https://doi.org/ 10.30526/2017.IHSCICONF.1809

Different Estimation Methods for System Reliability Multi-Components model: Exponentiated Weibull Distribution

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Abstract

In this paper, estimation of system reliability of the multi-components in stress-strength model $R_{(s,k)}$ is considered, when the stress and strength are independent random variables and follows the Exponentiated Weibull Distribution (EWD) with known first shape parameter θ and, the second shape parameter α is unknown using different estimation methods. Comparisons among the proposed estimators through Monte Carlo simulation technique were made depend on mean squared error (MSE) criteria.

Keywords: *Exponentiated Weibull Distribution (EWD), Reliability of multi-component Stress – Strength models $R_{(s,k)}$, Maximum likelihood estimator (MLE), Moment estimator (MOM), shrinkage method (Sh), Least Squares Estimator (LS), Rank Set Sampling (RSS). and mean squared error (MSE).*

 https://doi.org/ 10.30526/2017.IHSCICONF.1809

1. Introduction

The stress-strength (S-S) reliability of the system which contains one component is denoted by R=P(Y<X), where Y arise the stress random variable and X arise the strength random variable. And hence "the stress-strength (S-S) reliability is computed as $R = \int_0^\infty f_{\text{Strength}}(x).F_{\text{Stress}}(x)dx$. Its application has spread out into many fields; one of the most developed usual applications focuses on engineering-oriented problems. For example, the common tennis racket is composed of two basic subsystems, the frame, and string. During a tennis match, the random stress a racket is exposed to includes hitting the ball repeated times or falling on the ground accidently [29]. As well as (S-S) model is used in engineering devices, it has been usually determined, how long rang time of the system will be live. The stress-strength model has been studied by many authors like Ali, Pal, and Woo (2012), they estimated the (S-S) reliability of Generalized Gamma distribution with four parameters [3]. Hussian in 2013, considered the estimation of the reliability of stress-strength model for generalized inverted exponential distribution [11]. Ghitany et al. (2015), studied estimation of the reliability of stress-strength system from power Lindley distribution [7]. Najarzadegan et al. (2016), considered the estimation of P (Y<X) for the Levy distribution [18].

The reliability of the system model; s out of k (s-k) denoted by $R_{(s,k)}$ functioning when at least s (1≤s≤k) of components survive was introduced by Bhattacharyya and Johnson (1974). "They developed the reliability multi-component stress-strength system model $R_{(s,k)}$ which including k of a component and identical strength component put up with common stress function if s (1≤s≤k) or more of the components simultaneously operate";[6]. This system works successfully, if at least s out of k components resist the stress. Noted that, if s=1 and s=k corresponded respectively, to parallel and series systems. The mentioned model was used in many applications in physics and engineering such as strength failure and the system collapse [9].

In 2010, Rao & Kantam studied a system of k multi-component which have independently and identically strength $x_1, x_2, \ldots, x_k$ random variables distributed experiencing the random stress y when stress and strength follow the Log–Logistic distribution [21]. Srinivasa Rao (2012) estimated the multicomponent system of reliability for log-logistic distribution with different shape parameters [24]. As well as, in the same year, Srinivasa Rao studied estimation for the reliability of multi-component stress-strength model based generalized exponential distribution [23]. While Hassan & Basheikh (2012) studied reliability estimation of stress- strength model with non-identical component strengths by using the exponentiated Pareto distribution [9]. In 2016, Srinivasa Rao et al. estimated the multi-component stress-strength reliability of a system when stress and strength follow exponentiated Weibull distribution [22]. Recently, Hassan (2017) studied the estimation of multi-component of reliability (S-S) system model when each of stress and strength follows Lindley distribution [10].

Mudholkar and Srivastava (1993) introduced the exponentiated Weibull distribution (EWD) as an extension of the Weibull distribution, the obtained distribution is characterized by bathtub-shaped and model failure rates besides a broader class of monotone failure rates [16]. Mudholkar et al. (1995) explained also applications of the exponentiated Weibull distribution in reliability and survival [17]. Some properties and a flood data application of exponentiated Weibull family were studied by Mudholkar and

Huston (1996) [15]. Nassar and Eissa (2003) had done some studies on exponentiated Weibull distribution model [19]. Several authors have been used the (EWD) as a model of many fields like flood data, biological studies also in physical explanation …etc.

The aim of this paper is to estimate the multi-component system reliability of stress-strength model $R_{(s,k)}$ based on exponentiated Weibull distribution with known shape parameter θ and the other shape parameter α will be unknown via different estimation methods like Maximum Likelihood Estimator MLE, the Modified Thompson Shrinkage method Th, the Least Square LS, as well as some of the Ranked Set Sampling RSS. The comparisons between the proposed estimator methods by using simulation are performed, depends on a mean squared error as criteria.

Now, if the two random variables X and Y follows the EWD with parameter (α_1, θ) and (α_1,θ) as stress and strength respectively as below:

$$f(x;\alpha_1,\theta) = \alpha_1\theta x^{\theta-1}e^{-x^\theta}(1-e^{-x^\theta})^{\alpha_1-1} \qquad x>0, \qquad \alpha_1, \ \theta>0 \quad (1)$$

$$f(y;\alpha_2,\theta) = \alpha_2\theta y^{\theta-1}e^{-y^\theta}(1-e^{-y^\theta})^{\alpha_2-1} \qquad y>0, \qquad \alpha_2, \ \theta>0 \quad (2)$$

while the cumulative distribution function (CDF.) is:

$$F(x,\alpha_1) = (1-e^{-x^\theta})^{\alpha_1} \qquad x>0,\ \alpha_1,\theta>0 \quad (3)$$

$$F(y,\alpha_2) = (1-e^{-y^\theta})^{\alpha_2} \qquad y>0,\ \alpha_2,\theta>0 \quad (4)$$

In order to find a system reliability consisting of k^{th} identical components (when at least s out of k function) for the strength $X_1, X_2, \ldots, X_k$ which are random variables with EWD (α_1,θ), subjected to a stress Y which is a random variable follows EWD (α_2,θ) depend on Bhattacharyya and Johnson (1974), the reliability of a multi-component stress-strength model $R_{(s,k)}$;[5] can be calculated as

$R_{(s,k)}$ =P(at least s of the $X_1, X_2,\ldots, X_k$ exceed Y)

$$=\sum_{i=s}^{k}\binom{k}{i}\int_o^\infty(1-F_x(y))^i(F_x(y))^{k-i}dG(y)$$

$$=\sum_{i=s}^{k}\binom{k}{i}\int_o^\infty(1-(1-e^{-y^\theta})^{\alpha_1})^i((1-e^{-y^\theta})^{\alpha_1})^{k-i}\alpha_2\theta y^{\theta-1}e^{-y^\theta}(1-e^{-y^\theta})^{\alpha_2-1}dy$$

$$=\sum_{i=s}^{k}\binom{k}{i}\int_o^\infty(1-(1-e^{-y^\theta})^{\alpha_1})^i\alpha_2\theta y^{\theta-1}e^{-y^\theta}(1-e^{-y^\theta})^{\alpha_1k-i\alpha_1+\alpha_2-1}dy$$

And by some of the simplification, we get

$$R_{(s,k)}=\frac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\frac{k!}{(k-i)!}\left[\prod_{j=0}^{i}\left(k+\frac{\alpha_2}{\alpha_1}-j\right)\right]^{-1} \quad ; \text{ where k, i and j are integers} \quad (5)$$

2. Estimation methods of $R_{(s,k)}$:

2.1 Maximum Likelihood Estimator (MLE):

Suppose that k of components are put on life-testing experiment, in this case, we consider that $x_1, x_2, \ldots, x_n$ are a random sample of size n follows EWD (α_1, θ), and $y_1, y_2, \ldots, y_m$ be a random sample of size m follows EWD (α_2, θ),. Then the likelihood function of the mentioned system will be:

$$l \equiv L(\alpha_1, \alpha_2, \theta; x, y) = \prod_{i=1}^{n} f(x_i) \prod_{j=1}^{m} g(y_j)$$

$$= \prod_{i=1}^{n} \alpha_1 \theta x_i^{\theta-1} e^{-x_i^{\theta}} (1 - e^{-x_i^{\theta}})^{\alpha_1 - 1} \prod_{j=1}^{m} \alpha_2 \theta y_j^{\theta-1} e^{-y_j^{\theta}} (1 - e^{-y_j^{\theta}})^{\alpha_2 - 1}$$

$$= \alpha_1^{n} \theta^{n} \prod_{i=1}^{n} x_i^{\theta-1} e^{-\sum_{i=1}^{n} x_i^{\theta}} \prod_{i=1}^{n} (1 - e^{-x_i^{\theta}})^{\alpha_1 - 1} \alpha_2^{m} \theta^{m} \prod_{j=1}^{m} y_j^{\theta-1} e^{-\sum_{j=1}^{m} y_j^{\theta}} \prod_{j=1}^{m} (1 - e^{-y_j^{\theta}})^{\alpha_2 - 1}$$

Take logarithm for both sides, we get:

$$\text{Ln}(l) = n ln\alpha_1 + n ln\theta + (\theta - 1) \sum_{i=1}^{n} ln x_i^{\theta-1} - \sum_{i=1}^{n} x_i^{\theta} + (\alpha_1 - 1) \sum_{i=1}^{n} (1 - e^{-x_i^{\theta}})$$
$$m ln\alpha_2 + m ln\theta + (\theta - 1) \sum_{j=1}^{m} ln y_j^{\theta-1} - \sum_{j=1}^{m} y_j^{\theta} + (\alpha_2 - 1) \sum_{j=1}^{m} (1 - e^{-y_j^{\theta}})$$

Derive the above equation with respect to the unknown shape parameters α_i (i=1,2) and equating the result to zero, we get:

$$\frac{d\text{Ln}(l)}{d\alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} ln\left(1 - e^{-x_i^{\theta}}\right) = 0$$

$$\frac{d\text{Ln}(l)}{d\alpha_2} = \frac{m}{\alpha_2} + \sum_{j=1}^{m} ln\left(1 - e^{-y_j^{\theta}}\right) = 0$$

Thus, the maximum likelihood estimator of the parameter α_i (i=1,2) will be as follows:

$$\hat{\alpha}_{1_{MLE}} = \frac{-n}{\sum_{i=1}^{n} ln\left(1 - e^{-x_i^{\theta}}\right)}$$
(6)

$$\hat{\alpha}_{2_{MLE}} = \frac{-m}{\sum_{j=1}^{m} ln\left(1 - e^{-y_j^{\theta}}\right)}$$
(7)

By substituting $\hat{\alpha}_{i_{mle}}$ (i=1,2) in equation (5) we get the reliability estimation for $R_{(s,k)}$ model via Maximum Likelihood method as below:

$$\hat{R}_{(s,k)_{MLE}} = \frac{\hat{\alpha}_{2_{MLE}}}{\hat{\alpha}_{1_{MLE}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} (k + \frac{\hat{\alpha}_{2_{MLE}}}{\hat{\alpha}_{1_{MLE}}} - j) \right]^{-1}$$
(8)

2.2 Moment Method (MOM) $R_{(s,k)}$:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n for strength X follows EW (α_1, θ), and $y_1, y_2, \ldots, y_m$ be a random sample of size m for stress Y follows EW (α_2, θ). Let $\bar{X}$ and $\bar{Y}$ are the means of samples of strength and stress respectively, then the population moments of X, Y are given by; see[20].

$$E(X^r)=\begin{cases} \alpha \sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j} (-1)^j (j+1)^{-\frac{r}{\theta}-1} \Gamma\left(\frac{r}{\theta}+1\right) & , if \alpha \in N \\ \alpha \sum_{j=0}^{\alpha-1} \frac{\alpha-1 P_j}{j!} (-1)^j (j+1)^{-\frac{r}{\theta}-1} \Gamma\left(\frac{r}{\theta}+1\right) & , if\ \alpha \notin N \end{cases} \quad \text{for } r=1,2,3\ldots.$$

Where αP_j=α (α-1) (α-2)….. (α-j+1) and N is the set of natural number

Therefore, the population means of X and Y are respectively as below:

$$E(X)=\begin{cases} \alpha_1 \sum_{j=0}^{\alpha_1-1} \binom{\alpha_1-1}{j} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if \alpha_1 \in N \\ \alpha_1 \sum_{j=0}^{\alpha_1-1} \frac{\alpha_1-1^{P_j}}{j!} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if\ \alpha_1 \notin N \end{cases} \quad \ldots (9)$$

And,

$$E(Y) = \begin{cases} \alpha_2 \sum_{j=0}^{\alpha_2-1} \binom{\alpha_2-1}{j} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if \alpha_2 \in N \\ \alpha_2 \sum_{j=0}^{\alpha_2-1} \frac{\alpha_2-1^{P_j}}{j!} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if\ \alpha_2 \notin N \end{cases} \quad (10)$$

Equating the samples mean with the corresponding populations mean for both X and Y as follows:-

$$\bar{X} = \frac{\sum_{j=1}^{n} x_j}{n} = \alpha_1 \sum_{j=0}^{\alpha_1-1} \binom{\alpha_1-1}{j} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right)$$

$$\bar{Y} = \frac{\sum_{j=1}^{m} y_j}{m} = \alpha_2 \sum_{j=0}^{\alpha_2-1} \binom{\alpha_2-1}{j} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right)$$

By simplification, we obtain the estimation of unknown shape parameters α_i (i=1,2) using moment method as follows:

$$\hat{\alpha}_{1_{MOM}} = \frac{\bar{X}}{\sum_{j=0}^{\alpha_1-1} \binom{\alpha_1-1}{j} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right)} \quad (11)$$

$$\hat{\alpha}_{2_{MOM}} = \frac{\bar{Y}}{\sum_{j=0}^{\alpha_2-1} \binom{\alpha_2-1}{j} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right)} \quad (12)$$

Substitution $\hat{\alpha}_{i_{MOM}}$ (i=1,2) in equation (5), we conclude the reliability estimation for $R_{(s,k)}$ model via moment method as below:

$$\hat{R}_{(s,k)_{MOM}} = \frac{\hat{\alpha}_{2_{MOM}}}{\hat{\alpha}_{1_{MOM}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} (k + \frac{\hat{\alpha}_{2_{MOM}}}{\hat{\alpha}_{1_{MOM}}} - j) \right]^{-1}$$
(13)

2.3 Shrinkage Estimation Method [1, 4, 5, 25 & 28].

Thompson in 1968 has suggested the problem of shrink a usual estimator $\hat{\alpha}$ of the parameter α to prior information α_0 using shrinkage weight factor $\emptyset(\hat{\alpha})$, such that $0 \leq \emptyset(\hat{\alpha}) \leq 1$. Thompson says that "We are estimating α and we believe α_0 is closed to the true value of α or we fear that α_0 may be near the true value of α, that is mean something bad happens if $\alpha_0 \approx \alpha$ and we do not use α_0". Thus, the form of shrinkage estimator of α say $\hat{\alpha}_{sh}$ will be:

$$\hat{\alpha}_{sh} = \emptyset(\hat{\alpha})\hat{\alpha} + (1 - \emptyset(\hat{\alpha}))\hat{\alpha}_0$$
(14)

In this work, we apply the unbiased estimator $\hat{\alpha}_{ub}$ as a usual estimator and the moment estimator as a prior estimation of α in equation (14) above. Where $\emptyset(\hat{\alpha})$ denote the shrinkage weight factor as we mentioned above such that $0 \leq \emptyset(\hat{\alpha}) \leq 1$, which may be a function of $\hat{\alpha}_{ub}$; can be found by minimizing the mean square error of $\hat{\alpha}_{sh}$.Thus, the shrinkage estimator for the shape parameter α of EWD will be as follows:

$$\hat{\alpha}_{Sh} = \emptyset(\hat{\alpha})\hat{\alpha}_{Ub} + (1 - \emptyset(\hat{\alpha}))\hat{\alpha}_{MOM}$$
(15)

Note that,

$$\hat{\alpha}_{1_{Ub}} = \frac{n-1}{n}\hat{\alpha}_{1_{MLE}} = \frac{n-1}{-\sum_{i=1}^{n} ln\left(1-e^{-x_i^{\theta}}\right)},$$

Hence,

E($\hat{\alpha}_{1_{Ub}}$)= α_1 and $Var(\hat{\alpha}_{1_{Ub}})$=$\frac{\alpha_1^2}{n-2}$

And,

$$\hat{\alpha}_{2_{Ub}} = \frac{m-1}{m}\hat{\alpha}_{2_{MLE}} = \frac{m-1}{-\sum_{j=1}^{m} ln\left(1-e^{-y_j^{\theta}}\right)}$$

Implies, E($\hat{\alpha}_{2_{Ub}}$)= α_2 and $Var(\hat{\alpha}_{2_{Ub}})$=$\frac{\alpha_2^2}{m-2}$.

2.4 Modified Thompson type shrinkage weight function (Th)

In this subsection, we introduce and modify the shrinkage weight factor consider by Thompson in 1968 as below.

$$\gamma(\hat{\alpha}_i)=\frac{\left(\hat{\alpha}_{i_{Ub}}-\hat{\alpha}_{i_{MOM}}\right)^2}{\left(\hat{\alpha}_{i_{Ub}}-\hat{\alpha}_{MOM}\right)^2+Var\left(\hat{\alpha}_{i_{Ub}}\right)}(0.01) \quad \text{for} \quad i=1,2$$
(16)

Therefore, the shrinkage estimator of α_i(i=1,2) by using above modified shrinkage weight factor will be:

$$\hat{\alpha}_{i_{Th}}=\gamma(\hat{\alpha}_i)\hat{\alpha}_{i_{Ub}}+(1-\gamma(\hat{\alpha}_i))\hat{\alpha}_{i_{MOM}} \quad \text{for} \quad i=1,2$$
(17)

Substitute equation (17) in equation (5), we conclude the reliability estimation of $R_{(s,k)}$ based on modified Thompson type shrinkage weight factor as follows:

$$\hat{R}_{(s,k)Th}=\frac{\hat{\alpha}_{2_{Th}}}{\hat{\alpha}_{1_{Th}}}\sum_{i=s}^{k}\frac{k!}{(k-i)!}\left[\prod_{j=0}^{i}(k+\frac{\hat{\alpha}_{2_{Th}}}{\hat{\alpha}_{1_{Th}}}-j)\right]^{-1}$$
(18)

2.5 Least Squares Estimator Method (LS) [2 & 13]

In this subsection, we discuss the estimator of Least Squares method, this method used for many mathematical and engineering application [2]. The main idea for this method by minimizes the sum of squared error between the values and the expected value.

To find the parameters α_i (i=1,2) via mentioned least squares method, let:

$$S=\sum_{i=1}^{n}[F(x_i)-E(F(x_i))]^2 \quad i=1,2,3……n$$
(19)

Where, $F(x_i)$ refer to the CDF of two parameters EWD which is defined in equation (3) as follows:

$F(x_i)=(1-e^{-x_i^{\theta}})^{\alpha_1}$

And, $E(F(x_i))=P_i$

Such as; $P_i=\frac{i}{n+1}$, i=1,2,…,n

$F(x_i)=E(F(x_i))$

$$(1-e^{-x_i^{\theta}})^{\alpha_1}=\frac{i}{n+1}$$
(20)

Now, taking the natural logarithm of both sides in equation (20) as below:

$\alpha_1\text{Ln}(1-e^{-x_i^{\theta}})=\text{Ln}P_i$

$$\alpha_1\text{Ln}(1-e^{-x_i^{\theta}})-\text{Ln}P_i=0$$
(21)

Now, by putting equation (21) in equation (19) we obtain:

 https://doi.org/ 10.30526/2017.IHSCICONF.1809

$$S = \sum_{i=1}^{n}[\alpha_1 \text{Ln}(1 - e^{-x_i^{\theta}}) - \text{Ln}P_i\,]^{2}$$
(22)

And be finding the partial derivatives for the equation (22) with respect to α_1.

$$\frac{\partial S}{\partial \alpha_1} = 2\sum_{i=1}^{n}[\alpha_1 \text{Ln}(1 - e^{-x_i^{\theta}}) - \text{Ln}P_i\,]\,\text{Ln}(1 - e^{-x_i^{\theta}})$$
(23)

Then, equal the equation (23) to zero get as:

$$\sum_{i=1}^{n}\left[\alpha_1 \text{Ln}\left(1 - e^{-x_i^{\theta}}\right) - \text{Ln}P_i\right]\text{Ln}\left(1 - e^{-x_i^{\theta}}\right) = 0$$

$$\sum_{i=1}^{n}\alpha_1 \text{Ln}\left(1 - e^{-x_i^{\theta}}\right)\text{Ln}\left(1 - e^{-x_i^{\theta}}\right) = \sum_{i=1}^{n}\text{Ln}P_i\,\text{Ln}\left(1 - e^{-x_i^{\theta}}\right)$$

$$\hat{\alpha}_{1_{LS}} = \frac{\sum_{i=1}^{n}\text{Ln}P_i \text{Ln}\left(1-e^{-x_i^{\theta}}\right)}{\sum_{i=1}^{n}(\text{Ln}\left(1-e^{-x_i^{\theta}}\right))^2}$$
(24)

By same way, one can estimate the parameter α_2 for Y which is represent stress random variable and follows EWD (α_2,θ) with size m as follow:

$$\hat{\alpha}_{2_{LS}} = \frac{\sum_{j=1}^{m}\text{Ln}P_j \text{Ln}\left(1-e^{-y_j^{\theta}}\right)}{\sum_{j=1}^{m}(\text{Ln}\left(1-e^{-y_j^{\theta}}\right))^2}$$
(25)

; $P_j = \frac{j}{m+1}$, j=1,2,…,m.

Put $\hat{\alpha}_{1_{LS}}$ and $\hat{\alpha}_{2_{LS}}$ in equation (5), will be obtain the reliability estimation of $R_{(s,k)}$ based on LS as follows:

$$\hat{\boldsymbol{R}}_{(s,k)_{LS}} = \frac{\hat{\alpha}_{2_{LS}}}{\hat{\alpha}_{1_{LS}}}\sum_{i=s}^{k}\frac{k!}{(k-i)!}\left[\prod_{j=0}^{i}\left(k + \frac{\hat{\alpha}_{2_{LS}}}{\hat{\alpha}_{1_{LS}}} - j\right)\right]^{-1} \quad \text{i, j, k are integer}$$
(26)

Ranked Set Sampling Method (RSS) [8, 12, 26 & 27]

"Ranked set sampling was introduced and applied to the problem of estimating mean pasture yields by Mclntyre in 1952 this function was to improve the efficiency of the sample mean as an estimator of the population mean in situations in which the characteristic of interest was difficult or expensive to measure, but using ranked, it become cheaper" [27 & 12]

"The concept of rank set sampling is a recent development that enables one to provide more structure to the collected sample items" [26]

Let $x_1, x_2, \ldots, x_n$ be a random sample EWD, let $x_{(1)}, x_{(2)},\ldots, x_{(n)}$ be order statistics increasing order. The PDF of $x_{(q)}$ is:

 https://doi.org/ 10.30526/2017.IHSCICONF.1809

$$f(x_{(q)}) = \frac{n!}{(q-1)!(n-q)!}\left[F\left(x_{(q)}\right)\right]^{q-1}\left[1-F\left(x_{(q)}\right)\right]^{n-q} f(x_{(q)})$$
(27)

by substituting the PDF from equation (1) and the CDF from equation (3) in equation (27) will get:

$$f(x_{(q)}) = \frac{n!}{(q-1)!\,(n-q)!}\left[\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]^{q-1}\left[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]^{n-q}$$

$$\alpha_1\theta x_{(q)}{}^{\theta-1}e^{-x_{(q)}{}^{\theta}}(1-e^{-x_{(q)}{}^{\theta}})^{\alpha_1-1} \quad \ldots$$
(28)

$$f(x_{(q)}) = Q\alpha_1\theta x_{(q)}{}^{\theta-1}e^{-x_{(q)}{}^{\theta}}(1-e^{-x_{(q)}{}^{\theta}})^{q\alpha_1-1}[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}]^{n-q}$$

Such that; $Q = \frac{n!}{(q-1)!(n-q)!}$

the likelihood function of order sample $x_{(1)},x_{(2)},\ldots,x_{(n)}$ is:

$$L\left(x_{(1)},x_{(2)},\ldots,x_{(n)};\ \alpha_1,\theta\right) = Q^n\alpha_1{}^n\theta^n\prod_{q=1}^{n}x_{(q)}{}^{\theta-1}\prod_{q=1}^{n}e^{-x_{(q)}{}^{\theta}}\prod_{q=1}^{n}(1-e^{-x_{(k)}{}^{\theta}})^{k\alpha_1-1}\prod_{k=1}^{n}[1-\left(1-e^{-x_{(k)}{}^{\theta}}\right)^{\alpha_1}]^{n-k}$$
…(29)

Where, take (Ln) for both side of the equation (29) as below:

$$\mathrm{Ln}(l) = n\mathrm{Ln}Q + n\mathrm{Ln}\alpha_1 + n\mathrm{Ln}\theta + (\theta-1)\sum_{q=1}^{n}Lnx_{(q)}{}^{\theta} - \sum_{q=1}^{n}x_{(q)}{}^{\theta} + (q\alpha_1-1)\sum_{q=1}^{n}Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right) + (n-q)\sum_{q=1}^{n}Ln[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}]$$

Take the partial derivative with respect to the parameter α_1 get the following:

$$\frac{\partial\mathrm{Ln}(l)}{\partial\alpha_1} = \frac{n}{\alpha_1} + \sum_{q=1}^{n}q\,Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right) + \sum_{q=1}^{n}(n-q)\frac{-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right)}{[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}]}$$
(30)

Now, equal the equation (30) to zero, we get:

$$\frac{n}{\alpha_1} + \sum_{q=1}^{n}q\,Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right) - \sum_{q=1}^{n}(n-q)\frac{\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right)}{\left[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]} = 0$$

$$\frac{n}{\alpha_1} = \sum_{q=1}^{n}(n-q)\frac{\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right)}{[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}]} - \sum_{q=1}^{n}q\,Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right)$$

Then,

 https://doi.org/ 10.30526/2017.IHSCICONF.1809

$$\hat{\alpha}_{1_{RSS}} = \frac{n}{\sum_{q=1}^{n}(n-q)\frac{\left(1-e^{-x_{(q)}^{\theta}}\right)^{\alpha_1} Ln\left(1-e^{-x_{(q)}^{\theta}}\right)}{\left[1-\left(1-e^{-x_{(q)}^{\theta}}\right)^{\alpha_1}\right]} - \sum_{q=1}^{n} qLn\left(1-e^{-x_{(q)}^{\theta}}\right)} \quad (31)$$

And by same way assume the stress y random sample follows EWD with two parameters ($\alpha_2$and θ) with size m to find $\hat{\alpha}_{2_{RSS}}$ as below:

$$\hat{\alpha}_{2_{RSS}} = \frac{m}{\sum_{t=1}^{m}(m-t)\frac{\left(1-e^{-y_{(t)}^{\theta}}\right)^{\alpha_2} Ln\left(1-e^{-y_{(t)}^{\theta}}\right)}{\left[1-\left(1-e^{-y_{(t)}^{\theta}}\right)^{\alpha_2}\right]} - \sum_{t=1}^{m} tLn\left(1-e^{-y_{(t)}^{\theta}}\right)}$$

(32)

Substitution the estimation parameter $\hat{\alpha}_{i_{RSS}}$ (i=1,2) in equation (5) will be obtain the reliability estimation of $R_{(s,k)}$ based on RSS as follows:

$$\hat{R}_{(s,k)_{RSS}} = \frac{\hat{\alpha}_{2_{RSS}}}{\hat{\alpha}_{1_{RSS}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} \left(k + \frac{\hat{\alpha}_{2_{RSS}}}{\hat{\alpha}_{1_{RSS}}} - j\right) \right]^{-1}$$

(33)

Simulation study

In this section we numerical results were studied to compare the performance of the different estimators of reliability which is obtained in section 2, using different sample size =(25,50,75 and 100), based on 1000 replication via MSE criteria.

For this purpose, Monte Carlo simulation was used the following steps:-[14]

Step1: We generate the random sample which follows the continuance uniform distribution defined on the interval (0,1) as $u_1, u_2, \ldots, u_n$.

Step2: We generate the random sample which follows the continuance uniform distribution defined on the interval (0, 1) as $w_1, w_2, \ldots, w_m$.

Step3: Transform the uniform random samples in step1 to random samples follows EWD, applying the theorem that using the inverse cumulative probability distribution function (CDF) as below shown:

$$F(x) = (1 - e^{-x_i^{\theta}})^{\alpha}$$

$$U_i = (1 - e^{-x_i^{\theta}})^{\alpha}$$

$$x_i = [-\ln(1 - U_i^{\frac{1}{\alpha}})]^{\frac{1}{\theta}}$$

And, by the same method, we get:

$$y_j = [-\ln(1 - W_j^{\frac{1}{\lambda}})]^{\frac{1}{\theta}}$$

Step4: We recall the $R_{(s,k)}$ from equation (5).

Step5: We compute the Maximum Likelihood Estimator of the $R_{(s,k)}$ using equation (8).

Step6: We compute the Moment Method of $R_{(s,k)}$ using equation (13).

Step7: We compute the Modified Thompson types Shrinkage Estimator of $R_{(s,k)}$ using equation (18).

Step8: We compute the Least Squares method of $R_{(s,k)}$ using equation (26).

Step9: We compute the Ranked set sampling of $R_{(s,k)}$ using equation (33).

Step10: based on (L=1000) Replication, we calculate the MSE as follows:

$$\mathbf{MSE} = \frac{1}{L}\sum_{i=1}^{L}(\hat{R}_{(s,k)_i} - R_{(s,k)})^2$$

Where $\hat{R}_{(s,k)}$ refers the proposed estimators of real value of reliability $R_{(s,k)}$. Note that in this paper, we consider (s,k)=(2,3), (2,4) and (3,4), and all the results are put it in the tables (1-3) below:

Table (1): Shown estimation value of $R_{(s,k)}$, and MSE when (s,k)=(2,3), and α_1=2, α_2=5, θ=3;

$R_{(s,k)}$= 0.24242

	$\hat{R}_{(s,k)}$				MSE				
(n,m)	$\hat{R}_{(s,k)_{MLE}}$	$\hat{R}_{(s,k)_{Th}}$	$\hat{R}_{(s,k)_{LS}}$	$\hat{R}_{(s,k)_{RSS}}$	$\hat{R}_{(s,k)_{MLE}}$	$\hat{R}_{(s,k)_{Th}}$	$\hat{R}_{(s,k)_{LS}}$	$\hat{R}_{(s,k)_{RSS}}$	**Best**
(25,25)	0.24791	0.24283	0.49692	0.23294	0.00488	0.00024	0.08271	0.00029	Th
(25,50)	0.43344	0.24312	0.17813	0.43157	0.04141	0.00019	0.01516	0.03600	Th
(25,75)	0.55136	0.24343	0.01305	0.55219	0.09928	0.00018	0.05345	0.09613	Th
(25,100)	0.62907	0.24342	0.03564	0.63177	0.15264	0.00018	0.04533	0.15174	Th
(50,25)	0.10944	0.24014	0.58209	0.09797	0.01892	0.00016	0.12174	0.02090	Th
(50,50)	0.24405	0.24254	0.13274	0.23333	0.00253	0.00012	0.01865	0.00018	Th
(50,75)	0.35270	0.24296	0.14738	0.34225	0.01472	0.00010	0.01531	0.01007	Th
(50,100)	0.43686	0.24371	0.08694	0.42684	0.04038	0.00010	0.02523	0.03411	Th
(75,25)	0.06155	0.23740	0.38130	0.05372	0.03315	0.00016	0.02095	0.03561	Th
(75,50)	0.15610	0.24118	0.36197	0.14613	0.00864	0.00009	0.02494	0.00931	Th
(75,75)	0.24192	0.24214	0.15093	0.23350	0.00155	0.00007	0.00976	0.00013	Th
(75,100)	0.31836	0.24290	0.11883	0.30854	0.00755	0.00007	0.01784	0.00444	Th
(100,25)	0.03960	0.23560	0.72816	0.03387	0.04134	0.00019	0.24874	0.04349	Th
(100,50)	0.11040	0.24081	0.57515	0.09964	0.01807	0.00008	0.11484	0.02040	Th
(100,75)	0.17674	0.24118	0.17903	0.16956	0.00535	0.00007	0.00824	0.00534	Th
(100,100)	0.24444	0.24270	0.12698	0.23309	0.00119	0.00005	0.01604	0.00013	Th

 https://doi.org/ 10.30526/2017.IHSCICONF.1809

Table (2): Shown estimation value of $R_{(s,k)}$, and MSE when (s,k)=(2,4), and α_1=2, α_2=5, θ=3;

$R_{(s,k)}$= 0.33566

	$\hat{R}_{(s,k)}$				MSE				
(n,m)	$\hat{R}_{(s,k)_{MLE}}$	$\hat{R}_{(s,k)_{Th}}$	$\hat{R}_{(s,k)_{LS}}$	$\hat{R}_{(s,k)_{RSS}}$	$\hat{R}_{(s,k)_{MLE}}$	$\hat{R}_{(s,k)_{Th}}$	$\hat{R}_{(s,k)_{LS}}$	$\hat{R}_{(s,k)_{RSS}}$	**Best**
(25,25)	0.33902	0.33616	0.40165	0.32468	0.00612	0.00030	0.01323	0.00039	Th
(25,50)	0.53639	0.33595	0.14739	0.53480	0.04521	0.00027	0.03988	0.03988	Th
(25,75)	0.65045	0.33767	0.07510	0.64660	0.10217	0.00023	0.06972	0.09682	Th
(25,100)	0.71426	0.33733	0.02873	0.71654	0.14586	0.00025	0.09450	0.14518	Th
(50,25)	0.16928	0.33330	0.46971	0.15389	0.02996	0.00021	0.03347	0.03312	Th
(50,50)	0.33761	0.33591	0.34261	0.32479	0.00293	0.00014	0.01547	0.00023	Th
(50,75)	0.45225	0.33613	0.19639	0.44531	0.01634	0.00013	0.02012	0.01213	Th
(50,100)	0.53925	0.33703	0.16807	0.53043	0.04380	0.00013	0.03956	0.03803	Th
(75,25)	0.10250	0.33124	0.86866	0.08935	0.05540	0.00021	0.28612	0.06070	Th
(75,50)	0.23217	0.33458	0.43460	0.21818	0.01276	0.00013	0.01351	0.01387	Th
(75,75)	0.33637	0.33573	0.37699	0.32495	0.00206	0.00010	0.00330	0.00019	Th
(75,100)	0.42069	0.33658	0.30343	0.40879	0.00911	0.00009	0.00514	0.00542	Th
(100,25)	0.06675	0.32741	0.37284	0.05869	0.07278	0.00025	0.08851	0.07672	Th
(100,50)	0.16750	0.33292	0.60934	0.15680	0.02943	0.00012	0.07658	0.03203	Th
(100,75)	0.25687	0.33418	0.54797	0.24809	0.00760	0.00009	0.05578	0.00771	Th
(100,100)	0.33762	0.33584	0.43285	0.32471	0.00166	0.00008	0.01129	0.00018	Th

Table (3): Shown estimation value of $R_{(s,k)}$, and MSE when (s,k)=(3,4), and α_1=2, α_2=5, θ=3;

$R_{(s,k)}$= 0.14918

	$\hat{R}_{(s,k)}$				MSE				
(n,m)	$\hat{R}_{(s,k)_{MLE}}$	$\hat{R}_{(s,k)_{Th}}$	$\hat{R}_{(s,k)_{LS}}$	$\hat{R}_{(s,k)_{RSS}}$	$\hat{R}_{(s,k)_{MLE}}$	$\hat{R}_{(s,k)_{Th}}$	$\hat{R}_{(s,k)_{LS}}$	$\hat{R}_{(s,k)_{RSS}}$	**Best**
(25,25)	0.15829	0.14954	0.05402	0.14080	0.00367	0.00017	0.01398	0.00020	Th
(25,50)	0.33719	0.15013	0.04745	0.32714	0.04068	0.00014	0.01056	0.03190	Th
(25,75)	0.45906	0.15049	0.11451	0.45677	0.10095	0.00013	0.02620	0.09484	Th
(25,100)	0.54773	0.15079	0.02212	0.54573	0.16312	0.00013	0.01633	0.15744	Th
(50,25)	0.04997	0.14710	0.42192	0.04211	0.01034	0.00011	0.08331	0.01147	Th
(50,50)	0.15289	0.14934	0.12683	0.14133	0.00185	0.00008	0.00391	0.00013	Th
(50,75)	0.25085	0.14972	0.04628	0.23963	0.01302	0.00008	0.01138	0.00828	Th
(50,100)	0.33256	0.15000	0.17227	0.32360	0.03638	0.00007	0.01072	0.03053	Th
(75,25)	0.02290	0.14572	0.38050	0.01776	0.01607	0.00011	0.08070	0.01727	Th
(75,50)	0.08330	0.14853	0.28988	0.07343	0.00499	0.00006	0.02603	0.00575	Th
(75,75)	0.15287	0.14948	0.09863	0.14114	0.00125	0.00005	0.00343	0.00010	Th
(75,100)	0.21948	0.14977	0.02920	0.20747	0.00647	0.00005	0.01551	0.00345	Th
(100,25)	0.01210	0.14386	0.42068	0.00911	0.01883	0.00012	0.08416	0.01962	Th
(100,50)	0.04913	0.14751	0.34451	0.04319	0.01025	0.00006	0.04190	0.01124	Th
(100,75)	0.09971	0.14895	0.43593	0.09019	0.00300	0.00004	0.09557	0.00349	Th
(100,100)	0.15011	0.14905	0.32885	0.14152	0.00085	0.00004	0.03709	0.00008	Th

 https://doi.org/ 10.30526/2017.IHSCICONF.1809

3. Discussion Numerical Simulation Results:

For all *n*=(25,50,75, and100) and for all *m*=(25,50,75, and100), in this work, the minimum mean square error (MSE) for reliability estimation of $R_{(s,k)}$ model for the exponentiated Weibull distribution is held using Modified Thompson type shrinkage estimator $\hat{\boldsymbol{R}}_{(s,k)_{Th}}$. This implies that, the shrinkage for reliability estimation ($\hat{\boldsymbol{R}}_{(s,k)_{Th}}$) is the best and follows by using. For any *n*, some of the proposed estimator (MLE, LS, and RSS) with *m* the methods are vibration. Finally, for any n and m the second order best estimator is Ranked Set Sampling ($\hat{\boldsymbol{R}}_{(s,k)_{RSS}}$) or Maximum Likelihood Estimator ($\hat{\boldsymbol{R}}_{(s,k)_{MLE}}$).

4. Conclusion:

From the numerical results, one can find the proposal using Modified Thompson type shrinkage estimator ($\hat{\boldsymbol{R}}_{(s,k)_{Th}}$) which depends on unbiased estimator and prior estimate (moment method) as a linear combination, performance good behavior and it is the best estimator than the others in the sense of MSE.

Acknowledgment

The authors wanted to provide thanks to the referees and to the Editor for constructive suggestions and valuable comments which resulted in the improvement of this article.

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