On Rational – Valued Characters of Certain Types of Permutation Group

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Abstract

Tow results are proved. The first gives necessary and sufficient conditions for a permutation group to have the property that each of its rational – valued character can be written as (integral) linear combination of characters induced from the principal characters of certain subgroup. The other presents that this property is extendable to direct product of groups.
Examples give.

Introduction

Artin's induction theorem says that any rational - valued character of a finite group is a rational linear combination of the induced principal characters of cyclic subgroups (these characters are sometimes known as the Artin's characters). An analogous result for permutation groups is conjectured by R. Merris (1), but with characters induced from the principal characters of certain subgroups (larger) than the cyclic subgroups. In (2) F. C. Silva showed that Merris's conjecture is not in general true, and gave a criterion for the conjecture to hold in a given group. To fix the background, let G be a subgroup of the symmetric group S_n and let $\sigma \in G$. Then σ may be expressed in an essentially unique way as a product of disjoint cycles

$$\sigma = W_1 ... W_r,$$

where we include any cycle of length 1. For each i, let Ω_i denote the set of symbols appearing in the cycle W_i. Then $\Omega_1,...,\Omega_r$ are the orbits of the cyclic group $<\sigma>$ acting on the set $\{1, ..., n\}$. If W_i has order m_i then $(m_1,.....,m_r)$ is called the cycle - type of σ. If $\sigma,\tau \in S_n$ then σ and τ are said to be cycle – equivalent if $<\sigma>$ and $<\tau>$ have identical orbits on $\{1, ..., n\}$.

Let $Y(\sigma)$ be the subgroup of S_n defined by :

$Y(\sigma) = \{\tau \in S_n : \tau\ \Omega_i = \Omega_i \text{ for all } i\}$

Then $Y(\sigma)$ is clearly a Young subgroup of the partition of n give by $(m_1,.....,m_r)$.

Now define the closer of the cyclic group $<\sigma>$ to be

$$G_{[\sigma]} = G \cap Y(\sigma)$$

Certainly $G_{[\sigma]} \leq G$ and it is not hard to see that the definitions of $G_{[\sigma]}$ given here is equivalent to the both definitions in (1) and (3).

Now define ρ_σ to be the character of G obtained by inducing the principal character of $G_{[\sigma]}$:

$$\rho_\sigma = (1_{G_{[\sigma]}})^G .$$

Then Merris conjectured in (1) that every rational – valued character of G is a rational linear combination of the $\rho_\sigma, \sigma \in G$. In this context, F. C. Silva (2) proved:

Theorem Let $G \leq S_n$. Then the following statements are equivalent.

(a) Every rational – valued character of G is a rational liner combination of the characters $\rho_\sigma, \sigma \in G$.

(b) If τ, π are cycle – equivalent elements of G then $<\tau>$ and $<\pi>$ are conjugate in G.

In (4), T. Y. Lam showed that every rational – valued character of a finite group G is an (integral) linear combination of Artin's characters if and only if G is cyclic, it thus seemed of interest to investigate the situation for $G \le S_n$, with each rational – valued character expressable as (integral) linear combination of $\rho_\sigma, \sigma \in G$.

In the present paper we deal with permutation groups that satisfy the hypothesis (H) given in Theorem (1-1), (b) above.

1. Results on $G_{[\sigma]}$

In this section we establish some basic properties of $G_{[\sigma]}$ that we will use repeatedly.

Clearly $G_{[\sigma]} = G_{[\tau]}$ if and only if σ and τ are cycle – equivalent.

Lemma (1.1) $\tau \in G_{[\sigma]}$ if and only if $G_{[\tau]} \in G_{[\sigma]}$

Proof : sufficiency is clear. Conversely, if $\tau \in G_{[\sigma]}$ then $\tau \in Y(\sigma)$ which implies that $Y(\tau) \le Y(\sigma)$, whence the result.

Lemma (1.2)

If $\tau, \sigma \in G$, then $\tau^{-1} G_{[\sigma]} \tau = G_{[\tau^{-1}\sigma\tau]}$

Proof : $\tau^{-1}(G \cap Y(\sigma))\tau = G \cap (\tau^{-1} Y(\sigma)\tau) = G \cap Y(\tau^{-1}\sigma\tau)$, hence the result.

Let G be a finite group, and let A be a mapping from G to the set of subgroups of G such that the following conditions are satisfied :

(i) $x \in A(x)$,

(ii) if $y \in A(x)$ then $A(y) \le A(x)$, and

(iii) $y^{-1} A(x) y = A(y^{-1}xy)$ for all $x, y \in G$.

The mapping A satisfying the above conditions defines an equivalence relation on G, by setting $x - y$ wherever there exist $z \in G$ such that $z^{-1} A(x) z = A(y)$, and the equivalence classes are called A-classes of G.

Let $\overline{R}(G)$ be the group generated by the set of rational – valued character of G, and let $p(G)$ denote the subgroup of $\overline{R}(G)$ generated by the permutation characters.

$$(I_{A(x)})^G ; x \in G.$$

Now let $\{<x_1>,\ldots\ldots,<x_t>\}$ be the set of representative of the conjugacy classes of cyclic subgroup of G.

Let $\overline{G} = Gal(Q(\varepsilon)/Q)$, where ε is a primitive $|G|$ - th root of 1.

Then by Brauer's lemma on character table $\overline{G}$ permutes the set X = Irr (G) consisting of the absolutely irreducible characters of G and also permute the set Y consisting of the conjugacy classes of G, and let $\{X_1,\ldots\ldots,X_t\}$ and $\{Y_1,\ldots\ldots,Y_t\}$ denote the $\overline{G}$ -orbits of the action of $\overline{G}$ on the sets X and Y respectively.
Then by setting

$$\upsilon(G) = (\prod_{i=1}^{t} |X_i|)^{-1} (\prod_{i=1}^{t} |Y_i|),$$

We can state the following theorem due to Solomon (5).

Theorem (1.3) [Solomon]

Let G be a finite group, and let $\{A(x) : x \in G\}$ be a family of subgroup satisfying the conditions.

(i) – (iii) and assume that for all $x,y \in G$, $A(x)$ is conjugate to $A(y)$ implies

$<x>$ is conjugate to $<y>$. Then $P(G)$ is of finite index in $\overline{R}(G)$, and we have

$$\left|\overline{R}(G) : P(G)\right|^2 = \upsilon(G) \prod_{i=1}^{t} |N(A(x_i) : A(x_i))|^2 |C(x_i)|^{-1},$$

Where $N(A(x_i))$ is the normalizer of $A(x_i)$ in G and $C(x_i)$ is the centeralizer of x_i.

Returning to our subgroup, let A be the mapping given by $A(\sigma) = G_{[\sigma]}$, $\sigma \in G$, clearly $\sigma \in G_{[\sigma]}$ and by Lemma (1.1) and (1.2) this mapping satisfies the conditions (ii) and (iii).

2. Results

Here we meet our main results. First we have:

Theorem (2.1)

Let G be a subgroup of S_n, satisfying (H), then each rational – valued character of G can be written as Z-linear combination of $\rho_\sigma, \sigma \in G$, if and only if $\upsilon(G)\prod_\sigma \left|N(G_{[\sigma]}) : G_{[\sigma]}\right|^2 = \prod_\sigma |C(\sigma)|$,

Where the multiplication ranges over the set of representatives of the conjugacy classes of distinct cyclic subgroup $<\sigma>$ of G.

Proof :

$\sigma, \tau \in G$ are cycle equivalent implies that $G_{[\sigma]} = G_{[\tau]}$, so we have if $G_{[\sigma]} = G_{[\tau]}$ then $<\sigma>$ and $<\tau>$ are conjugate in G.

Hence each rational – valued character of G is Z- linear combination of $\rho_\sigma, \sigma \in G$ if and only if

$$\left|\overline{R}(G) : p(G)\right| = 1$$

And by Solomons theorem, if and only if

$$\upsilon(G)\prod_\sigma \left|N(G_{[\sigma]}) : G_{[\sigma]}\right|^2 = |C(\sigma)|$$

Examples :

(1) Let G be cyclic group of order P^k, P is a prime.

Then it can be easily seen that $\upsilon(G) = 1$.

$$\text{Now } \prod_\sigma \left|N(G_{[\sigma]}) : G_{[\sigma]}\right|^2 = (\prod_\sigma \frac{|G|}{<\sigma>})^2$$

$$= (\prod_{i=0}^{k} \frac{P^k}{P^i})^2 = P^{2\sum_{i=0}^{k} i} = P^{k(k+1)}$$

On the other hand the centeralizer of $\sigma \in G$ is equal to P^k and we have

(k+1) distinct subgroup of G whence $|C(\sigma)| = P^{k(k+1)}$, which implies that each rational-valued character of cyclic subgroup G can be written as Z-linear combination of $\rho_\sigma, \sigma \in G$.

(2) The situation for elementary abelian group $G = Z_2^n$ of exponent 2 and order

2^n can be easily verified .

In this case $|X_i| = 1$ for all i, since the irreducible characters are rational – value and clearly $|Y_i|$ are all i. Thus $\upsilon(G) = 1$ and we have

$$\upsilon(G)\prod_{\sigma}|N(G_{[\sigma]}):G_{[\sigma]}|^2 = (\prod_{i=0}^{n}\frac{2^n\binom{n}{i}}{2^i})^2$$

$$= \left(2^{\sum_{i=0}^{n-1}(n-i)\binom{n}{i}}\right)^2 = \left(2^{\sum_{i=0}^{n-1}n\binom{n-1}{i}}\right)^2 = 2^{n.2^n},$$

$$\text{and } \prod_{\sigma}|C(\sigma)| = \prod_{i=1}^{2^n}(2^n) = 2^{n.2^n}.$$

Therefore Z_2^n satisfy the condition in theorem (2.1).

(3) Another example is the dihedral group of order $2p$ where p is on odd prime $D_p = < a, b : a^2 = 1, b^p = 1, aba = b^{-1} >$,

The distinct non conjugate cyclic subgroups of D_p are

{<1>,<a>,<b>}, and we have $G_{[1]} = < 1 >, G_{[a]} = < a >$, and $G_{[b]} = D_p$.

$\upsilon(D_p) = 1$, and $\prod_{\sigma}|N(G_{[\sigma]}):G_{[\sigma]}|^2 = [\frac{2p}{1}\times\frac{2}{2}\times\frac{2p}{2p}]^2 = 4p^2$.

$\prod|C(\sigma)| = 2p.2.p = 4p^2$, hence D_p satisfy our property.

Next we show that in some circumstances this property holds for the direct product of groups. But before proving the theorem we need the following two lemmas.

Lemma (2.2)

Let $\sigma=(\sigma_1,\sigma_2)\in G=G_1\times G_2$, then

$$G_{[\sigma]}=G_{1[\sigma_1]}\times G_{2[\sigma_2]}$$

Proof $G_{[\sigma]}=(G_1\times G_2)\bigcap Y(\sigma)=(G_1\times G_2)\bigcap(Y(\sigma_1)\times Y(\sigma_2))$
$=(G_1\bigcap Y(\sigma_1))\times(G_2\bigcap Y(\sigma_2))=G_{1[\sigma_1]}\times G_{2[\sigma_2]}$.

Let G_1 and G_2 be two finite groups, $|G_1|=n_1,|G_2|=n_2$

with $(n_1,n_2)=1$, and let $l_1=\{\sigma_1,....,\sigma_s\}$ and $l_2=\{\tau_1,.....,\tau_t\}$ be the full sets of representatives of the conjugacy classes of distinct cyclic subgroups of G_1 and G_2 respectively, then $l_1\times l_2$ gives the full set of representatives of the conjugacy classes of the distinct subgroups of $G=G_1\times G_2$.

Let $\overline{G}=Gal(Q(\varepsilon)/Q)$ where ε is a primitive (n_1,n_2)-th root of 1, and for (i= 1,2), let $\overline{G_i}=Gal(Q(\varepsilon_i)/Q)$.

Where ε_i i is a primitive $|G_i|$-th root of 1. Then with the obvious abuse of notation we have

$$\overline{G}=\overline{G_1}\times\overline{G_2}$$

with component wise action :

$(\sigma,\tau)(g_1,g_2)=((\sigma)g_1,(\tau)g_2)$,

where $g_i\in\overline{G_i},\sigma\in G_1,\tau\in G_2$.

For each v, $1\le v\le st$, we not that the $\overline{G}-$orbit X_v on G has cardinality equal to $|X_{1,j}|^t|X_{2,k}|^s$, for some $1\le j\le s,1\le k\le t,$and $X_{1,j}$ is a $\overline{G_1}-$orbit on G_1 and $X_{2,k}$ is a $\overline{G_2}-$orbit on G_2.

On other hand, if ϕ_i is a character of G_i, i=1,2 then $\phi_1\times\phi_2$ is a character of $G_1\times G_2$and if Y_v, is a $\overline{G}-$orbit on the set Irr (G), then $|Y_v|=|Y_{1,j}|^t|Y_{2,k}|^s$, for some $1\le j\le s$, $1\le k\le t$, where $Y_{1,j}$ is a

$\overline{G}_1$ − orbit on Irr(G_1) and $Y_{2,k}$ is a $\overline{G}_2$ − orbit on Irr(G_2) . Hence we obtain the following

Lemma (2.3)

Let G_1 and G_2 be two finite groups of relatively prime orders, and let *s* and *t* be the number of conjugacy classes of cyclic subgroups of G_1 and G_2 respectively. Then

$$V(G) = (V(G_1))^t (V(G_2))^s .$$

Theorem (2.4)

Let G_1 and G_2 be two finite permutation groups of relatively prime orders satisfying (H) with $\left|\overline{R}(G_i) : P(G_i)\right| = 1$, for i = 1, 2 , then each rational – valued character of $G = G_1 \times G_2$ can be written as Z-linear combination of $\rho_\sigma, \sigma \in G$.

Proof :

Let $\sigma = (\sigma_1, \sigma_2), \tau = (\tau_1, \tau_2)$ be two elements of G, and let $G_{[\sigma]} = G_{[\tau]}$. Then by lemma (2.2).

$G_{1[\sigma_1]} = G_{1[\tau_1]}$ and $G_{2[\sigma 2]} = G_{2[\tau 2]}$, so by (H) $< \sigma_1 >$ is conjugate to $< \tau_1 >$ and $< \sigma_1 >$ is conjugate to $< \tau_2 >$ which implies that $< \sigma_1 > \times < \sigma_2 >$ is conjugate to $< \tau_1 > \times < \tau_2 >$.

Since G_1 and G_2 have relatively prime orders, then $< \sigma >$ and $< \tau >$ are conjugate and we have only to show that $\left|\overline{R}(G) : P(G)\right| = 1$.

Consider $\prod \left| N_G(G_{[\sigma]}) : G_{[\sigma]} \right|^2$

Where the multiplication is over all cyclic subgroups in $l_1 \times l_2$. Any such subgroups can be decomposed uniquely into $< \sigma_{1,j} > \times < \sigma_{2,k} >$, so this product is equivalent to

$$\prod_{j,k} \left| N_G(G_{1[\sigma_{1,j}]} \times G_{2[\sigma_{2,k}]}) : G_{1[\sigma_{1,j}]} \times G_{2[\sigma_{2,k}]} \right|^2$$

$$=\prod_{j,k}\left|N_1(G_{1[\sigma_{1,j}]}):G_{1[\sigma_{1,j}]}\right|^2 . \left|N_2(G_{2[\sigma_{2,k}]}):G_{2[\sigma_{2,k}]}\right|^2$$

$$=\prod_{\sigma_{1,j}}\left|N_1(G_{1[\sigma_{1,j}]}):G_{1[\sigma_{1,j}]}\right|^{2t} . \prod_{\sigma_{2,k}}\left|N_2(G_{2[\sigma_{2,k}]}:G_{2[\sigma_{2,k}]}\right|^{2s}$$

It follows from lemma (2.3) that

$$V(G)\prod\left|N_G(G_{[\sigma]}):G_{[\sigma]}\right|^2=\prod_j\left|C(\sigma_{1,j})\right|^t\prod_k\left|C(\sigma_{2,k})\right|^s$$

$$=\prod\left|C(\sigma)\right|$$

And this complete the proof.

The most important consequence of the result is the fact that if G is a nilpotent permutation group having the property that each rational – valued character of its sylow p-subgroup S_{p_i} can be written as Z-linear combination of $\rho_\sigma,\sigma\in S_{p_i}$, then each rational – valued character of G is a Z-linear combination $\rho_\sigma,\sigma\in G$.

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الخلاصة

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