Coclosed Rickart Modules

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Abstract

Let M be a right module over an arbitrary ring R with identity and $\mathrm{S} = \mathrm{End}_R(M)$. In this work, the coclosed rickart modules as a generalization of rickart modules is given. We say a module M over R coclosed rickart if for each $f \in \mathrm{End}_R(M)$, $\mathrm{Ker}\, f$ is a coclosed submodule of M. Basic results over this paper are introduced and connections between these modules and otherwise notions are investigated.

Keywords: coclosed rickart modules, relatively coclosed rickart modules, rickart modules, cosemisimple modules, modules with CCIP.

 https://doi.org/ 10.30526/2017.IHSCICONF.1818

1. Introduction

R is denoted a ring has unity and M is studied as a left S- right R-bimodule where $\mathrm{S} = \mathrm{End}_R(M)$ is the endomorphism ring of M. Also $r_M(Sf) = r_M(f) = \mathrm{Ker}\, f = \{ m \in M \mid f(m) = 0 \}$ is the right annihilator of each element $f \in S$.

In [2] is presented a generalization of rickart modules by using the concept of purity. Further, dual coclosed rickart modules is introduced in [3]. We say a submodule L of a module M is coclosed in M when $\frac{L}{K} \ll \frac{M}{K}$ then $L = K$ for each $K \subseteq L$ [1]. Equivalently, for each proper submodule $K \subset L$, there is a submodule N of M where $L + N = M$ while $K + N \neq M$ iff L is coclosed of M.

Our goal of this research is to present and discuss another generalization of rickart modules namely coclosed rickart modules. A module M over R is said coclosed rickart when each $f \in \mathrm{End}_R(M)$, $\mathrm{Ker}\, f$ is a coclosed of M. Other studies in [5],[6],[7],[8],[11] and [12] are related topics.

The paper is divided into four sections. Section two, includes introducing the concept of coclosed rickart modules and providing facts of this argument.While direct summands of coclosed rickart modules are shown to inherit the property (Proposition 2.6), this is not so for direct sums (Remark 2.7).We obtain a condition which allows direct sums of coclosed rickart modules is coclosed rickart (Proposition 2.8). Section three is devoted to look for any connection between coclosed rickart modules and other modules. We see that coclosed rickart modules and rickart modules coincide in lifiting modules (Proposition 3.4). In section four, we present and study the concept of relatively coclosed rickart modules. By using the CCIP, we will provide a condition for modules to be relatively coclosed rickart (for example, Theorem 4.7, Proposition 4.11) where an R-module M has the coclosed intersection property (in brief CCIP) if, the intersection of any two coclosed submodules of M is coclosed [4]. Many results are investigated, we see that the rings R for which each right module over R relatively rickart are precisely right cosemisimple rings (Theorem 4.13).

Coclosed Rickart Modules

A comprehensive study of coclosed rickart modules is given in this section. We provide several characterizations and some properties of this of modules are investigated. We begin with the next.

Definition 2.1. We call a module M over R is coclosed rickart when each $f \in \mathrm{End}_R(M)$, $\mathrm{Ker}\, f$ is a coclosed submodule of M.

Remarks and Examples Clearly each cosemisimple module is coclosed rickart, but not conversely, where An R-module M is called cosemisimple if each submodule of M is coclosed in M [1]. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is coclosed rickart because each $f \in \mathrm{End}_R(M)$, $\mathrm{Ker}\, f = 0$ is a coclosed submodule in M but it is not cosemisimple since the only proper coclosed submodule in the $\mathbb{Z}$-module $\mathbb{Z}$ is the zero submodule.

(1) It is obvious that rickart module is coclosed rickart while the reverse is not hold as follows. Consider the ring $= \prod_{i\geq 1} \mathrm{F}_i$, $\mathrm{F}_i = \mathrm{F}$ is a field and R is not semisimple then by [9, Theorem 2.25], there exists a right module M over R, it is not rickart. And, R is commutative regular (in sense Von Neumann) so by [1], R is cosemisimple (or V-ring) implies that $\mathrm{Rad}\left(\frac{M}{N}\right) = 0$ for each submodule N of M. So each submodule of M is coclosed in M implies it is a coclosed rickart module.

 https://doi.org/ 10.30526/2017.IHSCICONF.1818

(2) When M is a coclosed simple R-modul yeild M need not be coclosed rickart where an R-module M is called coclosed simple if $M \neq \{0\}$ and it has no coclosed submodules except $\{0\}$ and M [4]. Study $\mathbb{Z}_4$ as $\mathbb{Z}$-module is coclosed simple, while it is not coclosed rickart since there exists an endomorphism $f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$ defined by $f(m) = m2$ for each $m \in \mathbb{Z}_4$, implies Ker $f = \{\bar{0}, \bar{2}\}$ is not a coclosed submodule in $\mathbb{Z}_4$.

(3) When M is a quasi-Dedekind over R implies M is coclosed rickart, where M is quasi-Dedekind if every $0 \neq f \in \text{End}_R(M)$, Ker $f = 0$ [10]. The reverse is not hold. For example, $\mathbb{Z}_6$ as $\mathbb{Z}$-module is coclosed rickart while it is not quasi-Dedekind.

(4) If M is a coclosed rickart coclosed simple module, implies M is quasi-Dedekind.

Proof. When M is a coclosed rickart over R and $0 \neq f \in \text{End}_R(M)$, implies Ker f is coclosed in M. But M is coclosed simple with $f \neq 0$, thus Ker $f = 0$, as required.

(5) A homomrphic picture of a coclosed rickart module may not be coclosed rickart. Consider the natural epimorphism $f : \mathbb{Z} \longrightarrow \mathbb{Z}_4$. The $\mathbb{Z}$ -module $\mathbb{Z}$ is coclosed rickart, while Im $f = \mathbb{Z}_4$ is not a coclosed rickart over $\mathbb{Z}$.

(6) Every integral domain is coclosed rickart.

Proof. When R as in assumption, implies R is commutative implies that for each $a \in R$ and $f \in \text{End}_R(R) \cong R$, we can define $f : R \longrightarrow R$ by $f(r) = ra$ for each $r \in R$. It follows that Ker $f = \{r \in R | f(r) = 0\} = \{r \in R | ra = 0\} = ann_R(a) = 0$ is coclosed in R.

(7) The reverse of Remark (7) need not be true generally. For example, the ring $\mathbb{Z}_6$ is coclosed rickart but it is not integral domain.

Proposition Under an isomorphism, the coclosed rickart property is moved.

Proof. Let M_1 and M_2 be modules over R, where M_1 is coclosed rickart and $f : M_1 \rightarrow M_2$ an isomorphism. Let $g \in \text{End}_R(M_2)$, we prove that Ker g is a coclosed submodule in M_2. Let K be any proper submodule such that $K \subset$ Ker g implies that $f^{-1}(K) \subset f^{-1}(\text{Ker } g)$. But $f^{-1}(\text{Ker } g) = \text{Ker}(f^{-1}gf)$, to show this. Let $x \in f^{-1}(\text{Ker } g)$, $x = f^{-1}(y)$ and $y \in$ Ker g implies that $f(x) = y$ and $g(y) = 0$. Then $f^{-1}g f(x) = f^{-1}g(y) = 0$, implies $x \in \text{Ker}(f^{-1}gf)$. For the other, let $x \in \text{Ker}(f^{-1}gf)$, $f^{-1}g f(x) = 0$ and so $f(x) \in \text{Ker}(f^{-1}g)$. One can easily see that $\text{Ker}(f^{-1}g) = \text{Ker } g$, hence $f(x) \in$ Ker g, $x \in f^{-1}(\text{Ker } g)$, it follows that $f^{-1}(\text{Ker } g) = \text{Ker}(f^{-1}gf)$. This means that $f^{-1}(K) \subset \ker(f^{-1}gf)$, but $f^{-1}gf \in f \in \text{End}_R(M_1)$ and M_1 coclosed rickart then $\text{Ker}(f^{-1}gf)$ is a coclosed , there exists a submodule N of M_1, $f^{-1}(\text{Ker } g) + N = M_1$ but $f^{-1}(K) + N \neq M_1$. This means that Ker $g + f(N) = M_2$ but $K + f(N) \neq M_2$, it follows that Ker g is a coclosed submodule in M_2.

From a module to any of its submodules or inversely, coclosed rickart property does not always passed in general as follows

Examples If N is a submodule of a coclosed rickart module M over R , implies N may not be coclosed rickart. Study $M = \mathbb{Q} \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module. By [9, Ex 2.5], M is rickart, yield M is coclosed rickart. Now consider $N = \mathbb{Z} \oplus \mathbb{Z}_2$ a submodule of M. See that N may not be a coclosed rickart since there is $f : N \rightarrow N$ defined by $f(m, \bar{n}) = (0, \bar{m})$ where $m, n \in \mathbb{Z}$. Thus Ker $f = \{(m, \bar{n}) \in \mathbb{Z} \oplus \mathbb{Z}_2 \mid f(m, \bar{n}) = (0, \bar{0})\} = \{(m, \bar{n}) \mid \bar{m} = \bar{0}\} = 2\mathbb{Z} \oplus \mathbb{Z}_2$ is not a coclosed submodule in $\mathbb{Z} \oplus \mathbb{Z}_2$, because $2\mathbb{Z}_4 \cong \mathbb{Z}_2 \cong \frac{2\mathbb{Z} \oplus \mathbb{Z}_2}{4\mathbb{Z} \oplus \mathbb{Z}_2} \ll \frac{\mathbb{Z} \oplus \mathbb{Z}_2}{4\mathbb{Z} \oplus \mathbb{Z}_2} \cong \mathbb{Z}_4$.

 https://doi.org/ 10.30526/2017.IHSCICONF.1818

(1) If each proper submodule of M is a coclosed rickart, implies M may not be coclosed rickart. as the next, $\mathbb{Z}_4$ as $\mathbb{Z}$-module in which every proper submodule is semisimple module and hence they are coclosed rickart modules. But $\mathbb{Z}_4$ is not coclosed rickart.

We record the next from [1].

Lemma Let N, H be submodules of an R-module M and $K \subset N$. If K is coclosed in M, implies K is coclosed in N and the reverse is hold if N is coclosed in M.

Proposition. Each direct summand of a coclosed rickart module is coclosed rickart.

Proof. Let M be a coclosed rickart module over R and A a direct summand of M, then $M = A \oplus B$ for some submodule B of M. Let $f \in \mathrm{End}_R(A)$, then we have the following $M \xrightarrow{\rho} A \xrightarrow{f} A \xrightarrow{i} M$. Say $g = i\,f\,\rho$, $g \in \mathrm{End}_R(M)$. Therefore $\mathrm{Ker}\, g$ is a coclosed in M. See $\mathrm{Ker}\, g = \mathrm{Ker}\, f \oplus B$. To show this, let $m \in \mathrm{Ker}\, f + B$, $m = x + y$ where $x \in \mathrm{Ker}\, f$ and $y \in B$. Hence $g(m) = g(x + y) = (i\,f\,\rho)(x + y) = i(f(x)) = f(x) = 0$. Therefore $m \in \mathrm{Ker}\, g$ implies that $\mathrm{Ker}\, f + B \subseteq \mathrm{Ker}\, g$. Conversely, let $m \in \mathrm{Ker}\, g \subseteq M = A \oplus B$. let $m = x + y$, $x \in A$ and $y \in B$, then $g(m) = g(x + y) = (i\,f\,\rho)(x + y) = 0$. Thus $i(f(x)) = 0$ and hence $f(x) = 0$. That is $x \in \mathrm{Ker}\, f$ and since $y \in B$. Thus $m = x + y \in \mathrm{Ker}\, f + B$, $\mathrm{Ker}\, g \subseteq \mathrm{Ker}\, f + B$. That is, $\mathrm{Ker}\, g = \mathrm{Ker}\, f + B$. Clearly $\mathrm{Ker}\, f \cap B = 0$. Therefore $\mathrm{Ker}\, g = \mathrm{Ker}\, f \oplus B$ and hence $\mathrm{Ker}\, f$ is a coclosed submodule in $\mathrm{Ker}\, g$. But $\mathrm{Ker}\, g$ is coclosed in M, then $\mathrm{Ker}\, f$ is coclosed in M. But A is containing $\mathrm{Ker}\, f$, so by lemma 2.5, $\mathrm{Ker}\, f$ is coclosed in A. Therefore A is a coclosed rickart R-module.

Remark The direct sum of coclosed rickart modules need not be coclosed rickart, as follows. $\mathbb{Z} \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is not coclosed rickart while each of $\mathbb{Z}$ and $\mathbb{Z}_2$ is a coclosed rickart module.

Recall that a submodule N of an R-module M is said to be fully invariant if $f(N)$ is included in N for each $f \in \mathrm{End}_R(M)$ [13]. We call M is duo module if every submodule in M is fully invariant [13].

Proposition Let $M = \bigoplus_{i \in \Lambda} M_i$ be a duo module over R. M is a coclosed rickart module if and only if M_i is a coclosed rickart for each $i \in \Lambda$.

Proof. First side is already from Proposition 2.6. The other side, assume $f = (f_{ij}) \in \mathrm{End}_R(M)$ where $f_{ij} \in \mathrm{Hom}_R(M_i, M_j)$. Since M_i is fully invariant in $M = \bigoplus_{i \in \Lambda} M_i$, implies $\mathrm{Hom}_R(M_i, M_j) = 0$ for each $i \neq j$ [13, Lemma 1.9]. Further $f(M_i) \subseteq M_i$ for all $i \in \Lambda$, this implies that $\mathrm{Ker} f = \bigoplus_{i\in\Lambda} \mathrm{Ker}\, f_{ii}$. We claim that $\mathrm{Ker} f$ is a coclosed submodule in M. To show this, let $K \subset \bigoplus_{i\in\Lambda} \mathrm{Ker}\, f_{ii}$ be a submodule of M, then K is a fully invariant of M. By [7, Result 2.1], $K = \bigoplus_{i\in\Lambda} (K \cap M_i)$, let $K_i = K \cap M_i$ for each $i \in \Lambda$.Easily to check $K_i \subseteq \mathrm{Ker}\, f_{ii}$. Since M_i is coclosed Rickart implies that $\mathrm{Ker}\, f_{ii}$ is a coclosed submodule of M_i for all $i \in \Lambda$. Thus there is a submodule N_i of M_i, $\mathrm{Ker}\, f_{ii} + N_i = M_i$ but $K_i + N_i \neq M_i$. This implies that $(\bigoplus_{i\in\Lambda} \mathrm{Ker}\, f_{ii}) + (\sum_{i\in\Lambda} N_i) = \bigoplus_{i\in\Lambda} M_i$ but $(\bigoplus_{i\in\Lambda} K_i) + (\sum_{i\in\Lambda} N_i) \neq \bigoplus_{i\in\Lambda} M_i$. Put $N = \sum_{i\in\Lambda} N_i$. So we have $\mathrm{Ker} f + N = M$ but $K + N \neq M$ and hence $\mathrm{Ker} f$ is a coclosed submodule of M. Therefore M is coclosed rickart.

Proposition 2.9. Let R be a ring. The next statements are equivalently

(1) $\oplus_\Lambda R$ is a coclosed rickart R-module for any index set Λ.

 https://doi.org/ 10.30526/2017.IHSCICONF.1818

(2) All projective R-modules are coclosed rickart modules.

(3) All free R-modules are coclosed rickart modules.

Proof. $(1) \Rightarrow (2)$ Assume M is projective over R, we get a free module F over R with an epimorphism $f : F \longrightarrow M$. We have the short exact sequence $0 \to \operatorname{Ker} f \xrightarrow{i} \oplus_{\Lambda} R \xrightarrow{f} M \to 0$ where $F \cong \oplus_{\Lambda} R$ for some index set Λ. But M is projective then the sequence splits. Thus $\oplus_{\Lambda} R \cong \operatorname{Ker} f \oplus M$. Because $\oplus_{\Lambda} R$ is a coclosed rickart module, therefore from Proposition 2.3, $\operatorname{Ker} f \oplus M$ is also coclosed rickart. Hence from Lemma 2.5, M is coclosed rickart.

It is clear $(2) \Rightarrow (1)$ and $(2) \Leftrightarrow (3)$. Similarly, we can prove $(1) \Leftrightarrow (3)$.

Proposition Let M be a coclosed simple coclosed rickart module which has a nonzero maximal submodule N, then $M \oplus \frac{M}{N}$ is not a coclosed rickart module.

Proof. Suppose $M \oplus \frac{M}{N}$ is coclosed rickart, $f \in \operatorname{End}(M \oplus \frac{M}{N})$ defined by $f(m, \bar{n}) = (0, \bar{m})$ for all $m \in M$, $\bar{n} \in \frac{M}{N}$. Then $\operatorname{Ker} f = \left\{(m, \bar{n}) \in M \oplus \frac{M}{N} \mid f(m, \bar{n}) = (0, \bar{0})\right\} = \left\{(m, \bar{n}) \in M \oplus \frac{M}{N} \,\middle|\, m + N = N\right\} = N \oplus \frac{M}{N}$ is a coclosed of $M \oplus \frac{M}{N}$. But N is a coclosed of $N \oplus \frac{M}{N}$, then by Lemma 2.5, N is a coclosed of $M \oplus \frac{M}{N}$. Again by result 2.5, N is a coclosed of M, this implies a false because M is coclosed simple. Yeild $M \oplus \frac{M}{N}$ is not coclosed rickart.

Coclosed Rickart Modules and Other Topics

This part looks for any relationship between coclosed rickart modules and other concepts. We first study the next condition (*) for an R-module :

For any submodule N of M for which $\frac{M}{N} \cong H$ where H is a summand of M, yeild N is a coclosed submodule of M.

Proposition. Every coclosed rickart module satisfies the condition (*).

Proof. When N is a submodule of a coclosed rickart M over R with $\frac{M}{N} \cong H$ where H is a summand in M. Hence $f : \frac{M}{N} \to H$ is an isomorphism, implies $M \xrightarrow{\pi} \frac{M}{N} \xrightarrow{f} H \xrightarrow{i} M$. Then $if\pi \in \operatorname{End}_R(M)$ $\operatorname{Ker}(if\pi) = \pi^{-1}(\operatorname{Ker}(if)) = \pi^{-1}(0) = N$. Because M is coclosed rickart, then $N = \operatorname{Ker}(if\pi)$ is coclosed in M.

Corollary When M is a coclosed rickart module over R with the condition (*) for each submodule N of M, therefore M is a cosemisimple.

Proof. Obvious by Proposition 3.1.

The next result is to find a certain situation under which the reverse of Proposition 3.1 is hold.

Proposition If M a module has the condition (*) with $\operatorname{Im} f$ isomorphic to a summand of M for all $f \in \operatorname{End}_R(M)$, implies M is coclosed rickart.

Proof. When $f \in \operatorname{End}_R(M)$, from assumption, $\operatorname{Im} f \cong H$ where H is a direct summand of M, implies that $\frac{M}{\operatorname{Ker} f} \cong H$. So by the condition (*), $\operatorname{Ker} f$ is coclosed of M, as asserted.

Let M be a module over R. We record that, M lifiting when each submodule N of M, there is a direct summand K of M such that $K \subseteq N$ and $\frac{N}{K} \ll \frac{M}{K}$ [1].

Proposition **Every** lifiting coclosed rickart module is rickart module.

Proof. When M be a lifiting coclosed rickart over R and $f \in \mathrm{End}_R(M)$. Because M is lifting,
there exists a summand K of M, $K \subseteq \mathrm{Ker}\, f$ with $\frac{\mathrm{Ker}\, f}{K} \ll \frac{M}{K}$. But M is coclosed rickart, implies $\mathrm{Ker}\, f$ is a coclosed in M, hence $\mathrm{Ker}\, f = K$, as desired.

Examples Investigate $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is lifiting as $\mathbb{Z}$ - module. But $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is not coclosed rickart as $\mathbb{Z}$ - module, if otherwise, then by Proposition 2.5, each summand of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is coclosed rickart, whereas this contradicts that $\mathbb{Z}_4 \oplus 0 \cong \mathbb{Z}_4$ is not coclosed rickart as $\mathbb{Z}$ - module.

(1) $\mathbb{Z}$ as $\mathbb{Z}$ -module is coclosed rickart while not lifiting.

A module M is called Hopfian if every epimorphism $f \in \mathrm{End}_R(M)$ is an isomorphism [1].

Proposition When M is a coclosed rickart coclosed simple module over R implies M is Hopfian.

Proof. When $0 \neq f \in \mathrm{End}_R(M)$ is an epimorphism. Because M is coclosed rickart, implies $\mathrm{Ker}\, f$ is a coclosed of M. While M is coclosed simple and $f \neq 0$, it follows that $\mathrm{Ker}\, f = 0$, and hence M is Hopfian.

Record that, let M be a module over R, we call it scalar when each $f \in \mathrm{End}_R(M)$, there is $a \in R$ with $f(m) = ma$ for each $m \in M$ [14].

Proposition When M is a scalar coclosed rickart module, yield for each $0 \neq f \in \mathrm{End}_R(M)$, there is $0 \neq a \in R$ with $r_M(a)$ is coclosed of M.

Proof. When $0 \neq f \in \mathrm{End}_R(M)$ and M a scalar R-module yield there is $a \in R$, $f(m) = ma$ for every $m \in M$. But M is coclosed Rickart, therefore $r_M(a)$ is a coclosed of M.

Corollary. If R is a commutative via unity coclosed rickart over R, it follows for any $0 \neq f \in \mathrm{End}_R(R)$, there is $0 \neq a \in R$ and $r_R(a)$ is a coclosed submodule of R.

Proof. Due to every commutative ring R is a scalar, then result is already obtained by Proposition 3.7.

Proposition Assume M is a scalar faithful over R, implies R is coclosed rickart if only if $\mathrm{End}_R(R)$ is coclosed rickart.

Proof. Because M is a scalar module over R, then by [14, Lemma 6.2], $\mathrm{End}_R(R) \cong \frac{R}{r_R(M)}$. But M is faithful implies that $R \cong \mathrm{End}_R(R)$, as required.

 https://doi.org/ 10.30526/2017.IHSCICONF.1818

Remarks and Examples

(1) The factor module of a coclosed rickart module may not be coclosed rickart. Study $\mathbb{Z}$ as $\mathbb{Z}$ -module is coclosed rickart, whereas $\frac{Z}{4Z} \cong Z_4$ is not a coclosed rickart.

(2) When M is a coclosed rickart module, yeild $\frac{M}{N}$ is coclosed rickart for each summand N of M.

(3) If N is a submodule of M. When N and $\frac{M}{N}$ are coclosed rickart modules implies M need not be coclosed rickart. Study $M = \mathbb{Z} \oplus \mathbb{Z}_2$ over $\mathbb{Z}$ and $N = < 0 > \oplus \mathbb{Z}_2$. Thus N and $\frac{M}{N} \cong \mathbb{Z}$ are coclosed rickart $\mathbb{Z}$-modules while M is not coclosed rickart.

Relatively Coclosed Rickart Modules

In this part, we study relatively coclosed rickart modules. Main results of this type of modules are investigated. The family of rings R for which each right R-module is relatively coclosed rickart is right cosemisimple rings. Our concern is: When do modules have the relatively coclosed rickart property.

Definition. Let M and N be R-modules. M is called relatively coclosed rickart to N if for every $f \in \mathrm{Hom}_R(M, N)$, Ker f is a coclosed submodule of M.

As special case, M is coclosed rickart if and only if M is relatively coclosed rickart to M.

Remarks with Examples

(1) Obviously every cosemisimple R-module M is relatively coclosed rickart to any R-module N.

(2) Let M and N be R-modules. If M is relatively coclosed rickart to N, then N need not be relatively coclosed rickart to M. For example, let $\mathbb{Z}_n$ and $\mathbb{Z}$ as $\mathbb{Z}$-modules. Then $\mathbb{Z}_n$ is relatively coclosed rickart to $\mathbb{Z}$ for each positive integer n greater than one, in fact $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) = 0$. On the other hand, $\mathbb{Z}$ is not relatively coclosed rickart to $\mathbb{Z}_n$, since there exists the natural homomorphism $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_n)$ defined by $f(m) = \overline{m}$ for each $\in \mathbb{Z}$, for some $n > 1$, Ker $f = n\mathbb{Z}$ is not coclosed in $\mathbb{Z}$.

(3) When M is a coclosed rickart module over R, implies M needs not be relatively coclosed rickart to an R-module N. For example, $\mathbb{Z}$ as $\mathbb{Z}$-module is coclosed rickart. Whereas $\mathbb{Z}$ is not relatively coclosed rickart to $\mathbb{Z}_n$ as $\mathbb{Z}$-module for any $n > 1$.

(4) If M is relatively coclosed rickart to an R-module N, then M may not be coclosed rickart. For example, consider the $\mathbb{Z}$-module $\mathbb{Z}_4$ is relatively coclosed rickart to the $\mathbb{Z}$-module $\mathbb{Z}_3$, because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_4, \mathbb{Z}_3) = 0$. But $\mathbb{Z}_4$ is not coclosed rickart.

(5) When M is a coclosed simple or quasi-Dedekind over R, yeild M need not be relatively coclosed rickart to a module N over R. Discuss $\mathbb{Z}$ over $\mathbb{Z}$ is coclosed simple and quasi-Dedekind while not relatively coclosed rickart to the $\mathbb{Z}$-module $\mathbb{Z}_n$, for each $n > 1$.

Theorem Let M and N be R-modules. The next statements are equivalent

(1) M is relatively coclosed rickart to N.

(2) Each submodule B of N and each summand A of M is relatively coclosed rickart to B.

(3) Each direct A of M and for each coclosed B of N and each $f \in \mathrm{Hom}_R(M, B)$, Ker $f|_A$ is a coclosed in A.

Proof. (1) ⇒(2) Assume M is relatively coclosed rickart to N. When A is a direct summand of M and B of N. Let $f \in \operatorname{Hom}_R(A, B)$. Study the next $M = A \oplus H \xrightarrow{\rho} A \xrightarrow{f} B \xrightarrow{i} N$ where H is a submodule of M. Say $g = i\, f\, \rho \in \operatorname{Hom}_R(M, N)$. This implies that Ker g is a coclosed in M. By similar steps of result 2.6, we gain Ker f is a coclosed submodule in A. Therefore A is relatively coclosed Rickart to B.

(2) ⇒ (3) Let B be a coclosed submodule in N and A a direct summand of M. Let $f \in \operatorname{Hom}_R(M, B)$, then $f|_A \in \operatorname{Hom}_R(A, B)$. Since A is relatively coclosed Rickart to B, implies $\operatorname{Ker} f|_A$ is a closed of A.

(3) ⇒ (1) by taking $A = M$ and $B = N$.

The next two lemmas are proved in [4].

Lemma A module M has the CCIP iff every coclosed submodule in M gets the CCIP .

Lemma When M be a module over R with the CCIP, yeild each decomposition $M = A \oplus B$ and for every $f \in \operatorname{Hom}_R(A, B)$, ker f is a coclosed submodule of M.

We give the following results

Proposition Under an isomorphism, CCIP is transformed .

Proof. When M_1 and M_2 are modules over R with M_1 has The CCIP and $f : M_1 \to M_2$ an isomorphism. Assume A, B are coclosed of M_2. To show $A \cap B$ is again coclosed of M_2. Then there is submodules C, D of M_1 and $f(C) = A$, $f(D) = B$ implies $C = f^{-1}(A)$, $D = f^{-1}(B)$. It is easy to check that $C = f^{-1}(A)$, $D = f^{-1}(B)$ are coclosed submodules of M_1. It follows that $C \cap D = f^{-1}(A) \cap f^{-1}(B)$ is a coclosed of M_1. It is not hard to see that $f(C \cap D)$ is a coclosed submodule of M_2. On the other hand, $f(C \cap D) = A \cap B$ implies the result is gained.

Theorem If M is a module over R with the CCIP and $A \oplus B$ is a coclosed of M. Yeild A is relatively coclosed rickart module to B.

Proof. Assume that M has the CCIP. Then by lemma 4.4, every coclosed submodule of M has the CCIP implies that $A \oplus B$ has the CCIP. By lemma 4.5, for every $f \in \operatorname{Hom}_R(A, B)$, Ker f is a coclosed submodule in $A \oplus B$. But Ker $f \subseteq A$, then by lemma 2.5, Ker f is a coclosed in A. Therefore A is relatively coclosed Rickart to B.

As immediate consequences we have

Corollary When M and N be modules over R. If $M \oplus N$ has the CCIP, yeild M is relatively coclosed rickart to N.

Corollary If M be a module over . If $M \oplus M$ gets CCIP, implies M is coclosed rickart module.

Remark The reverse of Corollary 4.8 need not be hold generally. Study $\mathbb{Z}_2$ as $\mathbb{Z}$ -module is cosemisimple. Then it is relatively coclosed rickart for each R-module N. Let $= \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}_2$ be modules over $\mathbb{Z}$. We show that M does not have the CCIP. Let $A = (1, \bar{0})\,\mathbb{Z}$, and $B = (1, \bar{1})\,\mathbb{Z}$ be the submodules generated by $(1, \bar{0})$ and $(1, \bar{1})$ respectively. Obviously A and B are summands of M. Then , B are coclosed in M. But $A \cap B = 2\mathbb{Z} \oplus < \bar{0} >$ is not a coclosed submodule in M, since $< \bar{2} > \oplus < \bar{0} > \cong \frac{2\mathbb{Z} \oplus <\bar{0}>}{4\mathbb{Z} \oplus <\bar{0}>} \ll \frac{\mathbb{Z} \oplus \mathbb{Z}\, 2}{4\mathbb{Z} \oplus <\bar{0}>} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

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Proposition. Let $\{ M_i \}_{i\in\Lambda}$ be a family modules over R and N an R-module where $\Lambda = \{1,2,\dots,n\}$. The next are equivalent

(1) If N has the CCIP, then N is relatively coclosed Rickart to $\oplus_{i=1}^{n} M_i$.

(2) N is relatively coclosed rickart to M_i for all $i = 1,2,\dots,n$.

Proof. (1) $\Rightarrow$ (2) Via Theorem 4.3.

(2) $\Rightarrow$ (1) When N is relatively coclosed rickart to M_i for all $i = 1,2,\dots,n$ and N has the CCIP. To show that N is relatively coclosed rickart to $\oplus_{i=1}^{n} M_i$. Let $f \in \mathrm{Hom}_R(N, \oplus_{i=1}^{n} M_i)$. Let us consider the following sequence $N \xrightarrow{f} \oplus_{i=1}^{n} M_i \xrightarrow{\rho_i} M_i$, it is evident that $\mathrm{Im} f = \sum_{i=1}^{n} \mathrm{Im}\, \rho_i f$ where ρ_i is the natural projection map of $\oplus_{i=1}^{n} M_i$ onto M_i for $i = 1,2,\dots,n$. Yield $f = (\rho_1 f, \rho_2 f, \dots, \rho_n f)$ and $\mathrm{Ker}\, f = \cap_{i=1}^{n} \mathrm{Ker}\,(\rho_i f)$. But $\rho_i f$ $\mathrm{Hom}_R(N, M_i)$ and N is relatively coclosed rickart to M_i implies that $\mathrm{Ker}\, \rho_i f$ is a coclosed submodule in N. Because N has the CCIP, therefore $\mathrm{Ker}\, f = \cap_{i=1}^{n} \mathrm{Ker}(\rho_i f)$ is a coclosed submodule in N, and hence N is relatively coclosed rickart to $\oplus_{i=1}^{n} M_i$.

As an immediate result

Corollary. When $\{ M_i \}_{i\in\Lambda}$ is a family modules over R where $\Lambda = \{1,2,\dots,n\}$. The next are equivalent

(1) If M_j has the CCIP for all $j = 1,2,\dots,n$, then M_j is relatively coclosed rickart to $\oplus_{i=1}^{n} M_i$.

(2) M_j is relatively coclosed rickart to M_i for all $i = 1,2,\dots,n$.

A characterization of cosemisimple rings via relatively coclosed rickart modules is given as follows.

Theorem. The next are equivalent

(1) R is a cosemisimple right R-module.

(2) All R-modules are cosemisimple.

(3) All R-modules are relatively coclosed rickart to any R-module.

(4) All R-modules have the CCIP.

(5) All injective over R have the CCIP.

(6) All injective over $R$are cosemisimple.

(7) All quasi injective over R have the CCIP.

(8) All quasi injective over R are cosemisimple.

Proof. (1) $\Leftrightarrow$ (2) follows by [1, Theorem 1.12]. (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (4) $\Rightarrow$ (5), (6) $\Rightarrow$ (5)and (8) $\Rightarrow$ (7) are obvious..

(3) $\Rightarrow$ (1) When I is an ideal of R. Because all R-modules are relatively coclosed rickart to any R-module. Then the R-module R is relatively coclosed rickart to the module $\frac{R}{I}$ as R-module. Because there exists the natural epimorphism $\pi : R \to \frac{R}{I}$, hece $\mathrm{Ker}\, \pi = I$ is a coclosed right ideal of R. This implies that R is cosemisimple right R-module.

(4) $\Rightarrow$ (2) When M is an R-module and N a submodule of M, implies $M \oplus \frac{M}{N}$ has CCIP. Let $f : M \to \frac{M}{N}$ be the natural epimorphisim. By Lemma 4.5, $\mathrm{Ker}\, f$ is a coclosed submodule in M. Hence M is coclosed rickart.

(5) $\Rightarrow$ (4) when M is a module over an R, then there is an injective R-module N and a monomorphisim $f: M \to N$ implies that M is isomorphic to $\mathrm{Im} f$. When f is an epimorphism, then the result follows. Suppose not, so there is an injective R-module K and

 https://doi.org/ 10.30526/2017.IHSCICONF.1818

and a monomorphisim $g:\frac{N}{\mathrm{Im}f} \to K$. Let $h: N \to \frac{N}{\mathrm{Im}f}$ be the natural epimorphism. Let us consider $gh: N \to K$. Because $N \oplus K$ is injective then $N \oplus K$ has the CCIP. By Lemma 4.4, ker gh is a coclosed submodule of N and due to g is a monomorphisim, implies Ker gh = Ker h = Imf is coclosed of N. Because N has the CCIP, then by Lemma 4.4, Imf has the CCIP. But $M \cong \mathrm{Im}f$ and due to Proposition 4.6, therefore M has the CCIP.

(5) ⇒ (6) When M is an injective with N a submodule in M with $f: M \to \frac{M}{N}$ is the natural epimorphism. When $\frac{M}{N}$ is injective, then $M \oplus \frac{M}{N}$ is injective with the CCIP. By Lemma 4.5, Ker $f = N$ is a coclosed of M. Suppose not, when $E(\frac{M}{N})$ is the injective hull of $(\frac{M}{N})$ with $i: \frac{M}{\mathrm{N}} \to E(\frac{M}{N})$ is the inclusion function. Consider $if: M \to E(\frac{M}{N})$. Since $M \oplus E(\frac{M}{N})$ is injective, implies $M \oplus E(\frac{M}{N})$ has the CCIP. But i is a monomorphisim, then Ker if = Ker $f = N$ is a coclosed in M. Therefore M is cosemisimple.

(7) ⇒ (8) By the same way of the statement (5) ⇒ (6).

Theorem Let M be a module over R. The next are equivalent

(1) M is coclosed rickart with CCIP.
(2) The right annihilator in M of each finitely generated left ideal $I = < f_1, f_2, \dots, f_n >$ of $\mathrm{End}_R(M)$ is a coclosed. submodule of M.

Proof. (1) ⇒ (2) Let I be a nonzero left ideal of $\mathrm{End}_R(M)$ where M a coclosed rickart module over R with a finite number of generators $\{f_1, f_2, \dots, f_n\}$. Because M is coclosed Rickart, then $r_M(f_i)$ is a coclosed of M for every $i = 1,2,\dots,n$. But M with CCIP, it follows that $\bigcap_{i=1}^{n} r_M(f_i) = r_M(I)$ is a coclosed of M.

(3) ⇒ (1) When $f \in \mathrm{End}_R(M)$, then $< f >$ is a left ideal of $\mathrm{End}_R(M)$ with one generator. By hypothesis, $r_M(f)$ is coclosed of M, yeild M is coclosed rickart. Moreover, $\bigcap_{i=1}^{n} r_M(f_i) = r_M(I)$ for any finitely generated left ideal $I = < f_1, f_2, \dots, f_n >$ of $\mathrm{End}_R(M)$. Hence $\bigcap_{i=1}^{n} r_M(f_i)$ is a coclosed of M, thus M has the CCIP.

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