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Normalization Bernstein Basis for Solving Fractional Fredholm-Integro Differential Equation

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Abstract

In this work, we employ a new normalization Bernstein basis for solving linear Freadholm of fractional integro-differential equations nonhomogeneous of the second type (LFFIDEs). We adopt Petrov-Galerkian method (PGM) to approximate solution of the (LFFIDEs) via normalization Bernstein basis that yields linear system. Some examples are given and their results are shown in tables and figures, the Petrov-Galerkian method (PGM) is very effective and convenient and overcome the difficulty of traditional methods. We solve this problem (LFFIDEs) by the assistance of Matlab10.

Key words: Petrov-Galerkian method; Fractional Derivative; Caputo; Sense.

1.Introduction:

Integro-differential equations are encountered in various fields of sciences. It plays an important role in many branches of linear and non-linear functional analysis and their applications are in the theory of science, engineering and social sciences. Many problems can be modeled by fractional integro-differential equations from various science and engineering applications. Finding the approximate or exact solutions of fractional integro-differential equations is an important task. Save in a limited number, there is difficulty in finding the analytical solutions of fractional integro-differential equations. Therefore, there have been attempts to develop new methods for obtaining analytical solutions which reasonably approximate the exact solutions.

However, several numbers of algorithms for solving linear Fredholm of fractional integro-differential equation nonhomogeneous of the second type (LFFIDEs) have been investigated. Z. Taheri, Sh. Javadi and E. Babolian [1] employed shifted Legendre spectral collocation method to solve stochastic integro-differential equations (SFIDEs). [2] presented Bernstein polynomials basis for solving (LFFIDEs). Asma A., Adem Kılıc and Bachok M. [3] employed, homotopy perturbation and the variational iteration to approximate integro-differential equation of fractional (arbitrary) order. Li Huanga, Xian-Fang Li, Yulin Zhaoa and Xiang-Yang [4] used Taylor series approach for approximately a class of. Peter linzt [5] used Nystrom's method to establish numerical procedure for the approximate solution of linear integro-differential equations.

In this wrok, we presented the approximate solution of the (LFFIDEs).

$$D^{\alpha}u(x) = f(x) + \int_a^b k(x,t)u(t) \qquad a \le x,t \le b \qquad \ldots(1.a)$$

With the following supplementary conditions:

$$u^{i}(0) = \delta_i \qquad n-1 < \alpha \le n, \quad n \in N \qquad \ldots (1.b)$$

where $D^{\alpha}u(x)$ indicates the α the Caputo fractional derivative of $u(x)$; $f(x)$, $K(x,t)$ are given functions, x and t are real variables varying in the interval [a, b], and $u(x)$ is the unknown function to be determined.

2. Basic Definition

Definition1: A real function f(t), t > 0, is said to be in the space C_{μ}, $\mu \in R$, if there exists a real number $p > \mu$,such that $f(t) = t^{p}h_1(t)$; where $f_1(t) \in (0, \infty)$, and it is said to be in space C^{n}_{μ} if and only if $f^{n} C_{\mu}$, $n \in N$.

Definition2: The Riemann-Liouvill fractional integral operator of order α for a function in C_{μ}, where $\mu \ge -1$, is defined as $J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$, $\alpha > 0$

$J^{\alpha}f(x) = f(x)$.

Definition3: Let $f \in C^{m}_{-1}\, 1, m \in N \cup \{0\}$. Then the Caputo fractional derivatives of f(x) is defined as:

$$D^{\alpha} f(x) = \begin{cases} J^{m-\alpha} f^{m}(x), & m-1 < \alpha \le m, m \in N \\ \frac{D^{m} f(x)}{Dx^{m}}, & \alpha = m \end{cases}$$

Hence, we have following properties

1. $J^{\alpha} J^{v} f = J^{\alpha+v} f, \quad \alpha, v > 0, f \in C_{\mu}, \mu > 0$
2. $J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$
3. $J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0^{+}) \frac{x^{k}}{k!}, x > 0, m-1 < \alpha \le m$
4. $J^{\alpha} D^{\alpha} f(x) = f(x), x > 0, m-1 < \alpha \le m$
5. $D^{\alpha} C = 0$, C is constant
6. $D^{\alpha} x^{B} = \begin{cases} 0 & \beta \in N_0, < [\alpha] \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} & \beta \in N_0, \beta \ge [\alpha] \end{cases}$ where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0,1,2, \dots\}$.

2.1 The Derivative for Orthonormal Brnstein Polynomials:

The Bernstein polynomials of nth degree are defined on the interval [0,1] as[6].

$$B_{i,n}(x) = \binom{n}{i} x^{i} (1-x)^{n-i}, \quad \binom{n}{i} = \frac{n!}{i!(n-i)!} \quad for\ i = 0,1,2, \dots, n$$

The representation of the orthonormal Bernstein Polynomials, denoted by $\boldsymbol{b_{i,n}(x)}$ here, was discovered by analyzing the resulting orthonormal polynomials after applying the Gram-Schmidt process on sets of Bernstein polynomials of degree $B_{i,n}(x)$. Then the following sets of orthonormal polynomials $b_{i,n}(x)$, $0 \le i \le n$. For $n = 6$, the four orthonormal Bernstein polynomials are given as:

$$b_{0,6}(x) = \sqrt{13}\,(1-x)^{6},$$

$$b_{1,6}(x) = \sqrt{44}\,[6x(1-x)^{5} - \frac{1}{2}(1-x)^{6}]$$

$$b_{2,6}(x) = 11\,[15x^{2}(1-x)^{4} - 6x(1-x)^{5} + \frac{3}{11}(1-x)^{6}]$$

$$b_{3,6}(x) = \sqrt{252}[20x^{2}(1-x)^{3} - \frac{45}{2}x^{2}(1-x)^{4} + 5x(1-x)^{5} - \frac{11}{66}(1-x)^{6}]$$

$$b_{4,6}(x) = \frac{42}{\sqrt{5}}[15x^{4}(1-x)^{2} - 40x^{3}(1-x)^{3} + \frac{180}{7}x^{2}(1-x)^{4} - \frac{30}{7}x(1-x)^{5} + \frac{5}{42}(1-x)^{6}]$$

$$b_{5,6}(x) = \frac{28}{\sqrt{3}}[6x^{5}(1-x) - \frac{75}{2}x^{4}(1-x)^{2} + 60x^{3}(1-x)^{3} - 30x^{2}(1-x)^{4} + \frac{30}{7}x(1-x)^{5} - \frac{3}{28}(1-x)^{6}]$$

$$b_{6,6}(x) = 7[x^{6} - 18x^{5}(1-x) + 75x^{4}(1-x)^{2} - 100x^{3}(1-x)^{3} + 45x^{2}(1-x)^{4} - 6x(1-x)^{5} + \frac{1}{7}(1-x)^{6}]$$

3. Analysis of The Petrov-Galerkin Method (PGM):

In this section we introduce the (PGM) for Eq. (2). For the proof of all results in this section, we can use the same manner used in [7], but for Eq. (2). Let X be a Banach space with the norm $\|.\|$ and let X^* denote its dual space. Assume $K : X \to X$ is a compact linear operator. We rewrite this eq.(1) in operator from as:

$$D_*^{\alpha} u - Ku = f, \qquad f \in X \qquad ...(2)$$

where $u \in X$ is the unknown to be determined. The Peterov-Galerkin method (PGM) used for the numerical solutions of eq.(2). The Petrov-Galerkin methods (PGM) interpolate between the Galerkin method and the collocation method. For this purpose for each positive integer n, we assume that $X_n \subset X$, $Y_n \subset X^*$, and X_n, Y_n are finite dimensional vector spaces with dim X_n = dim Y_n , then X_n , Y_n satisfy condition (H) : for each $x \in X$ and $y \in X^*$, there exists $x_n \in X_n$ and $y_n \in Y_n$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. when the peterov-Galerkin method(PGM) for Eq.(2) is a numerical method for finding $u_n \in X$ such that

$$\langle D_*^{\alpha} u_n - K u_n , y_n \rangle = \langle f , y_n \rangle \qquad \text{for all } y_n \in Y_n \qquad ...(3)$$

It is clear that the Petrov-Galerkin method(PGM) is closely related to a generalized best approximation from X_n to $x \in X$ with respect to Y_n,. Given $x \in X$, an element $P_n x \in X_n$ is called a generalized best approximation from $X_n$to x with respect to Y_n if it satisfies the equation

$$\langle x - P_n x, y_n \rangle = 0 \quad \text{for all } y_n \in Y_n \qquad ...(4)$$

Similarly, given $y \in X^*$, an element $p_n' y \in Y_n$ is called best approxima tion from Y_n *to* y with respect Y_n to y if it satisfies the equatio

$\langle x_n, y - p_n' y \rangle = 0$ for all $x_n \in X_n$.

Proposition:

For each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if $Y_n \cap X_n^{\perp} = \{0\}$ ...(5)

Under this condition, P_n is a projection; i.e., $P_n^2 = P_n$

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Assume that, for each n, there is a linear operator $\prod_n : X_n \to Y_n$ with $\prod_n X_n = Y_n$, and satisfying the condition

(H-1) $\|x_n\| \le C_1 \langle x_n , \prod_n x_n \rangle^{\frac{1}{2}}$ for all $x_n \in X_n$,

(H-2) $\|\prod_n x_n\| \le C_2 \|x_n\|$ for all $x_n \in X_n$,

Where C_1 and C_2 are positive constants independent of n. if a pair of sequence $\{X_n\}$ and $\{Y_n\}$ satisfy (H-1) and (H-2), we call $\{ X_n , Y_n \}$ a regular pair.

For each $x \in X$, let $Q_n x$ be a best approximation from X_n to x, that is, $Q_n x \in X_n$ satisfies the equation $\|x - Q_n x\| = \inf_{x_n \in X_n} \|x - x_n\|$.

If a regular Pair $\{X_n, Y_n\}$ satisfies dim X_n = dim Y_n and condition (H) , then the corresponding generalized projection P_n satisfies:

(1) for all $x \in X$, $\|P_n x - x\| \to 0$ as n$\to \infty$

(2) there is a constant $C > 0$ such that, $\|P_n\| < C$, n = 1,2,..

(3) for some constant $C > 0$ independent of n, $\|P_n x - x\| \leq C\|Q_n x - x\|$ where $Q_n x$ is the best approximation from X_n to x.

if $\{X_n, Y_n\}$ a regular pair is with a linear operator $\prod_n : X_n \to Y_n$ with $\prod_n X_n = Y_n$, then eq. (3) may be rewritten

$$\langle D_*^\alpha u_n - Ku_n , \prod_n x_n \rangle = \langle f , \prod_n x_n \rangle \quad \text{for all } x_n \in X_n \qquad ...(6)$$

using the projection P_n defined earlier, eq.(3) is equivalent to

$$D_*^\alpha u_n - P_n K u_n = P_n f \qquad ...(7)$$

eq.(7) can also be derived from the fact that P_nx = 0 for an $x \in X$ if and only if $\langle x , y_n \rangle = 0$ for all $y_n \in Y_n$. This equation indicates that the Petrov-Galerkin method is a projection method.

Now, assume $u_n \in X_n$ and $\{b_j\}_{j=1}^n$ is a basis for X_n (trial space) and $\{b_i^*\}_{i=1}^n$ (test space) is a basis for Y_n. Therefore the (PGM) on $[a, b]$ for Eq. (2) is:

$$\langle D_*^\alpha u_n - K\, u_n, b_i^* \rangle = \langle f, b_i^* \rangle, \qquad i= 1,.. ,n \qquad ...(8)$$

4. Application of (PGM) for solving (LFFIDEs) Via Normalization Bernstein Basis:

In this section, the Petrov-Galerkian method (PGM) with aid of normalization Bernstein polynomials of six degree are interval [0, 1], is applied to study the approximation solution of the linear Fredholm fractional integro-differential eq(1) as the form:

$$D_*^\alpha u(x) = f(x) + \int_a^b k(x,t)u(t)dt, \qquad u(0) = \beta, \ x \in [a, b]$$

Our approach being by taking the fractional integration to both sides of eq. (1) we get

$$u(x) = u(0) + I^\alpha f(x) + I^\alpha(\int_a^b k(x,t)u(t)dt\,) \qquad ...(9)$$

To approximate solution of eq.(1),we use the normalization polynomial basis on $[a, b]$ as:

$$u(x) = \sum_{i=0}^n a_i\, b_{i,n}(x) \qquad ...(10)$$

Where ($a_i, i = 0,1, \ldots, n$) are unknown constants to be determined substituting eq.(10) in to eq.(9),we get

$$\sum_{i=0}^n a_i\, b_{i,n}(x) = u(0) + I^\alpha\, f(x) + I^\alpha \left(\int_a^b k(x,t) \sum_{i=0}^n a_i\, b_{i,n}(t)dt\right) \ ...(11)$$

Hence

$$\sum_{i=0}^{n} a_i\, b_{i,n}(x) - I^{\alpha}\left(\int_a^b k(x,t)\sum_{i=0}^{n} a_i\, b_{i,n}(t)dt\right) = u(0) + I^{\alpha} f(x) \qquad \ldots(12)$$

In the next step, apply Petrov-Galerkin method (PGM) for eq.(1) is a numerical method for finding $u(x) = \sum_{i=0}^{n} a_i\, b_{i,n}(x) \in X_n$, such that $\boldsymbol{a_i}$ is unknown and must be determined from eq.(12).

From eq.(8) it is clear that the eq.(12) can be written as :

$$<\sum_{i=0}^{n} a_i\, b_{i,n}(x) - I^{\alpha}\left(\int_a^b k(x,t)\sum_{i=0}^{n} a_i\, b_{i,n}(t)dt\right), b^*_{j,n}> = < u(0) + I^{\alpha} f(x), b^*_{j,n}> \qquad \ldots(13)$$

Thus

$$\int_0^1\{\sum_{i=0}^{n} a_i\, b_{i,n}(x) - I^{\alpha}\left(\int_a^b k(x,t)\sum_{i=0}^{n} a_i\, b_{i,n}(t)dt\right)\}b^*_{j,n} = \int_0^1\{u(0) - I^{\alpha} f(x)dx\}\, b^*_{j,n} \qquad \ldots(14)$$

Then, Eq.(14) is equivalent to linear system can be formed as follows :

$$\left.\begin{array}{c} L(x,a_i) = \int_0^1[\sum_{i=0}^{n} a_i\, b_{i,n}(x) - I^{\alpha}\,(\int_a^b k(x,t)\sum_{i=0}^{n} a_i b_{i,n}\,(t)dt)] \\ m_j = \int_0^1[u(0) + I^{\alpha} f(x)]b^*_{j,n} \end{array}\right] \qquad \ldots(15)$$

we can represent the system eq.(15) as a matrix form:

$$LA=M \qquad \ldots(16)$$

where

$$L = \begin{bmatrix} \int_0^1 L(x,a_0)b^*_{0,n}\,dt & \cdots & \int_0^1 L(x,a_n)b^*_{0,n}dt \\ \vdots & \ddots & \vdots \\ \int_0^1 L(x,a_0)b^*_{n,n}\,dt & \cdots & \int_0^1 L(x,a_n)b^*_{n,n}\,dt \end{bmatrix},\quad A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix},$$

$$M = \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \end{bmatrix}$$

Then we are solving the system to calculate the value a_i

5. Numerical Examples:

Example 1: Consider the following linear Fredholm fractional integro-differential equation:

$$D_x^{\alpha} u(x) = 3x^2 - \tfrac{x}{4} + \int_0^1 u(t)dt \quad , u(0)= 0,\ 0 < \alpha \le 1$$

Where the exact solution u(x) = x^3

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Table (1): Represents a comparison between the exact solution and approximate solution with different value $\alpha = 1,0.5,0.25$

x	Exact solution	Approximate solution		
		$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.25$
0	0	0	0.000606	0.002462
0.1	0.001	0.001	0.015119	0.052247
0.2	0.008	0.008	0.05891	0.15574
0.3	0.027	0.027	0.13789	0.31079
0.4	0.064	0.064	0.25797	0.52016
0.5	0.125	0.125	0.42506	0.78662
0.6	0.216	0.216	0.64507	1.1129
0.7	0.343	0.343	0.9239	1.5018
0.8	0.512	0.512	1.2675	1.9562
0.9	0.729	0.729	1.6817	2.4786
1	1.0	1.0	2.1725	3.072

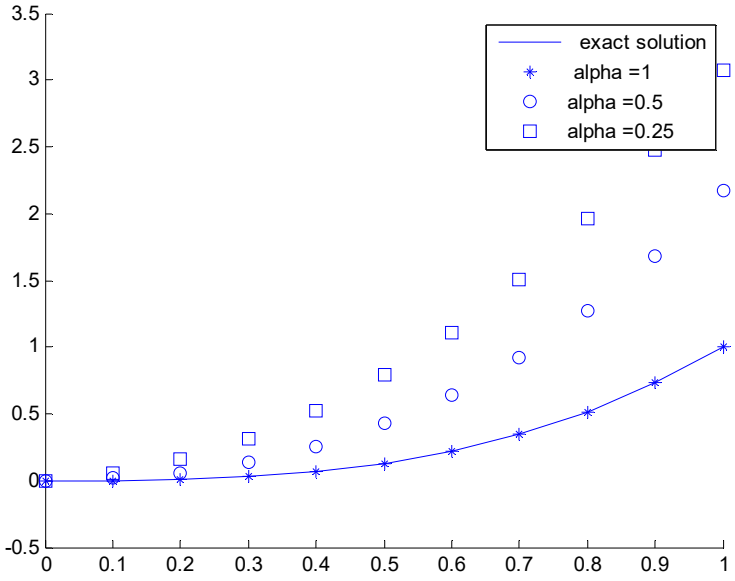

Figure (1): Comparison between the approximate solution and exact solution

Example 2: Consider the following linear Fredholm fractional integro-differential equation:

$D_x^{\alpha} y(x) = xe^x + e^x - x + \int_0^1 xy(t)dt$, y(0)=0, $0 < \alpha \leq 1$ The exact solution y(x)=xe^x

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Table (2): Represents a comparison between the exact solution and approximate solution with different value $\alpha = 0.25,0.5,1$

x	Exact solution	Approximate solution		
		$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.25$
0	0	0	0.14474	0.41705
0.1	0.11052	0.11044	0.43381	0.82922
0.2	0.24428	0.24397	0.73141	1.2522
0.3	0.40496	0.40424	1.0526	1.7024
0.4	0.59673	0.59537	1.4123	2.1962
0.5	0.82436	0.82151	1.8256	2.7500
0.6	1.0933	1.0868	2.3075	3.3801
0.7	1.4096	1.3953	2.8731	4.1030
0.8	1.7804	1.7512	3.5372	4.935
0.9	2.2136	2.1586	4.3151	5.8926
1	2.7183	2.6217	5.2216	6.9920

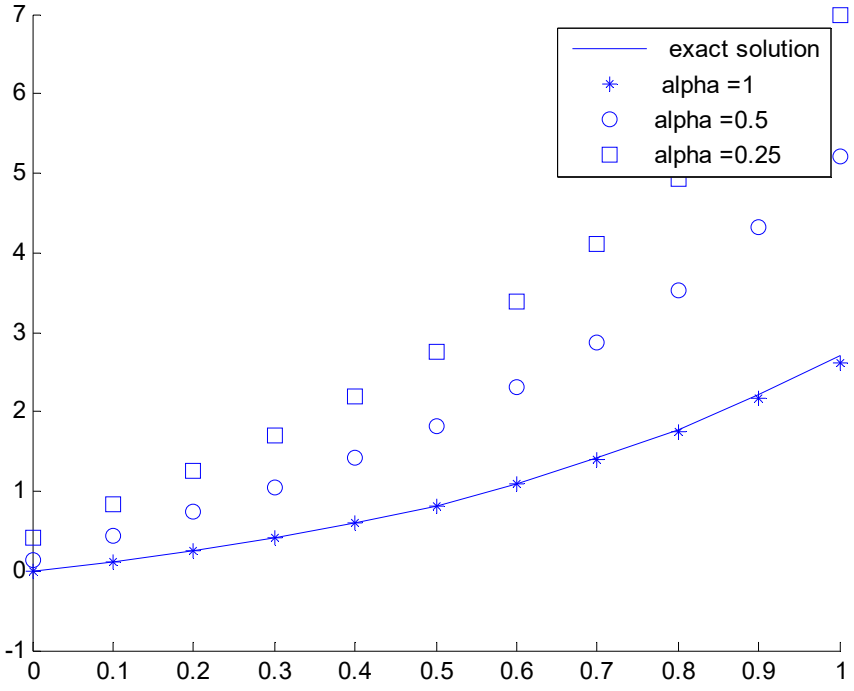

Figure (2): Comparison between the approximate solution and exact solution

Example 3:consider the following linear fredholm fractional integro-differential equation: $D_x^{\alpha} y(x) = \cos x + \cos 1 - 1 + \int_0^1 y(t)dt$, y(0)=0, $0 < \alpha \leq 1$ .The exact solution y(x)=sin x

Table(3): Represents a comparison between the exact solution and approximate solution with different value $\alpha = 1,0.5,0.25$

x	Exact solution	Approximate solution		
		$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.25$
0	0	0	0.56048	-4.5716
0.1	0.0998	0.0997	1.2655	-6.3967
0.2	0.19867	0.1984	1.7856	-7.6149
0.3	0.29552	0.29512	2.1787	-8.4416
0.4	0.38942	0.38889	2.5027	-9.0926
0.5	0.47943	0.47873	2.8152	-9.7833
0.6	0.56464	0.56369	3.1743	-10.729
0.7	0.64422	0.64279	3.6376	-12.146
0.8	0.71736	0.71507	4.263	-14.250
0.9	0.78333	0.77956	5.1083	-17.255
1	0.84147	0.8353	6.2313	-21.379

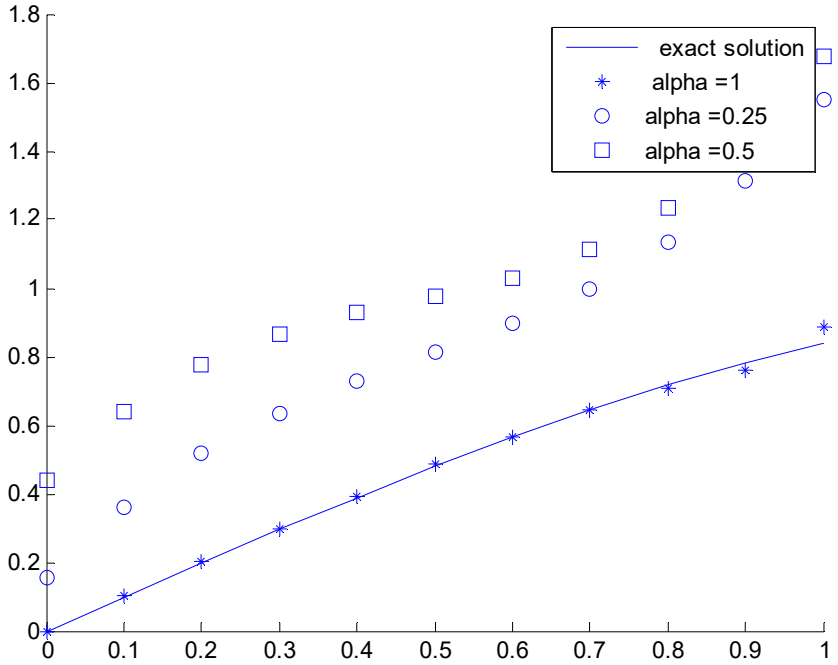

Figure (3): Comparison between the approximate solution and exact solution

6.Conclusions:

Integro-differential equations are usually difficult. It required to obtain the approximate solution. In this paper, Petrov-Galerkin method(PGM) has been successfully applied to find

the approximate solution of linear fractional Volterra integro-differential equation of the second type (LFFIDEs) via the normalization Bernstein basis. This method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear fractional Frdholm integro-differential equation of the second type (LFFIDEs), for the special case $\boldsymbol{\alpha = 1}$is shown in Figure 1, Figure 2 and Figure3. It can be seen from these figures that the solution obtained by the present method is identical with the exact solution. In our paper, we use the Matlab language to calculate the Petrov-Galerkin method via normalization Bernstein basis.

References

[1] Z. Taheri, Sh.Javadi and E.Babolian , Numerical solution of stochastic fractional integro-differential equation by the spectral collocation method, Journal of Computational and Applied Mathematics, 336-347. 2017

[2] M.D. Jani, Bhatta and S.Javadi, Numerical solution of fractional integro-differential equations with nonlocal conditions, applicaations and applid mathmatis, 98-111. 2017

[3] A. Asma, K. Adem and M. Bachok, Approximate solution of integro-differential equation of fractional (arbitrary) order, Journal of King Saud University –Science,.61-68. 2016

[4] H. Li ,F. Xian ,Z. Yulin and Y. Xiang, Approximate solution of fractional integro-differential equations by Taylor expansion method, computers and mathematics with applications,.1127-1134. ,2011

[5] L. Peter, A Method for the Approximate Solution of Linear Integro-Differential Equations,SIAM Journal on Numerical Analysis,.137-144. 2006

[6] M. Mayada, A New operational matrix of Derivative for orthonormal Bernstein polynomial's, Baghdad Science Journal,.1295-1300. 2014

[7] F. Shekarabi, M. Khodabin and K. Maleknejad, The Petrov-Galerkin Method for Numerical Solution of Stochastic Volterra Integral Equations, International Journal of Applied Mathematics,.170-176. 2014