se} W$ if D∩H= 0 with H is a small submodule of W implies H= 0.

Remarks and examples 1 . 2 :

1) It′s clear that every essential submodule is an s- essential submodule, hence every submodule of Z -module Z, Z_p^n (where P is a prime number, $n\in Z_+$) is s- essential.

2) If W is an L- module such that (0) is the only small submodule then every submodule is s- essential submodule in W.

In particular , for each submodule of semisimple module (or free Z -module) is s- essential.

Hence it′s clear that every submodule of Z -module Z_6 is s- essential, however they are not essential. Also every submodule of the Z- module Z ⊕ Z is s- essential submodule.

3) Let A be a submodule of an L -module W, then there exists a closed submodule H of W such that $A\leq_e H$, it is clear by [3, Exc.13, p.20], hence $A\leq_{se} H$.

4) In Z_{24} as Z- module, we have $<\bar{2}>$, $<\bar{3}>$, $<\bar{4}>$, $<\bar{6}>$, $<\overline{12}>$, and Z_{24} are s- essential submodules in Z_{24}, but $<\bar{8}>$ is not since $<\bar{8}>\cap<\bar{6}> =\{0\}$ while $<\bar{6}> \neq 0$ is a small submodule in Z_{24}.

5) For a nonzero R-module W, $W \leq_{se} W$.

6) The two concepts essential and s- essential are coincide under the class of hollow modules, by[1, Remark (2.3)], where "an L -module W is called Hollow if every proper submodule of W is small". [5]

Proposition 1 . 3 :

Let W be an L-module and let $S \leq_{se} T \leq M$ and $S' \leq_{se} T' \leq W$, then $S \cap S' \leq_{se} T \cap T'$.

Proof:

Let U << T ∩ T′ and (S ∩ S′) ∩ U = (0), hence S ∩ (S′ ∩ U) = 0.

But U << (T ∩ T′) implies U << T′ and U << T.

As S′∩U ⊆U<< T, then S′∩U<< T. But $S \leq_{se} T$, hence S′∩U= 0.

It follows that U = 0 since $S'\leq_{se} T'$ and U << T′.

The following result follows by Proposition 1.3 directly.

https://doi.org/10.30526/30.3.1618

Corollary 1 . 4 :

Let C, D be submodules of W such that $C\leq_{se}W$ and $D\leq_{se}W$. Then $C\cap D \leq_{se} W$, [4, proposition 2.7(1)(b)] .

Proposition 1 . 5 :

Let $W=W_1\oplus W_2$, and let $A = A_1\oplus A_2\leq_{se} B_1\oplus B_2$, where $B_1\leq W_1$ and $B_2\leq W_2$. Then $A_1\leq_{se} B_1$ and $A_2\leq_{se} B_2$.

Proof:

Suppose A_1 is not an s-essential submodule in B_1. So there exists a nonzero small submodule D_1 in B_1 such that $A_1\cap D_1=(0)$.

Since $D_1\oplus(0)$ is a small submodule in $B_1\oplus B_2$ and $(A_1\oplus A_2)\cap(D_1\oplus(0)) = (A_1\cap D_1)\oplus(A_2\cap(0)) = (0)$.

Then $A_1\oplus A_2$ is not an s-essential submodule in $B_1\oplus B_2$ which is a contradiction.

Thus $A_1\leq_{se} B_1$ and by the same way of proof that $A_2\leq_{se} B_2$.

Proposition 1 . 6 :

Let W be a faithful multiplication finitely generated (denoted by FMFG) L- module, and U a submodule of W. Then $U\leq_{se} W$ if and only if there exists an s -essential ideal E of L such that U = EW.

Proof:

(⇨) Let $U\leq_{se} W$. As W is a multiplication L- module, so U= EW for some $E\leq L$. To prove that $E\leq_{se} L$, assume J is a small ideal of L and $E\cap J = 0$, hence $(E\cap J)W = 0$. Then by [6, Th. 1.6(i), p. 759] $EW\cap JW = 0$, that is $U\cap JW = 0$.

But by [8, prop.1.1.8] JW is a small submodule of W and $U\leq_{se} W$, so JW = 0. Hence J= 0 (since W is a faithful module) . Thus $E\leq_{se} L$.

(⇦) To prove $U\leq_{se} W$. Assume V is a small submodule of W, hence V = JW for some $J \ll L$. if $U\cap V = 0$, then $EW\cap JW = 0$ and so $(E\cap J)W = 0$. Hence $E\cap J = 0$ since W is faithful. Thus J= 0 because $E\leq_{se} L$. It follows that V= 0 and $U\leq_{se} W$.

Theorem 1 . 7 :

Let W be a FMFG L- module. Then $I\leq_{se} J\leq L$ if and only if $IW\leq_{se} JW$.

Proof:

(⇨) Let U be a small submodule in $JW\leq W$, so $U\leq W$. Thus U= KW for some $K\leq L$. As $KW\leq JW$ then $K\leq J$, by [6, Th.3.1]

To prove K is a small submodule in J , let K+H = J, so KW + HW = JW. That is HW =JW (since KW = U which is a small submodule in JW). Hence HW =JW and so H=J, that is K is a small submodule in J.

If $IW\cap U = 0$, then $IW\cap KW = 0$. Thus $(I\cap K)W = 0$, so $I\cap K=0$ (since W is faithful multiplication).

But $I\leq_{se} J$ and K is a small submodule in J, hence K= 0. It follows U= 0, thus $IW\leq_{se} JW$.

(⇦) If $IW\leq_{se} JW$ to prove $I\leq_{se} J\leq L$. Let K be a small submodule of J.

Assume $I\cap K = 0$, then $(I\cap K)W = 0$, so $IW\cap KW = 0$.

Let KW +H = JW. Since W is a multiplication module , thus H = CW. Hence KW + CW = JW.

Since KW is a small submodule in JW , then CW = JW and hence C = J. Thus H = JW and KW is a small submodule of JW.

Now, $IW\cap KW = 0$ and KW is a small submodule in JW implies KW = 0 (since $IW\leq_{se} JW$) and so K=0. It follows $I\leq_{se} J$.

Recall that , “a non-zero L-module W is called small -uniform (shortly, by s -uniform) if every nonzero submodule of W is s -essential. A ring L is called s-uniform if L is an s-uniform L-module”. [9]

https://doi.org/10.30526/30.3.1618

Corollary 1 . 8 :

Let W be a FMFG L-module. Then W is s-uniform module if and only if L is s-uniform ring.

Definition 1 . 9 : [1]

A submodule D of an L -module W is called s-closed submodule denoted by $D \leq_{sc} W$, if D has no proper s -essential extension in W, that is , whenever $D \leq W$ such that $D \leq_{se} K \leq W$, then D = K. An ideal E of L is called an s-closed, if it′s an s- closed submodule in L. Where every s- closed submodule in W is closed in W but the converse is not true.

Examples 1 . 10 :

1) In Z_{24} as a Z-module. Z_{24} and $<\bar{8}>$ are the only s-closed submodules while $<\bar{2}>$,$<\bar{3}>$,$<\bar{4}>$,$<\bar{6}>$ and $<\overline{12}>$ are not because they have a proper s-essential submodule which is Z_{24}. All submodules of Z_{24} have the following properties.

$A \leq Z_{24}$	$A \ll Z_{24}$	$A \leq_{se} Z_{24}$	$A \leq_{sc} Z_{24}$
$<\bar{0}>$	✓	×	×
$<\bar{2}>$	×	✓	×
$<\bar{3}>$	×	✓	×
$<\bar{4}>$	×	✓	×
$<\bar{6}>$	✓	✓	×
$<\bar{8}>$	×	×	✓
$<\overline{12}>$	✓	✓	×
Z_{24}	×	✓	✓

Similarly, $<\bar{4}>$,$<\bar{6}>$ and $<\bar{8}>$ are not small submodules in $<\bar{2}>$ in Z_{24} but $<\overline{12}>$ is a small submodule in $<\bar{2}>$ and $<\bar{4}> \cap <\overline{12}> \neq \{0\}$ thus $<\bar{4}>$ is an s-essential submodule in $<\bar{2}>$, so it is not an s-closed submodule in $<\bar{2}>$.

2) If W is a simple module , then $<\bar{0}>$ and W are s- closed submodules.

3) Let W be an L-module. If every submodule of W is s-closed (hence every submodule is closed), then W is semisimple module, however the converse is not true, for example in Z_6 , Z_6 is a Z- module is semisimple but the submodules $<\bar{0}>$, $<\bar{2}>$, $<\bar{3}>$ are not s- closed.

Proposition 1. 11 :

Let W be an L-module such that the s-essential submodules satisfy transitive property. Then for each $A \leq W$, there exists an s-closed submodule such that $A \leq_{se} H$.

Proof:

Let $S=\{K \leq W : A \leq_{se} K\}$. $V \neq \emptyset$ since $A \in V$. So by " Zorn's Lemma" S has a maximal element say H.

To prove H is an s -closed submodule in W. Assume $H \leq_{se} D \leq W$.

Since $A \leq_{se} H$ and $H \leq_{se} D$, then $A \leq_{se} D$ (by transitive property), and so $D \in S$.

Hence H = D (by maximality of L). Thus H is an s-closed submodule.

The following proposition has been given in [1], we will mention it with its proof for the sake of completness.

Proposition 1 . 12 :

Lea A be a submodule of B, and let B an s-closed submodule of W, then (B/A) is an s -closed submodule of (W/A).

Proof:

Assume $(B/A) \leq_{se} (C/A)$ where $(C/A) \leq (W/A)$. Let $\pi : W \to (W/A)$ be a natural projection map.

Then $B=\pi^{-1}(B/A)$, and so by [4, prop.27(2), p.1054] $B \leq_{se} C$.

But B is an s- closed submodule in W. Thus B=C.

https://doi.org/10.30526/30.3.1618

It follows that (B / A) = (C/ A) and (B/ A) is an s-closed submodule in (W/A).

Proposition 1 . 13 :

Let A ≤ B ≤ W such that A is an s -closed submodule of an L-module W. Then $B \leq_{sc} W$ if and only if $\frac{B}{A} \leq_{sc} \frac{W}{A}$.

Proof : (⇨) See [1, coro.2.7]

(⇦) Suppose $\frac{B}{A} \leq_{sc} \frac{W}{A}$ and let $B \leq_{se} H \leq W$. Since $A \leq_{sc} W$ and A ≤ B then $\frac{B}{A} \leq_{se} \frac{W}{A}$ implies $B \leq_{se} W$ by [1, Remarks and Examples 2.2(6)]. That is $A \leq_{sc} B$ by [1, propo.2.8].

To prove $A \leq_{sc} H$, suppose that $A \leq_{se} C$ for some submodule C of H. As A is an s- closed submodule of W, thus A = C . Hence A is an s -closed submodule of H and $B \leq_{se} H$, that is $\frac{B}{A} \leq_{se} \frac{H}{A}$, by [1, Remarks and Examples 2.2(6)]. But $\frac{B}{A} \leq_{sc} \frac{W}{A}$, so $\frac{B}{A} = \frac{H}{A}$. Then B =H which means $B \leq_{sc} W$.

Proposition 1 . 14 :

Let W be a FMFG L-module, and C≤ W. C is an s-closed submodule in W if and only if C = HW for some s-closed ideal H in L.

Proof:

(⇨) Let C≤ W , then C = HW. To prove H is an s-closed ideal in L .

Assume $H \leq_{se} J$. Hence $HW \leq_{se} JM$ by (Th. 1.7) , thus $C \leq_{se} JW$ so C = JW that is HW =JW.

Since W is FMFG module so H = J , hence H is an s -closed ideal in L.

(⇦) Similarly.

2. Ascending (Descending) Chain Conditions on S-Closed Submodules

In this section, we study modules with chain conditions on s -closed submodules.

Definition 2 . 1 : An L-module W is said to have the ascending (descending) chain condition , briefly ACC (DCC) on s-closed submodules if every ascending (descending) chain $A_1 \subseteq A_2 \subseteq \dots$ ($A_1 \supseteq A_2 \supseteq \dots$) of s-closed submodules of W is finite. That is there exists $k \in Z_+$ such that $A_n = A_k$ for all $n \geq k$.

Recall that, “a Noetherian module is a module that satisfies the Ascending Chain Condition on its submodules. Also, an Artinian module is a module that satisfies the Descending Chain Condition on its submodules”. [3]

Remarks 2 . 2 :

1. Every noetherian (respectively artinian) module satisfies ACC (respectively DCC) on s -closed submodules.
2. If W satisfies ACC (respectively DCC) on closed submodules, then W satisfies ACC (respectively DCC) on s -closed submodules.

Proof: It is clear since every s -closed submodule in W is closed submodule in W.

The converse is true if W is hollow by Remark 1.2(6) or uniform module , where “ a uniform module is a nonzero module W which is every non-zero submodule of W is essential in W” . [3]

Recall that, “an L-module W is called chained if for all submodules C and D of W either C ≤ D or D ≤ C”. [7]

Proposition 2 . 3 : Let W be a chained L -module, and let A be an s-closed submodule of W. If W satisfied ACC (respectively DCC) on s -closed submodules, then A satisfies the ACC (respectively DCC) on s-closed submodules.

Proof: Assume W satisfies ACC on s-closed submodules and $A_1 \subseteq A_2 \subseteq \dots$ be ascending chain of s- closed submodules of A. Since A is an s-closed submodule of W and W satisfy chained condition, so by [1, prop.2.11, p.345] A_i is an s-closed submodule of W for each i = 1, 2, … . Hence $A_1 \subseteq A_2 \subseteq \dots$ be ascending chain of s-closed submodules of W. But W satisfies ACC on s- closed submodules , thus $k \in Z_+$ such that $A_n = A_k$ for all $n \geq k$. That is A satisfies ACC on s- closed submodules.

https://doi.org/10.30526/30.3.1618

Similarly, if W satisfies DCC on s- closed submodules, then A satisfies DCC on s- closed submodules of A.

Proposition 2. 4 : Let W = $W_1 \oplus W_2$ be an L-module satisfies ACC (respectively DCC) on s-closed submodules. Then W_1 and W_2 satisfy ACC (respectively DCC) on s- closed submodules.

Proof: Suppose W satisfies ACC (respectively DCC) on s-closed submodule and $A_1 \subseteq A_2 \subseteq \ldots$(respectively $A_1 \supseteq A_2 \supseteq \ldots$) be ascending (respectively descending) chain of s-closed submodules of W_1. Thus $A_1 \oplus W_2$, $A_2 \oplus W_2$, … are s-closed submodules of $W_1 \oplus W_2$, by [1, prop.2.5]. That is $A_1 \oplus W_2 \subseteq A_2 \oplus W_2 \subseteq \ldots$ (respectively $A_1 \oplus W_2 \supseteq A_2 \oplus W_2 \supseteq \ldots$) is a chain of s-closed submodules of W, but W satisfies ACC (respectively DCC) on s-closed submodules. So there exists $k \in Z_+$ such that $A_n \oplus W_2 = A_k \oplus W_2$ for all $n \geq k$. So $A_n = A_k$ for all $n \geq k$. Hence W_1 satisfies ACC (respectively DCC) on s-close submodules. By the same way of proof, W_2 satisfies ACC (respectively DCC) on s-closed submodules.

Recall that , " a submodule C is fully invariant in W if $f(C) \subseteq C$ for all $f \in End_R(W)$". [3]

Proposition 2. 5 : Let W = $W_1 \oplus W_2$ be an R-module where W_1 and W_2 are s-closed submodules of W . Then W satisfies ACC (respectively DCC) on nonzero s- closed submodules if and only if W_1 and W_2 satisfy ACC (respectively DCC) on nonzero s-closed submodules, provided that every s- closed submodule of W is a fully invariant.

Proof:

(⇨) See proposition 2.4.

(⇦) Suppose W_1 and W_2 satisfy ACC (respectively DCC) on s-closed submodules, to prove W satisfy ACC (respectively DCC) on s-closed submodules. Let and $A_1 \subseteq A_2 \subseteq \ldots$ (respectively $A_1 \supseteq A_2 \supseteq \ldots$) be ascending (respectively descending) chain of s-closed submodules of W.

Let $\pi_i : W \rightarrow W_i$ be a projection map for each i = 1, 2. Suppose that $A_i = (A_i \cap W_1) \oplus (A_i \cap W_2)$ by [10, Lemma.2.1].

Note that, A_i , W_1 and W_2 are s-closed submodules of W, for each i. Thus by [1, Remarks and Examples 2.2 (3)] $(A_i \cap W_1)$ and $(A_i \cap W_2)$ are s-closed submodules of W. Since $(A_i \cap W_1) \subseteq W_1 \subseteq W$, so by [1, prop.2.8, p.345] $(A_i \cap W_1)$ is an s-closed submodule of W_1 and $(A_i \cap W_2)$ is an s-closed submodule in W_2 for each i = 1,2, … . In fact if $A_i \cap W_j = 0$ for all i = 1,2, … and j = 1,2 then $A_i = (A_i \cap W_1) \oplus (A_i \cap W_2) = 0$ which is a contradiction with our assumption. That is $A_i \cap W_j$ are nonzero s-closed submodules in W for each i = 1, 2, … and j = 1, 2. So we have the following ascending (respectively descending) chain of nonzero s-closed submodules in W_j, $(A_1 \cap W_j) \subseteq (A_2 \cap W_j) \subseteq \ldots$ (respectively $A_1 \cap W_j \supseteq A_2 \cap W_j \supseteq \ldots$) for each j = 1, 2. But W_j satisfies ACC (respectively DCC) on s-closed submodules for each j = 1, 2. Thus there exists $k_j \in Z_+$ such that $A_n \cap W_j = A_{kj} \cap W_j$,for all $n \geq k_j$ and j = 1, 2. Let k = max { k_1, k_2 }, so $A_n = (A_n \cap W_1) \oplus (A_n \cap W_2) = (A_k \cap W_1) \oplus (A_k \cap W_2) = A_k$,for all $n \geq k$. Hence W satisfies ACC (respectively DCC).

Remark 2. 6 :

We can generalize proposition 2.5 for finite index I of the direct sum of L-modules.

Proposition 2. 7 : Let $A \leq B \leq W$ such that A is an s-closed submodule of an L- module W. W satisfies ACC (respectively DCC) on s-closed submodules if and only if $\frac{W}{A}$ satisfies ACC (respectivelyDCC) on s-closed submodules.

Proof: (⇨) Suppose W satisfied ACC on s- closed submodules, and let $\frac{B_1}{A} \subseteq \frac{B_2}{A} \subseteq \ldots$, be ascending chain of s-closed submodules of $\frac{W}{A}$, then B_i is an s-closed submodule of W by (proposition 1.12) . Thus there exists $k \in Z_+$ such that $B_n = B_k$ for all $n \geq k$. Hence $\frac{B_n}{A} = \frac{B_k}{A}$ for all $n \geq k$ .That is $\frac{W}{A}$ satisfies ACC on s-closed submodules.

https://doi.org/10.30526/30.3.1618

(⇦) Suppose $\frac{W}{A}$ satisfies ACC on s- closed submodules. Let $A \subseteq A_1 \subseteq A_2 \subseteq \dots$ be a chain of s-closed submodules of W. Since $A \subseteq A_1$ and $A \subseteq A_2$, … and A is an s-closed submodule of W, then by [1, coro.2.7, p.345] $\frac{A_i}{A}$ is an s-closed submodule of W for each i. Thus we have $\frac{A_1}{A} \subseteq \frac{A_2}{A} \subseteq \dots$ is an ascending chain of s-closed submodules of $\frac{W}{A}$, hence by our assumption $\frac{W}{A}$ satisfies ACC on s-closed submodules so there exists $k \in Z_+$ such that $\frac{A_n}{A} = \frac{A_k}{A}$ for all n≥k. That is $A_n = A_k$ for all n≥k which means W satisfied ACC on s-closed submodules.
By the same way we can prove that W satisfies DCC on s-closed submodules if and only if $\frac{W}{A}$ satisfies DCC on s-closed submodules.

Proposition 2. 8 : Let $W = W_1 \oplus W_2$ be an L-module and $L = ann(W_1) + ann(W_2)$. Then W satisfies ACC (respectively DCC) on s- closed submodules if and only if W_1 and W_2 satisfy A CC (respectively DCC) on s- closed submodules.

Proof: (⇨) see proposition 2.4.
(⇦) Let $E_1 \subseteq E_2 \subseteq \dots$ be an ascending chain of s- closed submodules of W (Since $L = ann(W_1) + ann(W_2)$, every submodule E_i of W has the form $N_i \oplus K_i$ for some $N_i \leq W_1$ and $K_i \leq W_2$). Hence by [1, prop.2.5] N_i is an s- closed submodule in W_1 ,and K_i is an s- closed submodule of W_2 for all i= 1, 2, … .So $N_1 \subseteq N_2 \subseteq \dots$ is an ascending chain of s- closed submodules of W_1 and $K_1 \subseteq K_2 \subseteq \dots$ is an ascending chain of s- closed submodules of W_2.
Since W_1 and W_2 satisfy ACC on s- closed submodules, then there exists $t, r \in Z_+$ such that $N_t = N_{t+i}$ and $K_r = K_{r+i}$, for each i = 1, 2, … . Take s = max {t, r}, hence $N_s \oplus K_s \subseteq N_{s+i} \oplus K_{s+i}$, for each i = 1, 2, … . That is W satisfies ACC on s- closed submodules.
By the same way we can prove that W satisfies DCC on s- closed submodules if and only if W_1 and W_2 satisfy DCC on s- closed submodules.

Proposition 2. 9 : Let W be an L-module such that the sum of any two s- closed submodules of W is again an s- closed submodule. If A is an s- closed submodule of W such that A and $\frac{W}{A}$ satisfy ACC (respectively DCC) on s-closed submodules, then W satisfies ACC (respectively DCC) on s- closed submodules.

Proof: Assume $B_1 \subseteq B_2 \subseteq \dots$ be ascending chain of s -closed submodules of an L-module W, then by [1, Remaks and Examples 2.2(3), p.343] $B_i \cap A$ is an s-closed submodule of W, for each i = 1, 2, … , but $(B_i \cap A) \subseteq A$, thus $B_i \cap A$ is an s-closed submodule of A, for each i = 1, 2, … , by [1, prop. 2.8, p.345].
Also, $B_i + A$ is an s - closed submodule of W (by our assumption), hence $\frac{B_i + A}{A}$ is an s - closed submodule of $\frac{W}{A}$, for each i = 1, 2, … , by proposition 1.12.
Now consider the two following two ascending chain of s-closed submodules of A and $\frac{W}{A}$:
$B_1 \cap A \subseteq B_2 \cap A \subseteq \dots$, and $\frac{B_1 + A}{A} \subseteq \frac{B_2 + A}{A} \subseteq \dots$, but A and $\frac{W}{A}$ satisfy ACC on s-closed submodules. Therefore , there exists $k_1, k_2 \in Z_+$ such that $B_n \cap A = B_{k1} \cap A$, for each $n \geq k_1$, and $\frac{B_n + A}{A} = \frac{B_{k2} + A}{A}$, for each $n \geq k_2$. By isomorphism theorem $\frac{B+A}{A} \cong \frac{B}{B \cap A}$ [2, Th. 3.4.3, p. 56] , so $\frac{B_n + A}{A} \cong \frac{B_n}{B_n \cap A}$.
Hence, $\frac{B_n}{B_n \cap A} = \frac{B_{k2}}{B_{k2} \cap A}$, which means $B_n \cap A = B_{k2} \cap A$, for each $n \geq k_2$. Let k = max{ k_1, k_2 }, thus $B_n \cap A = B_k \cap A$ for each n ≥ k and $B_n \cap A = B_k \cap B_n$ for each n ≥ k.
Now, for each n ≥ k , $B_n = B_n \cap (B_n + A) = B_n \cap (B_k + A) = B_k \cap (B_k + A) = B_k$.
Thus, M satisfies ACC on s-closed submodules.
By a similarly proof W satisfies DCC on s-closed submodules.

https://doi.org/10.30526/30.3.1618

Proposition 2. 10 : Let W be a FMFG L-module. Then W satisfies ACC (respectively DCC) on s- closed submodules if and only if L satisfies ACC (respectively DCC) on s- closed ideals.

Proof: (⇨) Suppose W satisfies ACC (respectively DCC) on s- closed submodules. To prove L satisfies ACC (respectively DCC) on s- closed ideals. Let $I_1 \subseteq I_2 \subseteq \ldots$ ($I_1 \supseteq I_2 \supseteq \ldots$) be an ascending (respectively descending) chain of s-closed ideals of L. Thus by (proposition 1.14) $A_1 = I_1W \subseteq A_2 = I_2W \subseteq \ldots$ (respectively $A_1 = I_1W \supseteq A_2 = I_2W \supseteq \ldots$) is an ascending (respectively descending) chain of s-closed submodules of W. But W satisfies ACC (respectively DCC) on s-closed submodules, so there exists $k \in Z_+$ such that $A_n = A_k$ for all $n \geq k$, hence $I_nW = I_kW$ for all $n \geq k$, that is $I_n = I_k$ for all $n \geq k$. So L satisfies ACC (respectively DCC) on s- closed ideals.

(⇦) Similarly.

Recall that, "an L- module W is called a scalar module if every L- endomorphism of W is a scalar homomorphism, that is for each $0 \neq f \in End(W)$, there exists $0 \neq s \in L$ such that $f(a)= sa$ for all $a \in W$". [11]

Corollary 2. 11 : Let W be a FMFG L-module. Then W satisfies ACC (respectively DCC) on s- closed submodules if and only if End(W) satisfies ACC (respectively DCC) on s- closed ideals.

Proof: (⇨) Since W be a FMFG L-module, then W is a scalar module by [11, Coro.1.1.11], $End(W) \cong \frac{L}{ann(M)}$ by [12, Lemma 6.2]. But $ann(W) = 0$, so $End(W) \cong L$. Hence the result follows by proposition 2.10.

(⇦) Similarly.

Future works:

1. Give an example shows that every noetherian (respectively artinian) module satisfies ACC (respectively DCC) on s -closed submodules.
2. Give an example shows that the converse of (Remark 2.2(2)) is not true in general.
3. Give an example shows that the converse of (Proposition 2.4) is not true in general.

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