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Strongly C_{11}-Condition Modules and Strongly T_{11}-Type Modules

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Abstract

In this paper, we introduced module that satisfying strongly C_{11}-condition modules and strongly T_{11}-type modules as generalizations of t-extending. A module M is said strongly C_{11}-condition if for every submodule of M has a complement which is fully invariant direct summand. A module M is said to be strongly T_{11}-type modules if every t-closed submodule has a complement which is a fully invariant direct summand. Many characterizations for modules with strongly C_{11}-condition for strongly T_{11}-type module are given. Also many connections between these types of module and some related types of modules are investigated.

Keywords. C_{11}-condition, strongly C_{11}-condition modules, T_{11}-type modules strongly T_{11}-type modules, t-semisimple modules, strongly t-semisimple modules, strongly extending modules, t-extending modules and strongly t-extending modules.

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1. Introduction

First, a few concepts and some results which are relevant for our work are recalled. Throughout, all rings are associative rings with unity and all modules are right unitary modules. A submodule N of M is called closed in M if has no proper essential extension in M[8], that means if N is essential in W, where $W \leq M$,then $N = W$, where a submodule N is essential in an R-module M (briefly ($N \leq_{ess} M$) if $N \cap K = (0)$, $K \leq M$ implies $K = (0)$[8] .

As a generalization of essential submodule, Asgari in [2], introduced the notion of t-essential submodule, where a submodule N of M is called t-essential (denoted by $N \leq_{tes} M$) if whenever $W \leq M$, $N \cap W \leq Z_2(M)$ implies $W \leq Z_2(M)$. $Z_2(M)$ is called the second singular submodule and is defined by $Z\left(\frac{M}{Z(M)}\right) = \frac{Z_2(M)}{Z(M)}$[8], where $Z(M) = \{x \in M: xI = (0)$ for some essential ideal of $R\}$. Equivalently $Z(M) = \{x \in M: ann(x) \leq_{ess} R\}$ and $ann(x) = \{r \in M: xr = 0\}$. M is called singular (nonsingular) if $Z(M) = M(Z(M) = 0)$. Note that $Z_2(M) = \{x \in M: xI = (0)$ for some t-essential ideal I of $R\}$.M is called Z_2-torsion if $Z_2(M) = M$[8]. A submodule N is called t-closed (denoted by $N \leq_{tc} M$) if N has no proper t-essential extension in M[2]. It is clear that every t-closed submodule is closed, but the convers is not true. However, under the class of nonsingular, the two concepts are equivalent. Recall that ″ a module M is called extending if for every submodule N of M then there exists a direct summand $W(W \leq^{\oplus} M)$ such that $N \leq_{ess} W$ ″ [6]. Equivalently ″M is extending module if every closed submodule of M is a direct summand″. As a generalization of extending module, Asgari [2] introduced the concept t-extending module, where ″a module M is t-extending if every t-closed submodule is a direct summand″. Equivalently,″M is t-extending if every submodule of M is t-essential in a direct summand ″[2]. The notion of a strongly extending module is introduced in [13], which is a subclass of the class of extending module, where ″an R-module M is called strongly extending if each submodule of M is essential in a fully invariant direct summand of M ″, where ″a submodule N is called fully invariant if for each $f \in \text{End}(M)$, $f(\text{N}) \leq \text{N}$″.″An R-module is called strongly t-extending if every submodule N of M, there exists a fully invariant direct summand W of M such that $N \leq_{tes} W$″[7].Equivalently “M is strongly t-extending if every t-closed submodule of M is fully invariant direct summand″[7]. A module M is called duo if every submodule of M is fully invariant [12]. Hence the two concepts strongly extending and extending are equivalent in the class of duo modules. Asgari and Haghany introduced the concept of t-semisimple modules and t-semisimple rings. A module M is called t-semisimple if every submodule N contains a direct summand K of M such that K is t-essential in N [3]. In [9] Inaam and Farhan introduced and studied strongly t-semisimple, where an R-module is called strongly t-semisimple if for each submodule N of M there exists a fully invariant direct summand K such that $\text{K} \leq_{\text{tes}} \text{N}$ [9]. Inaam and Farhan in [10] introduced and studied FI-t-semisimple and strongly FI-t-semisimple. ″An R-module M is called FI-t-semisimple if for each fully invariant submodule N of M, there exists $\text{K} \leq^{\oplus} \text{M}$ such that $\text{K} \leq_{\text{tes}} \text{N}$. An R-module M is called strongly FI-t-semisimple if for each fully invariant submodule N of M, there exists a fully invariant direct summand K such that $\text{K} \leq_{\text{tes}} \text{N}$ ″[10].

Recall that:″ An R-module M is said to be satisfy C_{11}-condition if every submodule of M has a complement which is a direct summand″[15]. Asgari [4], restricted C_{11} condition to t-closed condition of M. She defined the following. ″An R-module M said to be T_{11}-type module (or M satidfy T_{11}-condition) if every t-closed submodule has a complement which is a direct summand. A ring is said to be right T_{11}-type ring if R_R is a T_{11}-type module″ [4].

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This paper consists of three sections. In section two we deal with certain known results which are worthwhile throughout the paper. In section three we investigate certain types of module namely strongly C_{11}-condition and strongly T_{11}-type modules. An R-module M is said to be satisfy strongly- C_{11}-condition if every submodule of M has a complement which is a fully invariant direct summand. Also; an R-module M is called strongly T_{11}-type module if every t-closed submodule has a complement which is a fully invariant direct summand. We give several characterizations of strongly C_{11}-condition and strongly T_{11}-type module. In particular M is strongly T_{11}-type module if and only if $M = Z_2(M) \oplus M'$, where M' satisfies strongly C_{11}-condition. Every module with strongly C_{11}-condition is strongly T_{11}-type module, but not conversely (see Remark 3.6(1)). The two concepts are equivalent under certain classes of module is given. Furthermore, many connections between strongly T_{11}-type modules strongly t-semisimple module, strongly FI-t-semisimple module, FI-t-semisimple module, strongly extending module, strongly t-extending module are presented.

2. Preliminaries

Proposition (1.1)[2]: ″The following statements are equivalent for a submodule A of an R-module M

A is t-essential in M ;

$(A+Z_2(M))/Z_2(M)$ is essential in $M/Z_2(M)$;

$A+Z_2(M)$ is essential in M;

M/A is $Z_2 - torsion$.″.

Lemma (1.2)[2]:″ Let M be an R- module. Then

If $\leq_{tc}$ M, then $Z_2(M) \leq C$.

$0 \leq_{tc} M$ if and only if M is nonsingular.

If $A \leq C$, then $C \leq_{tc}$ M if and only if $\frac{C}{A} \leq_{tc} \frac{M}{A}$.″

Theorem (1.3)[9]: ″ The following statements are equivalent for an R -module M:

M is strongly t-semisimple,

$\frac{M}{Z_2(M)}$ is a fully stable semisimple and isomorphic to a stable submodule of M,

$M = Z_2(M) \oplus M'$ where $M'$is a nonsingular semisimple fully stable module and M' is stable in M,

Every nonsingular submodule is stable direct summand,

Every submodule of M which contains $Z_2(M)$ is a direct summand of M and $\frac{M}{Z_2(M)}$ is fully stable and isomorphic to a stable submodule of M″.

Proposition (1.4)[10]:″ Let M be an R-module with the property, complement of any submodule of M is stable. The following statements are equivalent.

M is strongly FI-t-semisimple;

M is FI-t-semisimple;″.

Let (⁎) means the following: For an R-module M, the complement of $Z_2(M)$ is stable in M[10].

Proposition (1.5)[10]:″ Let M be an R-module which satisfies (⁎). If M is strongly FI-t-semisimple, then $\frac{M}{Z_2(M)}$ is FI-semisimple, and hence it is strongly FI-t-semisimple.″

Corollary (1.6)[10]:″ For an R-module M which satisfies (⁎) M is strongly FI-t-semisimple if and only if for every fully invariant submodule N of M such that $N \supseteq Z_2(M)$, is strongly FI-t-semisimple,″.

Proposition (1.7)[2]:″ Let M be a nonsingular module. Then M is strongly extending if and only if M is strongly t-extending″.

2. T-semisimple modules and T_{11}-type modules

Remarks and Examples (2.1):

It is clear that ″module satisfying C_{11} implies T_{11}-type-module″ [4].

″Every t-extending module (hence every extending module) is a T_{11}-type module″ [4].

″Every Z_2-torsion is T_{11}-type module and every finite generated abelian group is a T_{11}-type module″ [4].

Proposition (2.2): Every t-semisimple module is T_{11}-type module.

Proof: By [3, Proposition 2.16], every t-semisimple is t-extending, hence by Remarks and Examples 2.1(2) it is T_{11}-type module. □

Remark (2.3)[4]:″ The class of T_{11}-type modules properly contains the modules satisfying C_{11} condition and the class of t-extending modules″ .

Examples (2.4):

Let $R = Z[X]$, R_R is uniform, nonsingular R-module. By [13, Theorem 2.4] $R⊕R$ satisfies C_{11}-condition. Hence $R⊕R$ is T_{11}-type module. But $R⊕R$ is not t-semisimple ,because it is so, then $R⊕R$ is t-extending, which is a contradiction since by [5,Example 2.4] $R⊕R$ is not extending, hence not t-extending, since $R⊕R$ is nonsingular see Remarks and Examples 2.1(2).

The Z-module Z is not t-semisimple. But Z is indecomposable and nonsingular uniform, so Z is T_{11}-type module see[4, Corollary 2.8].

An Z-module Q is indecomposable, nonsingular, uniform so Q is T_{11}-type module[4, Corollary 2.8]

″Any direct summand of uniform is C_{11}-type module, so is T_{11}-type module ″[15].

(5) it is clear the Z module $Q⊕Z$, $Z_4⊕Z_8$, $Z_8⊕Z_2$ are T_{11}-type module. Also notice that $Q⊕Z$ is not t-semisimple.

3. Strongly C_{11}-modules and strongly T_{11}-type modules.

In this section, we generalize modules with C_{11}-condition and T_{11}-type modules into modules with strongly C_{11}- conditions and strongly T_{11}-type modules. We study these concepts and their connection with strongly t-semisimple modules and other related classes of modules.

Definition (3.1): An R-module M said to be satisfy strongly C_{11}- condition (M is strongly C_{11}) if every submodule has a complement which is fully invariant direct summand.

Lemma (3.2)[15]:″ Let $N \leq M$, let K be a direct summand of M . K is a complement of N if and only if $K \cap N = 0$ and $K⊕N \leq_{ess} M$ ″.

The following Lemma is clear.

Lemma (3.3): If $N \leq M$ and K is a fully invariant direct summand then K is a fully invariant complement of N if and only if $K \cap N = 0$ and $K⊕N \leq_{ess} M$.

The following Proposition gives characterizations for module with strongly C_{11}-condition.

Proposition (3.4): The following statements are equivalent for a module M

M satisfies strongly C_{11}-condition;

For any complement submodule L in M, there exists a fully invariant direct summand K of M such that K is a complement of L in M;

For any submodule N of M, there exists a fully invariant direct summand K of M such that $N \cap K = 0$ and $N⊕K$ is an essential submodule of M;

For any complement submodule L in M, there exists a fully invariant direct summand K of M such that $L \cap K = 0$ and $L⊕K \leq_{ess} M$.

Proof: (1) ⇒ (2) For any complement submodule L in M. By strongly C_{11}-condition, there exists a fully invariant direct summand K of M which is a complement of L in M.

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(3) ⇒ (4) and (2) ⇔ (4) are obvious.
(1) ⇒ (3) It is clear by Lemma (3.3).
(4) ⇒ (1) Let A be any submodule of M. Then there exists a complement (so closed submodule B in M) such that $A \leq_{ess} B$, by [8,Excerces 13,P.20]. By hypothesis, there exists a fully invariant direct summand K of M such that $B \cap K = 0$ and $B \oplus K \leq_{ess} M$. Hence by Lemma (3.2) K is a complement of B in M. So $B \cap K = 0$, which implies $K \cap A = 0$. Suppose that $K' \leq M$ and, $K' > K$. Therefore $K' \cap B \neq 0$ and hence $(K' \cap B) \cap A \neq 0$ (since $A \leq_{ess} B$), so that $K' \cap A \neq 0$. Thus K is a complement of A in M. □

As we have seen t-semisimple is T_{11}-type module. We claim that strongly t-semisimple modules imply modules which are strongly than module with T_{11}-type module. Hence this leads us to define the following:

Definition (3.5): An R-module is said to be strongly T_{11}(or strongly T_{11}-type modules) if for each t-closed submodule, there exists a complement which is a fully invariant direct summand.

Remarks (3.6):

(1) It is clear that every module, which satisfies strongly C_{11}-condition, is a strongly T_{11}-type module, but the converse is not true in general, as the following example shows:

Let $M = Z_8 \oplus Z_2$ as Z-module M is T_{11}-type module by [4, Corollary 2.6]. M is strongly T_{11}-type module, but it is not strongly C_{11}-type module. If $N = (\bar{2}) \oplus (\bar{0}) = \{(\bar{2},\bar{0}),(\bar{4},\bar{0}),(\bar{6},\bar{0}),(\bar{0},\bar{0})\}$, $N \cap ((\bar{0}) \oplus Z_2) = (\bar{0},\bar{0})$ and $N \oplus W = (\bar{2}) \oplus Z_2 \leq_{ess} M$ where $W = ((\bar{0}) \oplus Z_2)$ and $W \leq^{\oplus} M$, then W is a complement of N. Also $N \cap K = (\bar{0})$, where $K = \{(\bar{4},\bar{1}),(\bar{0},\bar{0})\}$. But $\oplus K = (\bar{0},\bar{0}),(\bar{2},\bar{0}),(\bar{4},\bar{0}),(\bar{6},\bar{0})\} \oplus \{\bar{4},\bar{1}),(\bar{0},\bar{0})\}$
$=\{(\bar{4},\bar{1}),(\bar{0},\bar{0}),(\bar{6},\bar{1}),(\bar{2},\bar{0}),(\bar{0},\bar{1}),(\bar{4},\bar{0}),(\bar{2},\bar{1}),(\bar{6},\bar{0})\} = U$ and $U \leq_{ess} M$, hence is a complement of N, but $K \nleq^{\oplus} M$. Thus W is a unique complement of N which is a direct summand, but W is not fully invariant submodule as there exists $f: W = \{(\bar{0},\bar{0}),(\bar{0},\bar{1})\} \mapsto M$ defined by $f(\bar{0},\bar{0}) = (\bar{0},\bar{0})$ $f(\bar{0},\bar{1}) = (\bar{4},\bar{1})$, f is Z-homorphism and $f(W) \nleq W$. Thus M does not satisfy strongly C_{11}-type module. Also M is singular, so M is the only t-closed submodule and has a complement which is the zero submodules and it is clear direct summand fully invariant submodule. Thus M is strongly T_{11}-type.

(2) Let M be an R-module which satisfies strongly C_{11}-condition. Then M is strongly FI-t-semisimple if and only if M is FI-t-semisimple.

Proof: It follows directly by Proposition 1.4, and Definition 3.1. □

Proposition (3.7): Let M be a nonsingular R-module. M is strongly C_{11}-condition module if and only if M is strongly T_{11}-type module.

Proof:⇒ It is clear.

⇐ Let $A \leq M$, by [8,Exercies 13,P.20], there exists a closed submodule W of M such that $A \leq_{ess} W$. Since M is nonsingular, W is t-closed in M. Hence there exists a fully invariant direct summand D of M such that $W \oplus D \leq_{ess} M$. It follows that $A \oplus D \leq_{ess} W \oplus D \leq_{ess} M$. Thus $A \oplus D \leq_{ess} M$. So that M satisfies strongly C_{11}-condition. □

Recall that "an R-module M is called multiplication module if for each $N \leq M$, there exists an ideal I of R with $N = MI$"[17]. Equivalently "an R-module M is a multiplication module if for each $N \leq M, N = (N:_R M)M$, where $(N:_R M) = \{r \in R: Mr \leq N\}$".[17]

Proposition (3.8): Let M be a multiplication (hence M is duo or fully stable). Then
M is T_{11}-type module if and only if M is strongly T_{11}-type module.
M is C_{11}-type if and only if M is strongly C_{11}-type module.

We will give some properties of strongly T_{11}-type modules.

Theorem (3.9): Consider the following statements for a module M
M is strongly T_{11}-type module;

$M = Z_2(M) \oplus M'$, where M' is a fully invariant submodule in M and satisfies strongly C_{11}-condition;

For every submodule A of M, there exists a fully invariant direct summand D of M such that $A \oplus D \leq_{tes} M$.

For every t-closed submodule C of M, there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{tes} M$.

For every t-closed submodule C of M, there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{ess} M$.

Then (1), (3), (4) and (5) are equivalent, (1) ⇒(2) if $\frac{L}{Z_{2(M)}}$ is fully invariant of $\frac{M}{Z_{2(M)}}$ for each fully invariant submodule L of M containg $Z_2(M)$, and (2) ⇒ (5).

Proof: (1)⇒(5) Let C be a t-closed submodule of M. By condition (1) there exists a complement D to C such that $D \leq^{\oplus} M$, D is fully invariant. Thus $C \oplus D \leq_{ess} M$.

(3)⇒ (1) Let C be a t-closed submodule of M. By hypothesis there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{ess} M$. Let E be a complement of C, then $C \cap E = 0$ and $C \oplus E \leq_{ess} M$. We claim that $C \oplus D \leq_{ess} C \oplus E$. Let $(C \oplus D) \cap X = (0)$, where $X \leq C \oplus E$. $(C \oplus D) \cap X = (0) \leq Z_2(M)$. Thus implies $X \leq Z_2(M)$ since $C \oplus D \leq_{tes} M$. But $Z_2(M) \leq C$ (since C is t-closed) hence $X \leq C$. It follows that $(C \oplus D) \cap X = X = (0)$. Thus $(C \oplus D) \leq_{ess} C \oplus E$. It follows that $D \leq_{ess} E$. However, $D \leq^{\oplus} M$ and so D is closed in M. which implies $D = E$, that is E a complement of C, which is a fully invariant direct summand. Thus M is a strongly T_{11}-type module.

(4) ⇒ (3) Let $A \leq M$. By [4, Lemma 2.3], there exists a t-closed C of A such that $A \leq_{tes} C$. By hypothesis, there exists a fully invariant direct summand D such that $C \oplus D \leq_{tes} M$. But $A \leq_{tes} C$, we conclude that $A \oplus D \leq_{tes} C \oplus D$ and hence $A \oplus D \leq_{tes} M$.

(5) ⇒(4) The implication is clear since every essential submodule is t-essential submodule.

(2) ⇒(5) Let C be a t-closed submodule of M. Hence by Lemma (1.2), $Z_2(M) \leq C$ and so $C = Z_2(M) \oplus (C \cap M')$. Moreover, $C \cap M'$ is a t-closed submodule of M' by [2, proposition 2.9]. Since M satisfies strongly C_{11} condition, there exists a fully invariant direct summand D of M' such that $(C \cap M') \oplus D \leq_{ess} M'$. But $D \leq^{\oplus} M'$ and $M' \leq^{\oplus} M$, then $D \leq^{\oplus} M$ and $C \oplus D = [Z_2(M) \oplus (C \cap M')] \oplus D = Z_2(M) \oplus [(C \cap M')) \oplus D] \leq_{ess} Z_2(M) \oplus M' = M$. Hence $C \oplus D \leq_{ess} M$, but D is fully invariant in M' and M' is fully invariant in M. Hence D is fully invariant in M.

(1) ⇒(2) Since M is strongly T_{11}-type module and $Z_2(M)$ is a t-closed submodule of M, there exists a complement M' to $Z_2(M)$ which is a fully invariant direct summand, say $M = L \oplus M'$. Since M' is nonsingular, we have $Z_2(M) = Z_2(L)$. But $Z_2(M) \oplus M' \leq_{ess} M$ since M' is complement to $Z_2(M)$, so by Proposition (1.1) $\frac{M}{M'}$ is Z_2-torsion, thus L is Z_2-torsion (since $L \simeq \frac{M}{M'}$). So $L = Z_2(L) = Z_2(M)$ and hence $L = Z_2(M)$. Therefore $M = Z_2(M) \oplus M'$. Now to show that $M' \simeq \frac{M}{Z_2(M)} \simeq \bar{M}$ satisfies strongly C_{11} condition. Let $\bar{C} = \frac{C}{Z_2(M)}$ be a closed submodule of $\bar{M}$ so $\bar{C}$ is t-closed in $\bar{M}$ and C is t-closed submodule of M by Lemma 1.2(3).But M is a strongly T_{11}-type, so there exists a complement D of C in M which is a fully invariant direct summand of M. Say $M = D \oplus D'$ for some $D' \leq M$. Since $Z_2(M) = Z_2(D) \oplus Z_2(D')$ we get $\bar{M} = \frac{M}{Z_2(M)} = \frac{D \oplus D'}{Z_2(D) \oplus Z_2(D')} \simeq \frac{D}{Z_2(D)} \oplus \frac{D'}{Z_2(D')} = \bar{D} \oplus \overline{D'}$. Cleary $\bar{D} \cap \overline{D'} = 0$ and $\bar{C} \oplus \bar{D} \leq_{ess} \bar{M}$. But $D, Z_2(M)$ are fully invariant in M and by hypothesis $\frac{D + Z_2(M)}{Z_2(M)} = \bar{D}$ is fully invariant in $\bar{M}$. □

Recall that. "A submodule N of R-module M is called stable, if $f(N) \leq N$ for each R-homomorphism $f: N \to M$. An R-module M is fully stable if every submodule of M is stable".[1]

Remarks (3.10):

If an R-module M is fully stable (where an R-module M is fully stable if every submodule of M is stable) and semisimple, then M is strongly C_{11}-condition module.

Proof: Let $N \leq M$, then $N \leq^{\oplus} M$, and so there exists $W \leq M$ such that $N \oplus W = M$, hence W is a complement of N. But M is fully stable, so W is a fully invariant, moreover $W \leq^{\oplus} M$. Thus M is strongly C_{11}-condition module. □

Let M be a strongly T_{11}-type module. Then M is strongly-FI-t-semisimple if and only if every fully invariant t-closed submodule of M is strongly-FI-t-semisimple.

Proof: Since M is strongly T_{11}-type module and $Z_2(M)$ is t-closed, then there exists a $Z_2(M)$ has a complement which is a fully invariant direct summand, that is $Z_2(M)$ has a complement which is stable direct summand (so condition $(*)$hold). Then by Corollary 1.6 every fully invariant submodule N of M, $N \supseteq Z_2(M)$ is strongly FI-t-semisimple. Thus every fully invariant t-closed submodule of M is strongly FI-t-semisimple. □

Let M be a strongly T_{11}-type module. If M strongly FI-t-semisimple, then $\frac{M}{Z_2(M)}$ is FI-semisimple and hence it is strongly FI-semisimple.

Proof: Since M is strongly T_{11}-type and $Z_2(M)$ is t-closed, then there exists a complement of $Z_2(M)$ which is a fully invariant direct summand. Thus (condition $(*)$ hold). Hence the result is followed by Proposition 1.5. □

Proposition (3.11): If an R-module M strongly t-semisimple, then M strongly T_{11}-type module.

Proof: By Theorem 1.3, $M = Z_2(M) \oplus M'$, where M' is nonsingular semisimple fully stable and M' is stable in M'. But M' is fully stable semisimple then $M'$is strongly C_{11}-condition module by Remarks 3.10(1). Hence M satisfies condition (2) of Theorem 3.9. where (2→5→1). Thus M is strongly T_{11}-type module. □

Theorem (3.12): Every strongly extending module is strongly T_{11}-type module.

Proof: Let N be a t-closed submodule of M. Hence N is a closed submodule. As M is strongly extending, N is a fully invariant direct summand. Then $M = N \oplus W$ for some $W \leq M$ and so W is a complement of N. To see this let $W' \leq M$ and $W \leq W' \leq M$ and $N \cap W' = (0)$, then $M = N \oplus W \subseteq N \oplus W'$, so $M = N \oplus W' = N \oplus W$. Assume $x \in W'$ then $x = n + y, n \in N, y \in W \leq W'$, then $x - y = n \in N \cap W' = 0$, hence $x - y = 0$ implies $x = y \in W$. Hence $W' = W$, moreover $W \leq^{\oplus} M$, so W is closed submodule and hence W is a fully invariant direct summand. Thus M is strongly T_{11}-type module. □

Proposition (3.13): If M is a strongly t-extending R-module then is strongly T_{11}-type module and every complement to a nonsingular direct summand is fully invariant direct summand.

Proof: Since M is strongly t-extending, then $M = Z_2(M) \oplus M'$, M' is strongly extending module[7]. Hence, M' is strongly T_{11}-type module by Proposition (3.12). But M' is nonsingular, so M' satisfies strongly C_{11}-condition module by Proposition (3.7). Thus M satisfies condition (2) of Theorem 3.9 so M is a strongly T_{11}-type module. Now let C be a complement of a nonsingular submodule of M, so by [2, Proposition 2.6(5⇔2)] C is a t-closed submodule of M. Hence C is a fully invariant direct summand of M by definition of strongly t-extending Proposition (1.7).

Not that if every complement of nonsingular submodule of an R-module is fully invariant direct summand implies M is strongly t-extending, since by [2, Proposition 2.6(5⇔2)] every t-closed

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is a complement of nonsingular submodule and so that every t-closed submodule is fully invariant direct summand (that is M is strongly t-extending). □

Proposition (3.14): Let $M = M_1 \oplus M_2$, M_2 is a fully invariant submodule in M. Then the following conditions are equivalent:

M_1 is strongly T_{11}-type module;

For every submodule A of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $A \oplus D \leq_{tes} M$.

For every t-closed submodule C of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{tes} M$;

For every t-closed submodule C of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{ess} M$.

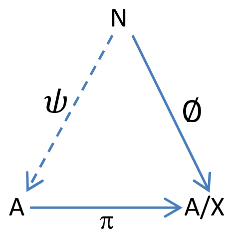
Proof: (1) ⇒ (2) Since M_1 is strongly T_{11}-type module, then by condition (3) of Theorem 3.9 for each $A \leq M_1$, there exists a fully invariant direct summand D of M_1 such that $A \oplus D \leq_{tes} M_1$. But $D \leq^{\oplus} M_1$ implies that $D \oplus M_2 \leq^{\oplus} M$. Also, we can show that $D \oplus M_2$ is fully invariant in M. Let $f \in End(M) = \begin{pmatrix} End(M_1) & Hom(M_2, M_1) \\ Hom(M_1, M_2) & End(M_2) \end{pmatrix}$. But M_2 is fully invariant in M by hypothesis so $\mathrm{Hom}(M_2, M_1) = 0$, then $f = \begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}$ for some $f_1 \in End(M_1), f_2 \in Hom(M_1, M_2), f_3 \in End(M_2)$.Hence $f(D \oplus M_2) \simeq \begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix} \begin{pmatrix} D \\ M_2 \end{pmatrix} \leq \begin{pmatrix} D \\ M_2 \end{pmatrix}$, hence $D \oplus M_2$ is fully invariant in M. Moreover, $A \oplus D \leq_{tes} M_1$ implies that $(A \oplus D) \oplus M_2 \leq_{tes} M_1 \oplus M_2 = M$.

(2) ⇒ (3) It is obvious

(3) ⇒ (4) For every t-closed submodule C of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{tes} M$. Then $C \oplus D + Z_2(M) \leq_{ess} M$ by Proposition (1.1). But $Z_2(M) = Z_2(M_1) \oplus Z_2(M_2)$. As C is t-closed in M_1,$C \supseteq Z_2(M_1)$ by Lemma 1.2(1). Also as $M_2 \leq D$, then $Z_2(M_2) \leq Z_2(D) \leq D$. It follows that $\oplus D + Z_2(M) = C \oplus D + Z_2(M_1) \oplus Z_2(M_2) = C \oplus D \leq_{ess} M$.

(4) ⇒ (1) Let C be a t-closed of M_1. By condition (4) there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{ess} M$. But D is a fully invariant submodule in M implies, $D = (D \cap M_1) \oplus (D \cap M_2)$, such that $D \cap M_1$ is fully invariant in M_1 and $D \cap M_2 = M_2$ since $M_2 \leq D$. Hence $D = (D \cap M_1) \oplus M_2$ and $D \cap M_1 \leq^{\oplus} M_1$. Now $C \oplus D = C \oplus [(D \cap M_1) \oplus M_2] \leq_{ess} M = M_1 \oplus M_2$. Hence$[C \oplus [(D \cap M_1)] \leq_{ess} M_1$. Thus M_1 satisfies condition (5) of Theorem (3.9), which implies M_1 is strongly type-T_{11} module. □

For R-modules N and A. N is said to be A- projective, if every submodule X of A, any homomorphism $\emptyset: N \longmapsto \frac{A}{X}$ can be lifted to a homorphism, $\psi: N \longmapsto A$, that is if $\pi: A \longmapsto \frac{A}{X}$, be the-natural epiomorphism, then there exists a homorphism $\psi: N \longmapsto A$ such that $\pi \circ \psi = \emptyset$.



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M is called projective if M is N-projective for every R-module N. If M is M-projective, M is called self-projective". [11]

Proposition (3.15): If an R-module M is strongly T_{11}-type module and L is fully invariant direct summand of M, then L is strongly T_{11}-type module and $\frac{M}{L}$ is strongly T_{11}-type module if M is self- projective.

Proof: To prove L is strongly T_{11}-type module. Let A be a submodule of L, hence A is a submodule of M, and so by condition (3) of Theorem 3.9, there exists a fully invariant direct summand D of M, such that $A \oplus D \leq_{tes} M$. Hence $A \oplus D \cap L \leq_{tes} L$ and so $A \oplus (D \cap L) \leq_{tes} L$. On the other hand, $D \leq^{\oplus} M$ implies $M = D \oplus D'$ for some $D' \leq M$. As L is fully invariant submodule in M, $L = (D \cap L) \oplus (D' \cap L)$, where $D \cap L$ is fully invariant in D, $D' \cap L$ is fully invariant submodule in D' .Now $D \cap L$ is fully invariant in D and D is fully invariant in M, so $D \cap L$ is fully invariant in M. Also $D \cap L \leq^{\oplus} L$, $L \leq^{\oplus} M$, so $D \cap L \leq^{\oplus} M$. Thus by [4, Lemma 2.3] $D \cap L$ is fully invariant direct summand of M.

Let $\frac{C}{L}$ be a t-closed submodule in $\frac{M}{L}$. Then C is a t-closed in M. As M is strongly T_{11}-type module there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{ess} M$ by Theorem 3.9. Let $M = D \oplus D'$ for some $D' \leq M$ and since L is fully invariant in M, $L = (D \cap L) \oplus (D' \cap L)$ such that $D \cap L$ is fully invariant in D, $D' \cap L$ is fully invariant in D'. Then $\frac{M}{L} = \frac{D \oplus D'}{(D \cap L) \oplus (D' \cap L)} \simeq \frac{D}{D \cap L} \oplus \frac{D'}{D' \cap L} \simeq \frac{D+L}{L} \oplus \frac{D'+L}{L}$. But it is easy to see that $\frac{C}{L} \oplus \frac{D+L}{L} \leq_{ess} \frac{M}{L}$. As $L \leq^{\oplus} M$, L is closed and this implies that $\frac{C \oplus D}{L} \leq_{ess} \frac{M}{L}$ by [8,Proposition 1.4,P.18]. Thus $\frac{C}{L} \oplus \frac{D+L}{L} \leq_{ess} \frac{M}{L}$. On the other hand, since D is a fully invariant submodule in M and, L is fully invariant in M, then $D \oplus L$ is fully invariant in M. Hence $\frac{D+L}{L}$ is fully invariant in $\frac{M}{L}$ by [16, Lemma 1.1.20(2)] (since M is self-projective). Thus $\frac{D+L}{L}$ is a fully invariant direct summand of $\frac{M}{L}$ and $\frac{C}{L} \oplus \frac{D+L}{L} \leq_{ess} \frac{M}{L}$. Therefore $\frac{M}{L}$ is strongly T_{11}-type module by Theorem 3.9(1⇔3). □

Corollary (3.16): If R is a commutative strongly T_{11}-type module and $L \leq^{\oplus} R$, then $\frac{R}{L}$ is a strongly T_{11}-type module.

Corollary (3.17): Let M be a multiplication strongly T_{11}-type module and $L \leq^{\oplus} M$. Then $\frac{M}{L}$ is strongly T_{11}-type modu

References

[1] M. S. Abas On Fully Stable Modules Ph.D Thesis College of Science University of Baghdad. 1991

[2] Sh. Asgari and A. Haghany t-Extending modules and t-Baer modules Comm.Algebra **39** 1605-1623. 2011

[3] Sh. Asgari, A. Haghany and Y Tolooei . T-semisimple modules and T-semisimple rings comm Algebra **41** 5 .1882-1902. 2013

[4] Sh. Asgari, A. Haghany and A .R .Rezaei . Module Whose t-closed submodules have a summand as a complement comm.Algebra **42** pp. 5299–5318 . 2014

[5] A. W. Chatters and S. M. Khuri .Endomorphism rings of modules over nonsingular CS rings *J. London Math. Soc.* **21** . 434-444.

[6] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer . Extending Modules Pitman

Research Notes in Mathematics 313 Longman Harlow, 1994

[7] S. Ebrahimi, Dolati Pish Hesari and M. Khoramdel. strongly t-extending and strongly t-Baer *International Electronic Journal of Algebra* **20** . 86-98. 2016

[8] K. R. Good earl . Ring *Theory Non Singular Rings and Modules Marcel Dekker* Inc. New York and Basel. 1976

[9] M. A. Inaam and Farhan D Shyaa . Strongly t-semisimple modules and Strongly t-semisimple Rings *International Journal of Pure and Applied Mathematics* **15** 1 . 27-41. 2017

[10] M. A. Inaam and Farhan D Shyaa . FI-semisimple Fi-t-semisimple and strongly FI-t-semisimple modules *International Journal of Pure and Applied Mathematics* **15** 2 . 285-300. 2017

[11] S. H. Mohamed and B. J. Muller .Continuous and Discrete Modules Cambridge University Press Cambridge. 1990

[12]A.C. Özcan, A. Harmanci and F. Smithp . Duo Modules *Glasgow Math.J.* **48** . 533-545. 2006

[13] A .Saad Al-Saadi . S-Extending modules and related Concepts Ph.D Thesis College of Science Al- Mustansiriyah University. 2007

[14] Saad Abulkadhim Al-Saadi and Tamadher Arif Ibrahim 2014 Strongly Rickart Modules, Journal of advances in mathematics **19** 4 . 2507-2514

[15] P. F. Smith and A. Tercan .Generalizations of CS-modules Comm Algebra **21** . 1809–1847. 1993

[16] I. E. Wijayanti . Coprime modules and Comodules Ph D Thesis University of Dusseidorf. 2006

[17] Zeinb Abd EL-bast and Parrick F Smith . Multiplication modules comm Algebra **16** 4 . 755-779. 1988