Bounded Modules

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Abstract

Let R be a commutative ring with identity, and let M be a unitary (left) R-module. The ideal $ann_R M = \{r \in R; rm = 0 \ \forall \ m \in M\}$ plays a central role in our work. In fact, we shall be concerned with the case where $ann_R M = ann_R(x)$ for some $x \in M$ such modules will be called ***bounded*** modules.It turns out that there are many classes of modules properly contained in the class of bounded modules such as cyclic modules, torsion-free modules,faithful multiplication modules, prime modules and cyclic modules over their endomorphism rings. Also, using boundedness of modules, we showed that :

- The classes of injective modules modulo annihilator and quasi-injective modules are equivalent.
- The classes of faithful modules and compactly faithful modules are equivalent.

Introduction

Let R be a commutative ring with identity, and let M be a unitary (left) R-module. M is call***ed bounded*** R-module provided that there exists an element $x \in M$ such that $ann_R M = ann_R(x)$,where $ann_R M = \{r \in R; rm = 0 \ \forall \ m \in M\}$. (11, p.70).

Our objective is to investigate some of the properties of bounded modules and to examine in particular when such modules are cyclic or torsion-free or prime or cyclic module over their endomorphism ring. In the first section of this paper, we give necessary and sufficient conditions in order that a bounded module is cyclic or torsion-free or prime. The submodules of bounded modules are studied in this section. It is shown that the class of bounded modules is not closed under submodules in general.

Section two of this paper is devoted to discuss the concept of cyclic module over its endomorphism ring and the concept of compactly faithful module. Our main concern in this section will be studying the relation between injective modules modulo annihilator and quasi-injective modules in view of bounded modules, and the relation between faithful module and compactly faithful module namely by using bounded modules.

§ 1: Bounded Modules

1.1 Definition: An module M is R- said to be *bounded* module if there exists an element $x \in M$ such that $ann_R M = ann_R(x)$. Where $ann_R M = \{r \in R; rm = 0 \ \ \forall \ \ m \in M\}$. (11, p.70).

FIRST , we state and prove the following lemma:

1.2 Lemma: If R is commutative ring with 1 and M is an R-module then $ann_R(x) = ann_R(Rx)$.

Proof: Since $1 \in R$, then $x \in Rx$, hence $ann_R(Rx) \subseteq ann_R(x)$. Let $t \in ann_R(x)$, then $tx = 0$. To prove $t \in ann_R(Rx)$ since $tx = 0$, then $r(tx) = 0 = t(rx) \ \forall r \in R$, hence $t(Rx) = 0$, thus $t \in ann_R(Rx)$, which completes the proof.#

The following result is an immediate consequence of Lemma (1.2).

1.3 Corollary: Every cyclic R-module is bounded.

But the converse is not true in general, for example: The Z-module Q is bounded but not cyclic.

Recall that an R-module M is said to be *fully stable* if $ann_M(ann_R(x)) = (x)$ for each $x \in M$. (1, corollary 3.5).

In the next proposition, we give a necessary condition for bounded module to be cyclic.

1.4 Proposition: If M is a fully stable bounded R-module, then M is cyclic R-module.

Proof: Since M is bounded R-module, then there exists $x \in M$ such that $ann_R M = ann_R(x)$, therefore, $ann_M(ann_R M) = ann_M(ann_R(x))$, thus, $M = ann_M(ann_R(x))$. But, M is fully stable, implies that $M = (x)$, which completes the proof. .#

Let R be an integral domain and M be an R-module. An element $x \in M$ is called a *torsion element* of M if $ann_R(x) \neq 0$. The set of torsion elements denoted by $T(M)$ is a submodule of M. If $T(M) = 0$, the R-module M is said to be *torsion-free* (6, p.45).

1.5 Proposition: Every torsion-free R-module (where R is an integral domain) is bounded.

Proof: Let M be a torsion-free R-module, then $ann_R(m) = 0 \quad \forall 0 \neq m \in M$. But, $ann_R M = \bigcap_{m \in M} ann_R(m)$. So, $ann_R M = 0$. Therefore, M is bounded.#

But, the converse of (1.5) is not true in general, for example: For each positive integer $n>1$, Z_n as a Z-module is bounded module but not torsion- free. . However, we shall give, in the end of this section, a condition under which the converse is true.

Recall that an R- module Mis called multiplication if for every submodule N of M , there exists an ideal I in R such that N=IM, (14) . It is proved in (8,lemma 4.1) that a faithful multiplication module is torsion- free. We use this fact to give the following consequence of (1.5) .

1.6 Corollary: Let R be an integral domain and M be a faithful multiplication R-module, then M is bounded.

1.7 Remark: Boundedness of the quotient module M/N does not yield boundedness of M itself in general as the following example shows: Let $M = \bigoplus_{P \text{ is prime}} Z_P$,then M is not bounded Z-module. Let

$N = \bigoplus_{P>2 \text{ is prime}} Z_P$ is a submodule of *M* and $M = Z_2 \oplus N$ .Note that $M/N \approx Z_2$ is a bounded Z-module.

However, we shall, in the following proposition, give some restrictions to treat such a case, but first we need to recall the following concept: A submodule *N* of an R-module *M* is said to be ***pure*** if $IM \cap N = IN$,for every ideal *I* of *R*. In case *R* is a principal ideal domain (PID) or *M* is cyclic, then *N* is pure if and only if $rM \cap N = rN \ \forall r \in R$, (13). The residual of N in M denoted by $[N : M] = \{r \in R; rM \subseteq N\}$.

1.8 Proposition: Let *N* be a pure submodule of an R-module *M* such that *M/N* is a bounded R-module and $ann_R M = [N : M]$ .Then, *M* is bounded.

Proof: Since *M/N* is a bounded R-module, then there exists $\overline{x} \in M / N$ such that $ann_R M / N = ann_R(\overline{x})$ where $\overline{x} = x + N$. But, $ann_R M / N = [N : M]$ and $[N : M] = ann_R M$ by hypothesis. Hence, $ann_R M = ann_R(\overline{x})$.

It is left to show that $ann_R(\overline{x}) = ann_R(m)$ for some $m \in M$

Let $r \in ann_R(\overline{x})$,then $r\overline{x} = \overline{0}$ implies that $rx \in N$,so $rx \in N \cap rM$ and by the purity of *N*, we get that $rx \in rN$. Therefore, $rx = ry$ for some $y \in N$ that is $r(x - y) = 0$. Let $m = x - y$, then $rm = 0$ yields $r \in ann_R(m)$ and hence $ann_R(\overline{x}) \subseteq ann_R(m)$ .Also, if $t \in ann_R(m)$, then $tm = 0 = t(x - y)$, implies that $tx - ty \in N$ and hence $tx \in N$,so $t\overline{x} = \overline{0}$, thus $t, \in ann_R(\overline{x})$. #

1.9 Lemma(8): Let *R* be a (PID) and *M* be an R-module. If *M/N* is a torsion-free, then *N* is pure.

Proof: Assume that *N* is not pure in *M*, that is, there exists $r \in R$ such that $rM \cap N \neq rN$ that means $rM \cap N \not\subset rN$ then there exists

$x \in rM \cap N$ and $x \notin rN$,thus, $x = rm \in N$ and $m \notin N$ hence $rm + N = N$ then $r(m+N) = N$ implies that $m + N \in T(M/N) = \overline{0} = N$ .Thus, $m \in N$ which is a contradiction. Therefore, N is pure. #

1.10 Corollary :Let M be a faithful R-module (where R is a (PID)) and N be a submodule of M such that M/N is torsion-free. Then M is bounded.

A submodule N of an R-module M is called ***bounded*** if N is a bounded R-module, that is, N is bounded if there exists $x \in N$ such that $ann_R N = ann_R(x)$.

It is not necessary in general that every proper submodule of a bounded module be bounded, as it is shown in the following example:

1.11 Example :Let $R \models \{f \; f: R \to R \; is \; map\}$ we define + and . on R as follows:

$$(f+g)(x) = f(x) + g(x) \text{ and } \forall x \in R \; (f.g)(x) = f(x).g(x)$$

$(R,+,\cdot)$ $\forall f, g \in R,$ is a commutative ring with identity where $I : R \to R$ such that $I(x) = 1 \quad \forall x \in R$ is the identity element of R.

Let M=R as an R-module, then M is bounded R-module by (1.3). Let $\forall x \notin [-n, n]\, N = \{f \in R; f(x) = 0 \}$ where $n \geq 0$ is an integer depending on f. To prove N

is a submodule of M. $N \subseteq M$ and $N \neq \Phi$ since the zero map is in N. Let $f, g \in N$ then there exists n,m non-negative integers such that $\forall x \notin [-n, n]\; f(x) = 0$ $\forall x \notin [-m, m]\; g(x) = 0$.

If $n > m$ then $\forall x \notin [-n, n]\,(f - g)(x) = f(x) - g(x) = 0$.

Thus, $f - g \in N$

Let $h \in R$ and $f \in N$ then there exists an integer $n \geq 0$ such that $\forall x \notin [-n, n]\; (h \cdot f)(x) = h(x) \cdot f(x) = 0$ ،$\forall x \notin [-n, n]\, f(x) = 0$ then $h \cdot f \in N$, thus, N is a submodule of M.

We claim that $ann_R N = \{\mathbf{0}\}$ .L et $h \in R$ and، $h \neq 0$ then $h(a) \neq 0$ for some $a \in R$ .Define $f : R \to R$ such that:

$$f(x)=\begin{cases}0 & if\ x\neq a\\ b & if\ x=a\end{cases} \qquad \text{where} \quad b\neq 0$$

hence $f\in R$ and $\forall x\notin[-n,n]\ f(x)=0$ where $n>a$ .Therefore, $f\in N$ and $(h\cdot f)(a)=h(a)\cdot f(a)\neq 0$ .Hence $ann_R N=\{0\}$

While for each $f\in N$, $ann_R f\neq\{0\}$ .For if $f\in N$ then $f(x)=0$ $\forall x\notin[-n,n]$ n is non-negative integer. Define $h:R\to R$ such that:

$$x\in R\ ,\quad h(x)=\begin{cases}0 & \forall x\in[-n,n]\\ x & if\ x\notin[-n,n]\end{cases} \qquad \text{and } x\neq 0$$

then $h\in R$ and، $(h\cdot f)(x)=h(x)\cdot f(x)=0$ implies that $h\in ann_R f$. Therefore, *N* is not bounded R-module. #

However, we give in the next proposition a condition under which the class of bounded modules is closed under submodules. But, first we need to recall some definitions: An R-module *M* is said to be ***uniform*** module if every non-zero submodule of *M* is essential (2), Where *N* is called ***essential*** in M provided that $N\cap K\neq 0$ for every non-zero submodule *K* of *M* (2).

An ideal I of R is called prime if for each $a,b\in R, ab\in I$ implies that $a\in I$ or $b\in I$ (13,P.38)

1.12 Proposition: Let M be an R-module and $0\neq x\in M$ such that:

- *Rx* is an essential submodule of *M*.
- $ann_R(x)$ is a prime ideal of *R*, and
- $ann_R M=ann_R(x)$. Then every submodule of *M* is bounded.

Proof: Let *N* be a submodule of *M*. Then, there exists $0\neq t\in R$, $tx\in N$ and $tx\neq 0$. Hence $ann_R(x)\subseteq ann_R N\subseteq ann_R(tx)$

Let $r\in ann_R(tx)$ then $r(tx)=0=(rt)x$ implies that $rt\in ann_R(x)$.

But, $t\notin ann_R(x)$, therefore, $r\in ann_R(x)$. Implies that

$r\in ann_R N$ .Therefore $ann_R N=ann_R(tx)$ which completes the proof.#

1.13 Corollary: If M is a bounded uniform R-module such that $ann_R M$ is a prime ideal of R, then every submodule of M is bounded.

Recall that an R-module M is said to be a ***prime*** module if $ann_R M = ann_R N$ for every non-zero submodule N of M (7). It is clear that every prime R-module is, bounded module, but the converse need not be true in general, for example :Let $M = Z \oplus Z_n$ as a Z-module M is a bounded module but not prime module since $ann_Z M = 0$, but $ann_Z(0 \oplus Z_n) = nZ \neq (0)$.

However, we give the following partial converse :

1.14 Proposition : Let M be an R-module and $0 \neq x \in M$ such that:

- Rx is an essential submodule of M.
- $ann_R(x)$ is a prime ideal of R, and
- $ann_R M = ann_R(x)$. Then, M is a prime R-module.

Proof : Following what's in the proof of proposition (1.12), we get that :

$ann_R N = ann_R(tx) \subseteq annn_R(x) = ann_R M$ .Hence

$ann_R N = ann_R M$ for every non-zero submodule N of M. So, M is a prime R-module. #

1.15 Corollary :If M is a bounded uniform R-module such that $ann_R M$ is a prime ideal of R, then M is a prime R-module.

The condition $ann_R M$ is a prime ideal of R in corollary (1.3.2) cannot be dropped. For example: Z_8 as a Z-module is a uniform bounded Z-module, which is not, prime Z-module. In fact, $ann_Z Z_8 = 8Z$ is not a prime ideal of Z.In the following proposition, we give a condition under which the converse of proposition (1.5) is true:

1.16 Proposition :If R is an integral domain and M is a faithful uniform bounded R-module, then M is torsion-free.

Proof: By corollary (1.15) and [5,remark 1.1(2)]. #

Next ,we study the direct sum of bounded modules .

1.17 Proposition: Let M_1 and M_2 be two bounded R-modules. Then, $M_1 \oplus M_2$ is a bounded R-module.

There exists $x \in M_1$ such that $ann_R M_1 = ann_R(x)$. Also, there exists $y \in M_2$ such that $ann_R M_2 = ann_R(y)$. So, $(x,y) \in M_1 \oplus M_2$. We claim that $ann_R(M_1 \oplus M_2) = ann_R((x,y))$. Let $r(x,y)=(0,0)$, so $(rx,ry)=(0,0)$. It follows that $rx=0$ and $ry=0$, that is $r \in ann_R(x)$ and $r \in ann_R(y)$, therefore, $r \in ann_R M_1$ and $r \in ann_R M_2$. Now, if $(m,m') \in M_1 \oplus M_2$, then $r(m,m')=(rm,rm')=(0,0)$ implies that $r \in ann_R(M_1 \oplus M_2)$.Therefore, $ann_R(M_1 \oplus M_2) = ann_R((x,y))$, which completes the proof. #

1.18 Corollary: A finite direct sum of bounded R-modules is bounded.

However, an infinite direct sum of bounded R-modules need not be bounded, for example: Z_P as a Z-module is bounded for all primes P, but $\bigoplus_{P \text{ is prime}} Z_P$ is not a bounded Z-module. In addition, direct summand of a bounded module need not be bounded in general. For example:

Let $M = Z \oplus Z_{P^\infty}$ as a Z-module, M is bounded since $ann_Z M = 0 = ann_Z((1,0))$, but Z_{P^∞} is not bounded Z-module.

1.19 Proposition: Let M be an R-module and let I be an ideal of R, which is contained in $ann_R M$. Then, M is a bounded R-module, if and only if M is a bounded R/I-module.

Proof: If M is a bounded R-module, then there exists $x \in M$ such that $ann_R M = ann_R(x)$, we claim that $ann_{R/I} M = ann_{R/I}(x)$.

Let $r+I \in ann_{R/I}(x)$, so $(r+I)x=0$, but $(r+I)x = rx = 0$, that is $r \in ann_R(x)$, therefore, $r \in ann_R M$, implies that $rm = 0 \ \forall \ m \in M$. Then, $(r+I)m = 0 \ \forall \ m \in M$, therefore, $r+I \in ann_{R/I} M$, thus, M is bounded R/I-module.

Next, if M is a bounded R/I-module, then there exists $x \in M$ such that $ann_{R/I}M = ann_{R/I}(x)$, we claim that $ann_R M = ann_R(x)$.
Let $r \in ann_R(x)$, so $rx = 0$, but $rx = (r+I)x = 0$, that is $r+I \in ann_{R/I}(x)$, therefore, $r+I \in ann_{R/I}M$, implies that $(r+I)m = 0 \; \forall \; m \in M$. Then, $rm = 0 \; \forall \; m \in M$, therefore, $r \in ann_R M$, so M is a bounded R-module. #
we discuss the localization of boundedness in the following result:

1.20 Proposition: If M be a finitely generated bounded R-module and S be a multiplicatively closed subset of R, then M_S is a bounded R_S-module.
Proof: Since M is a bounded R-module, then there exists $x \in M$ such that $ann_R M = ann_R(x)$, so $(ann_R M)_S = (ann_R(x))_S$. But, M is finitely generated, thus, $ann_{R_S} M_S = ann_{R_S}((x)_S)$ by (7, proposition3.14, p.43).
It is left to show that the localization of a cyclic R-module is cyclic R_S-module.

Let $k \in (x)_S$, then $k = \frac{a}{s}; a \in (x)$ and $s \in S$, that is $k = \frac{rx}{s} = \frac{r}{s} \cdot \frac{x}{1} \in \left(\frac{x}{1}\right)$, implies that $(x)_S \subseteq \left(\frac{x}{1}\right)$.

Also, if $h \in \left(\frac{x}{1}\right)$, then $h = \frac{u}{t} \cdot \frac{x}{1} = \frac{ux}{t} \in (x)_S$ (since $ux \in (x)$ and $t \in S$), that is $\left(\frac{x}{1}\right) \subseteq (x)_S$. Implies that $(x)_S = \left(\frac{x}{1}\right)$. Therefore, $ann_{R_S} M_S = ann_{R_S}((x)_S) = ann_{R_S}\left(\frac{x}{1}\right)$, hence, M_S is a bounded R_S-module. #

1.21 Corollary: If P is a prime ideal of R and M is finitely generated bounded R-module, then M_P is a bounded R_P-module.

S: 2 Boundedness and Injecativity:

1.22 Definition :An R-module M is called ***cyclic over its endomorphism ring*** provided that there exists an element $x \in M$ such that each element $m \in M$ can be written as $m = f(x)$ for some $f \in End_R(M)$ (11).

In the following two remarks we show that theise are plenty of such modules:

2.2 Remark :Every cyclic R-module M is cyclic over $End_R(M)$.

Proof: Let M be a cyclic R-module. Then, $M = (x) = Rx$ for some $x \in M$. It can be easily checked that $M \approx R / ann_R(x)$.Let $\overline{R} = R / ann_R(x)$,then= $\overline{R}x = \{(r + ann_R(x))x : r \in R\}$. $\{rx : r \in R\} = Rx = M$. So, M is a cyclic $\overline{R}$ -module. On the other hand, $End_R(M) \approx \overline{R}$ (15), so the result follows. #

2.3 Remark :If $M = R \oplus A$ ،A is any R-module, then M is a cyclic $End_R(M)$-module generated by (1,0) where 1 is the identity element of R.

Proof :Let $m \in M$ .Since R is a free R-module with basis {1}, then there exists an R-homomorphism $h : R \to R \oplus A$ such that $h(1) = m$ by (17 ،Propo.4,p.162). Let $\rho : R \oplus A \to R$ be the natural projection. Let $f = h \circ \rho$ then $f \in End_R(M)$ and $f(1,0) = h(\rho(1,0)) = h(1) = m$ which was what we wanted. #

In the following proposition we shows that the class of bounded modules contains the class of cyclic modules over their endomorphism rings:

2.4 Proposition :Let M be an R-module, if M is cyclic over $End_R(M)$ .Then, M is a bounded R-module.

Proof: Since M is cyclic over $End_R(M)$ then, there exists $x \in M$ such that for each $m \in M$ there exists $f \in End_R(M)$ such that $m = f(x)$. We claim that $ann_R M = ann_R(x)$.

Let $r \in ann_R(x)$ then $rx = 0$. So, $f(rx) = 0 = rf(x) = rm$ .Therefore, $rm = 0 \;\; \forall m \in M$ that is $r \in ann_R M$.thus, M is a bounded R-module. #

But, the converse of (2.4) is not true in general. However, we were not able to give an example of bounded module, which is not cyclic over its endomorphism give an example . On the other hand, we give in the following remark a condition under which the converse is true:

2.5 Remark : Every bounded fully stable R−module M is cyclic over $End_R(M)$.

Recall that M is called an ***injective*** R-module if for any two R-modules A, B and for any monomorphism $f : A \to B$ and any homomorphism $g : A \to M$ ،there exists a homomorphism $h : B \to M$ such that $hof = g$ [16, p.219]. An R-module M is called ***quasi-injective*** if for every submodule N of M, every R-homomorphism of N into M can be extended to an R-endomorphism of M (12). The following lemma is needed:

2.6 Lemma (10) : Let M be an R-module. If $x \in M$ and ,$ann_R M = ann_R(x)$, then for every $y \in M$ there exists an R-homomorphism $f : Rx \to Ry$ such that $f(x) = y$.

Proof: We define $f : Rx \to Ry$ such that $f(rx) = ry \forall r \in R$ .f is well-defined for if $r_1 x = r_2 x$ for some $r_1, r_2 \in R$, then, $(r_1 - r_2)x = 0$ that is,$r_1 - r_2 \in ann_R(x)$, hence $r_1 - r_2 \in ann_R M$, therefore,$(r_1 - r_2)y = 0$ ،$\forall y \in M$ that is، $r_1 y = r_2 y$ thus .

Let $r_1 x, r_2 x, rx \in Rx, t \in R$.

$$f(r_1 x + r_2 x) = f((r_1 + r_2)x) = (r_1 + r_2)y = r_1 y + r_2 y = f(r_1 x) + f(r_2 x)$$

$$f(t(rx)) = f((tr)x) = (tr)y = t(ry) = tf(rx).$$

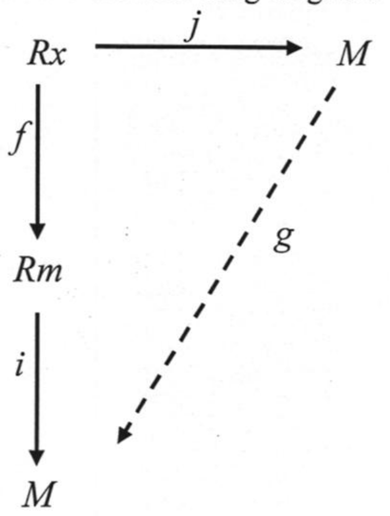
Thus, f is an R-homomorphism

$f(x)=f(1.x)=1.y=y$ which completes the proof. #

Now , we state and prove the following result :

2.7 Proposition : Let M be a quasi-injective bounded R-module, then M is cyclic over $End_R(M)$.

Proof: Since M is bounded R-module, then there exists $x \in M$ such that $ann_R M = ann_R(x)$, therefore, for every $m \in M$ there exists an R-homomorphism $f: Rx \to Rm$ such that $m = f(x)$ by Lemma (2.6). Since M is quasi-injective, we have the following diagram:



Such that $g \circ j = i \circ f$ where $i: Rm \to M$ and $j: Rx \to M$ are inclusion maps .Let $y \in Rx$ then $(g \circ j)(y) = (i \circ f)(y)$ so $g(j(y)) = i(f(y))$, therefore, $g(y) = f(y)$,thus, $g / Rx = f$ implies that $f \in End_R(M)$.#

Using boundedness, we get the following result :

2.8 Proposition: If M is a bounded faithful R-module, then R can be embedded (as an R-module) in M.

Proof: Since M is bounded faithful R-module, then $ann_R M = 0 = ann_R(x)$ for some $x \in M$ Let $\propto: R \to M$ be such that

$\propto(r) = rx$ for all $r \in R$ .It is easily checked that $\propto$ is a well-defined R-homomorphism.

Also, Ker $\propto = \{r \in R; \propto(r) = 0\} = \{r \in R; rx = 0\} = ann_R(x) = 0$

Thus , $\propto$ is a monomorphism. Hence, the result follows# .

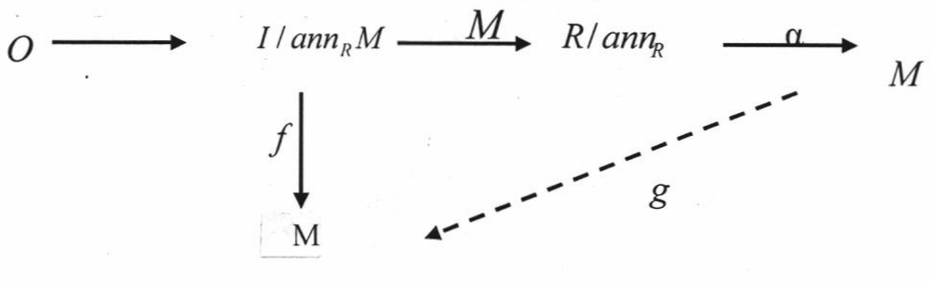
It is well known that if M is injective $-R/ann_R M$ module, then M is quasi-injective R-module (10, p.149). However, the converse is not true in general, for example: Let $M = \bigoplus_{P \text{ is prime}} Z_p$ as a Z-module ،M is semisimple module and hence is quasi-injective, but not injective module over $Z/ann_Z(\bigoplus_{P \text{ is prime}} Z_P) \approx Z$.

In the following proposition we show that the converse of the above fact is true in the class of bounded modules:

2.9 Proposition: If M is a bounded quasi-injective R-module, then M is an injective $-R/ann_R M$ module.

Proof: Since M is a bounded R-module, then M is a bounded $R/ann_R M-$ module by proposition (1.19) and since M is faithful $R/ann_R M-$module, hence there exists a monomorphism $\propto: R/ann_R M \to M$ such that $\propto(\bar{r}) = \bar{r}x \ \forall \bar{r} \in R/ann_R M$, $\bar{r} = r + ann_R M$ by proposition (2.8).

Let $I/ann_R M$ be an ideal of $R/ann_R M$ where I is an ideal of R containing $ann_R M$ and $f : I/ann_R M \to M$ be an $R/ann_R M-$ homomorphism. Consider the following diagram:



Since M is a quasi-injective R-module, then M is a quasi-injective $R/ann_R M$ – module by (3 ،Lemma 2). Therefore, there exists an $R/ann_R M$ – homomorphism say $g: M \to M$ such that the preceding diagram is commutative, that is, . $g \circ \propto \circ i = f$ .Now, let، $\bar{r} \in I/ann_R M$, then $= \bar{r}g(x) = \bar{r}m\ f(\bar{r}) = (g \circ \propto oi)(\bar{r}) = g(\propto (i(\bar{r}))) = g(\bar{r}x)$, where $g(x) = m$. Thus, M is injective $R/ann_R M$ -module by (9, Theorem 6, p.5).#

2.10 Corollary: A faithful bounded R-module M is quasi-injective if and only if M is an injective R-module.

Recall that an R-module M is called ***compactly faithful*** if R can be embedded in M^n for some positive integer n, where M^n is a direct sum of n-copies of M, (11, p. 67). Clearly,
every compactly faithful module is faithful. However, the converse is not true in general as seen in the following example:

Let $M = \bigoplus_{p \ is \ prime} Z_p$ as a Z-module M is faithful, but not compactly faithful because the ring Z cannot be embedded in any finite sum of copies of the Z-module $\bigoplus_{p \ is \ prime} Z_p$.

In the the following proposition we show that the converse holds in the class of bounded modules:

2.11 Proposition :Every bounded faithful module is compactly faithful.

Proof: Follows immediately proposition (2.8). #

Under the circumference of finite generation of modules, the two concepts of faithful and compactly faithful are equivalent.

2.12 Proposition :Every finitely generated faithful R-module is compactly faithful.

Proof: Let M be a finitely generated faithful R-module. Let $\{x_1,\cdots,x_n\}$ be a generating set for M. Therefore, $0 = ann_R M = \bigcap_{i=1}^{n} ann_R(x_i)$.

Let $m = (x_1,\cdots,x_n) \in M^n$,then $ann_R(m) = 0$. But Rm is a submodule of M^n and $R/ann_R(m) \approx Rm$ .So, $R \approx Rm$ which completes the proof.#

An R-module M is called ***Noetherian*** if and only if every submodule of M is finitely generated [6, proposition 6.2, p.75].

Corollary : If M is a Noetherian faithful R-module, then M is compactly faithful.

References

1. Abbas M.S. (1990). On fully stable module, Ph. D. Thesis, University of Baghdad.
2. Ahmed, Abdul-Rahman, A. and Al-Hashimi, B. (1994). On Submodules of Multiplication Modules, Iraqi J. Sci., Vol. 35, No.
3. Ahsan, J. (1973). Rings All of Whose Cyclic Modules are Quasi-Injective, Proc., London Math. Soc. (3) :27, 425-439.
4. Al-Aubaidy, M.A. On Regular Modules, to appear. Iraqi J. of Science.
5. Al-Bahrany, B.(1996). A Note on Prime Modules and Pure Submodules, J. Sci., 37 :1431-1441.
6. Atiyah, M.F. and Macdonald, I.G. (1969). Introduction to commutative algebra, University. of Oxford.
7. Desale, G. and Nicholson, W.K. (1981). Endoprimitive Rings, J. Algebra, 70: 548-560.
8. El-Bast, Z. A. and Smith, P.F. (1988). Multiplication Modules, Comm. In Algebra, 16: 755-779.
9. Faith, C. (1967).Lectures on injective module and quotient rings, Springer-Verlag, Berlin Heidelberg, New York.
10. Faith, C. Modules finite over endomorphism rings, Department of Mathematics Rutger, the State University New Brunswick, New Jersey.
11. Faith, C. (1976). Algebra II: Ring theory, Springer-Verlag, Berlin Heidelberg, New York.

12. Harada, M. (1965). Nota on Quasi– Injective Modules' Osaka J. Math., 2: 351-356.
13. Kasch, F. (1972). Modules and rings, Academic Press, London.
14. Naoum, A.G. and Hasan, M.A.(1986). The Residual of finitely Generated Multiplication Modules, Arch. Math., vol.46: 225-320.
15. Naoum, A.G. (1995). Regular Multiplication Modules", Periodica Mathematica Hungarica,vol.31 :(2)155-162.
16. Sahai, V. and Bist, V. Algebra, Narosa publishing Gouse New Delhi Madras Bombay calculate.
17. Solin, A. (1977). Theory of modules, AWiley-Intersciense Publication, London, New York, Sydny, Toronto.

الموديولات المقيده

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الخلاصة

لتكن R حلقه ابداليه ذات محايد ، وليكن M موديولا احادياً (ايسر) على الحلقــه R . المثالي

$$ann_R M = \{r \in R; rm = 0 \quad \forall \ m \in M\}$$

يلعــب دورا اساســياً فــي عملنــا . فــي الحقيقــه ســنهتم بالحالــه التــي يكــون فيهــا $ann_R M = ann_R(x)$ لبعض x في M ، مثل هذه الموديولات تسمى موديولات مقيده ، فقد تبين لنا انه هناك عدة اصناف من الموديولات التي تكون محتواة فعليا في صنف الموديــولات المقيده مثل الموديولات الدواره ، الموديولات عديمة الالتواء ، الموديولات الجدائيــه المخلصــه ، الموديولات الاوليه ، الموديولات الدوريه على حلقات تشاكلاتها .

كما ظهر لنا وباستخدام خاصيه التقييد للموديولات ما يأتي

- ان صنفي الموديولات الاغماريه على الحلقه R/ annM والموديولات شبه الاغماريــه علــى الحلقه R متكافئين.
- ان صنفي الموديولات المخلصه والموديولات المرصوصه الاخلاص متكافئين .