Generalized Spline Approach for Solving System of Linear Fractional Volterra Integro-Differential Equations

Nabaa Najdi Hasan
Doaa Ahmed Hussien
Dept. of Mathematics/ College of Science/ University of Al-Mustansiriyah
doaaa92@yahoo.com
alzaer1972@uomustansiriyah.edu.iq
Received in: 20/November/2017, Accepted in:2/January/ 2018

Abstract

In this paper generalized spline method is used for solving linear system of fractional integro-differential equation approximately. The suggested method reduces the system to system of linear algebraic equations. Different orders of fractional derivative for test example is given in this paper to show the accuracy and applicability of the presented method.

Keywords: Linear system of fractional Volterra integro-differential equations, generalized spline functions.

Introduction

The concept of fractional or non-integer order derivative and integration can be traced back to the genesis of integer order calculus itself. Almost every mathematical theory applicable to the study of non-integer order calculus was developed through the end of 19^{th} century.
However, it is in the past hundred year that the most intriguing leaps in engineering and scientific application have been found. The calculation technique has to change in order to meet the requirement of physical reality in some cases [1].
The use of fractional differentiation for the mathematical modeling of real world physical problems has been wide spread in recent years,e.g. the modeling of earthquake , the fluid dynamic of viscoelastic material properties, etc, [2].
There are several approaches to the generalization of the notation of differentiation of fractional orders, e.g. Riemann-Liouville, Grunwald-Letnikov, Caputo and generalized functions approach [3].
In this paper, we present numerical solution of the system of integro-differential equation with fractional derivative.

$$D^{\alpha_i} y_i(t) = f_i(t) + \sum_{j=1}^{n} (a_{ij}(t) y_j(t) + \int_0^t K_{ij}(t,s) y_j(s) ds \quad , i = 1,2, \dots n \qquad \dots (1)$$

With initial conditions:

$$y_i^{(k)}(0) = b_{ik} \qquad , k = 0,1, \dots, \alpha - 1.$$

Generalized spline function:

Consider the linear differential operator, [4]:

$$L = k_n(w) D^n + k_{n-1}(w) D^{n-1} + \cdots + k_1(w) D + k_0(w).$$

Where $a_j(w) \in C^n[a,b]$, (class of all function which are n continuously differentiable defined on $[a,b], j = 0,1, \dots, n \; and \; a_n(w) \neq 0 \; on \; [a,b]$, $D = d/dw$ and associate with L its formula adjoint operator

$$L^* = (-1)^n D^n \{k_n(w)\} + (-1)^{n-1} D^{n-1} \{k_{n-1}(w)\} + \cdots + (-1) D \{k_1(w)\} + k_0(w)$$

Definition (1), [5]: Let $\Delta : a = w_0 < w_1 < \dots < w_N = b$, $N \in \mathbb{N}$ be a partition on $[a,b]$. A real function S, defined on $[a,b]$ is said to be generalized spline with partition Δ if :

$S \in \mathcal{K}^{2n}[w_{i-1}, w_i]$; $i=1,2, \dots, N$.

1- $L^* LS(x) = 0; \quad \forall\, w \in [w_{i-1}, w_i]$; $i=1,2, \dots, N$.
2- $S \in C^{2n-2}[a,b]$

where $\mathcal{K}^{2n}[w_{i-1}, w_i]$ class of all functions defined on $[w_{i-1}, w_i]$ has derivative of order 2n.

Definition (2), [5]: Let $S: [a,b] \to \mathbb{R}$ is an interpolating generalized spline function of f associated with the partition Δ and the operator L, if in addition to the conditions of definition (2.1), $S(w) = f(w)$ on Δ and $S^{(g)}(w_i) = f^{(g)}(w_i)$ for $i = 0, \dots$, N and $g = 1,2, \dots, n-1$.

Definition (3): The Caputo fractional derivative operator $D_\theta{}^{v}$ of order v is defined in the following from:

https://doi.org/10.30526/31.1.1842

$$D_\theta{}^v f(w) = \frac{1}{\Gamma(m_* - v)} \int_0^w \frac{f^{(m_*)}(t)}{(w-t)^{v-m_*+1}} dt \qquad , w > 0 \qquad \ldots(2)$$

Where $m_* - 1 < v < m_*$, $m_* \in N$.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operatin

$$D_\theta{}^v\big(\lambda f(w) + \mu g(w)\big) = \lambda D_\theta{}^v f(w) + \mu D_\theta{}^v g(w).$$

Where, λ and μ are constants, For the Caputo's derivative we have [6]:

$D_\theta{}^v c = 0$, c is constant.

$$D_\theta{}^v w^n = \begin{cases} 0 & for\ n \in N_0\ and\ n < \lceil v \rceil ; \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-v)} w^{n-v} & for\ n \in N_0\ and\ n \geq \lceil v \rceil \end{cases} \qquad \ldots(3)$$

We use the ceiling function $\lceil v \rceil$ to denote the smallest integer greater than or equal to v , and $N_0 = \{1,2, \ldots\}$. Recall that for $v \in N$, the Caputo differential operator coincides with the usual differential operator of linear order.

2Theorem,[7]: Let $\alpha \in R, n-1 < \alpha < n, n \in N, \lambda \in \mathbb{C}$. Then the Caputo fractional derivative of the exponential function has the form:

$$D_\theta{}^\alpha e^{\lambda w} = \sum_{\varkappa=0}^{\infty} \frac{\lambda^{\varkappa+n} w^{\varkappa+n-\alpha}}{\Gamma(\varkappa+1+n-\alpha)} = \lambda^n w^{n-\alpha} E_{1,n-\alpha+1}(\lambda w)$$

where$E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-leffler type.

Proof:

To prove the theorem, the relation between Caputo and Riemann-Liouville fractional derivative:

$$D_\theta{}^\alpha f(w) = D^\alpha f(w) - \sum_{\varkappa=0}^{n-1} \frac{w^{\varkappa-\alpha}}{\Gamma(\varkappa+1-\alpha)} f^{(\varkappa)}(0)$$

as well as the well-known Riemann-Liouville fractional derivative of the exponential function, namely,

$$D^\alpha e^{\lambda w} = w^{-\alpha} E_{1,1-\alpha}(\lambda w)$$

could be used. Then for the Caputo fractional derivative holds

$$D_\theta{}^\alpha e^{\lambda w} = D^\alpha e^{\lambda w} - \sum_{\varkappa=0}^{n-1} \frac{w^{\varkappa-\alpha}}{\Gamma(\varkappa+1-\alpha)} (e^{\lambda w})^{(\varkappa)}(0)$$

$$= w^{-\alpha} E_{1,1-\alpha}(\lambda w) - \sum_{\varkappa=0}^{n-1} \frac{w^{\varkappa-\alpha}}{\Gamma(\varkappa+1-\alpha)} \lambda^\varkappa$$

$$= \sum_{\varkappa=0}^{\infty} \frac{(\lambda w)^\varkappa w^{-\alpha}}{\Gamma(\varkappa+1-\alpha)} - \sum_{\varkappa=0}^{n-1} \frac{w^{\varkappa-\alpha}}{\Gamma(\varkappa+1-\alpha)} \lambda^\varkappa$$

$$= \sum_{\varkappa=n}^{\infty} \frac{\lambda^{\varkappa} w^{\varkappa-\alpha}}{\Gamma(\varkappa+1-\alpha)}$$

$$= \sum_{\varkappa=0}^{n-1} \frac{\lambda^{\varkappa+n} w^{\varkappa+n-\alpha}}{\Gamma(\varkappa+n+1-\alpha)}$$

$$= \lambda^n w^{n-\alpha} E_{1,n-\alpha+1}(\lambda w).$$

Solution of system of linear fractional integro-differential equation

In this paper, the generalized spline function is applied to study the approximate solution of system of fractional integro-differential of equations are given in equation (1)

Let

$$S_i(w) = \sum_{j=1}^{2n} c_{ij} q_j(w) \quad , i = 1, \ldots, n \qquad 0 \le w \le 1 \qquad \ldots(4)$$

Be the generalized spline function to approximate the solution of equation (1) Where q_j , $j = 1, \ldots, 2n$ be the basis function of generalized spline S_i and $2n$ is order of $L^* Lu = 0$ and c_{ij} be the coefficients, $i = 1, \ldots, n, n \in N$, $j = 1, \ldots, 2n$.

Now, substituting equation (4) in equation (1),

$$D^{\alpha} \sum_{j=1}^{2n} c_{ij} q_j(t) = f_i(t) + \sum_{j=1}^{n} a_{ij}(t) \left(\sum_{j=1}^{2n} c_{ij} q_j(t) + \int_0^t K_{ij}(t,s) \left(\sum_{j=1}^{2n} c_{ij} q_j\right)(s) ds\right. \qquad \ldots(5)$$

then

$$\sum_{j=1}^{2n} c_{ij} D^{\alpha} q_j(t) = f_i(t) + \sum_{j=1}^{n} a_{ij}(t) \left(\sum_{j=1}^{2n} c_{ij} q_j(t) + \int_0^t K_{ij}(t,s) \left(\sum_{j=1}^{2n} c_{ij} q_j(s)\right) ds\right. \qquad \ldots(6)$$

$$\sum_{j=1}^{2n} c_{ij} \left[D^{\alpha}\left(q_j(t)\right) - \sum_{j=1}^{n} c_{ij} \left(\sum_{j=1}^{2n} c_{ij} q_j(t) - \int_0^t K_{ij}(t,s)\, q_j(s)\right] ds = f_i(t)\right.$$

Let

$$\varpi_{ij}(t) = D^{\alpha}\left(q_j(t)\right) - \sum_{j=1}^{n} c_{ij} \left(\sum_{j=1}^{2n} c_{ij} q_j(t)\right) - \int_0^t K_{ij}(t,s)\, q_j(s) ds \, , j = 1, \ldots, 2n$$

Adding the initial conditions of equation (1) as a new raw in the following matrices:

$$\varpi = \begin{bmatrix} \varpi_1(w_0) & \varpi_2(w_0) & \cdots & \varpi_{2n}(w_0) \\ \vdots & \vdots & \vdots & \vdots \\ \varpi_1(w_N) & \varpi_2(w_N) & \cdots & \varpi_{2n}(w_N) \\ y_1{}'(0) & y_2{}'(0) & \cdots & y_{2n}{}'(0) \\ \vdots & \vdots & \vdots & \vdots \\ y_1{}^{\alpha-1}(0) & y_2{}^{\alpha-1}(0) & \cdots & y_{2n}{}^{\alpha-1}(0) \end{bmatrix}, c = \begin{bmatrix} c_{11} \\ \vdots \\ c_{12n} \\ c_{21} \\ \vdots \\ c_{22n} \\ c_{31} \\ \vdots \\ c_{32n} \\ \vdots \\ c_{n1} \\ \vdots \\ c_{n2n} \end{bmatrix}, F = \begin{bmatrix} f_1(w_0) \\ f_2(w_0) \\ \vdots \\ f_n(w_0) \\ \vdots \\ f_1(w_N) \\ f_2(w_N) \\ \vdots \\ f_n(w_N) \\ b_{10} \\ \vdots \\ b_{1\alpha-1} \\ b_{20} \\ \vdots \\ b_{2\alpha-1} \\ \vdots \\ b_{n0} \\ \vdots \\ b_{n\alpha-1} \end{bmatrix} \quad \ldots(7)$$

or in the system form:

$$\varpi c = F \quad \ldots(8)$$

ϖ and F are constant matrices with dimensions (N+α)×2n and (N+α)×1 respectively .

The system will construct has N+α equations and c_{ij} coefficients where $i = 1, \ldots, n$ and $j = 1, \ldots, 2n$.

By solving the above system, we obtain the values of the unknown coefficients and the approximate solution of equation (1).

To demonstrate the accuracy and applicability of the presented method illustrative example for solving linear system of fractional integro-differential equations with different values of fractional derivative is provided.

Illustrative Example: Consider the following system of integro-differential equations of fractional order,,

$$D^{0.25} y_1\ (z^*) = f_1(z^*) + \int_0^{z^*} \big(\sin y_1(\varsigma) + y_2(\varsigma)\big) d\varsigma \quad \ldots(9)$$

$$D^{0.5} y_2(z^*) = f_2(z^*) + \int_0^{z^*} (1 - \tfrac{3}{4}\big(y_2(\varsigma) + y_3(\varsigma)\big) d\varsigma, \quad \ldots(10)$$

$$D^{0.75} y_3(z^*) = f_3(z^*) - \int_0^{z^*} \big(y_1(\varsigma) + y_2(\varsigma) + y_3(\varsigma)\big) d\varsigma \quad \ldots(11)$$

The exact solution of this system is ,[8]:

$y_1(z^*) = z^*$, $\qquad y_2(z^*) = z^{*2}$, $\qquad y_3(z^*) = z^{*3}$.

and $f_1(z^*) = \frac{1}{\Gamma(1.75)} z^{*3/4} + z^* cosz^* - sinz^* - \frac{z^{*3}}{3}$,

$$f_2(z^*) = \frac{2}{\Gamma(2.5)} z^{*3/2} - \frac{z^{*3}}{3} + \frac{3}{16} z^{*5} \ ,$$

$$f_3(z^*) = \frac{6}{\Gamma(3.25)} z^{*9/4} + \frac{z^{*2}}{2} + \frac{z^{*3}}{3} + \frac{z^{*4}}{4} \, .$$

Let Δ be the partition, $\Delta = 0 < z^*_0 < z^*_1 < z^*_2 < z^*_3 < z^*_4 < z^*_5 = 1$.

where $h = 0.2$,then $z^*_0 = 0$, $z^*_1 = 0.2$, $z^*_2 = 0.4$, $z^*_3 = 0.6$, $z^*_4 = 0.8$, $z^*_2 = 1$.

Applying the generalized spline function to the fractional integro-differential equation (9), (10) and (11)

Let $LL^*u = D^4u - 13D^2u + 36u$, with basic functions:

$u_1(z^*) = e^{3z^*}$, $u_2(z^*) = e^{-3z^*}$, $u_3(z^*) = e^{2z^*}$, $u_4(z^*) = e^{-2z^*}$

By equation (4), let $y_1(z^*)$ will approximate by $S_1(z^*)$,where

$$S_1(z^*) = c_{11}e^{3z^*} + c_{12}e^{-3z^*} + c_{13}e^{2z^*} + c_{14}e^{-2z^*} \qquad \dots(12)$$

and $y_2(z^*)$ will approximate by $S_2(z^*)$, where

$$S_2(z^*) = c_{21}e^{3z^*} + c_{22}e^{-3z^*} + c_{23}e^{2z^*} + c_{24}e^{-2z^*} \qquad \dots(13)$$

and $y_3(z^*)$ will approximate by $S_3(z^*)$, where

$$S_3(z^*) = c_{31}e^{3z^*} + c_{32}e^{-3z^*} + c_{33}e^{2z^*} + c_{34}e^{-2z^*} \qquad \dots(14)$$

To find the unknown coefficients c_{ij} , $i = 1,2,3$ and $j = 1,2,3,4$ twelve algebraic equations are needed.

Substituting equation (12), (13) and (14) in the initial conditions $y_1(0) = 0, y_2(0) = 0 and y_3(0) = 0$,respectively ,yield:

$$c_{11} + c_{12} + c_{13} + c_{14} = 0 \qquad \dots(15)$$

$$c_{21} + c_{22} + c_{23} + c_{24} = 0 \qquad \dots(16)$$

$$c_{31} + c_{32} + c_{33} + c_{34} = 0 \qquad \dots(17)$$

Now, substituting equation (12), (13) and (14) in equation (9), (10) and (11) respectively, get:

$$D^{0.25}\left(c_{11}e^{3z^*} + c_{12}e^{-3z^*} + c_{13}e^{2z^*} + c_{14}e^{-2z^*}\right) - \int_0^{z^*} (sin(c_{11}e^{3\varsigma} + c_{12}e^{-3\varsigma} + c_{13}e^{2\varsigma} + c_{14}e^{-2\varsigma}) + (c_{21}e^{2\varsigma} + c_{22}e^{-2\varsigma} + c_{23}e^{2\varsigma} + c_{24}e^{-2\varsigma}))d\varsigma = f_1(z^*) \qquad \dots(18)$$

$$D^{0.5}\left(c_{21}e^{3z^*} + c_{22}e^{-3z^*} + c_{23}e^{2z^*} + c_{24}e^{-2z^*}\right) - \int_0^{z^*} (1 - \frac{3}{4}(c_{21}e^{3\varsigma} + c_{22}e^{-3\varsigma} + c_{23}e^{2\varsigma} + c_{24}e^{-2\varsigma} + c_{31}e^{3\varsigma} + c_{32}e^{-3\varsigma} + c_{33}e^{2\varsigma} + c_{34}e^{-2\varsigma})d\varsigma = f_2(z^*) \qquad \dots(19)$$

$$D^{0.75}\left(c_{31}e^{3z^*}+c_{32}e^{-3z^*}+c_{33}e^{2z^*}+c_{34}e^{-2z^*}\right)$$
$$+\int_0^{z^*}(c_{11}e^{3\varsigma}+c_{12}e^{-3\varsigma}+c_{13}e^{2\varsigma}+c_{14}e^{-2\varsigma}+c_{21}e^{3\varsigma}+c_{22}e^{-3\varsigma}+c_{23}e^{2\varsigma}+c_{24}e^{-2\varsigma}$$
$$+c_{31}e^{3\varsigma}+c_{32}e^{-3\varsigma}+c_{33}e^{2\varsigma}+c_{34}e^{-2\varsigma})d\varsigma$$
$$=f_3(z^*) \qquad \dots(20)$$

where

$f_1(z^*)=\frac{1}{\Gamma(1.75)}z^{*3/4}+z^*cosz^*-sinz^*-\frac{z^{*3}}{3}$ $f_2(z^*)=\frac{2}{\Gamma(2.5)}z^{*3/2}-\frac{z^{*3}}{3}+\frac{3}{16}z^{*5}$ $f_3(z^*)=\frac{6}{\Gamma(3.25)}z^{*9/4}+\frac{z^{*2}}{2}+\frac{z^{*3}}{3}+\frac{z^{*4}}{4}$.

Then by solving the system:

$$\varpi c=F \qquad \dots(21)$$

Where ϖ is constant matrix and:

$c=[c_{11}\ \ c_{12}\ \ c_{13}\ \ c_{14}\ \ c_{21}\ \ c_{22}\ \ c_{23}\ \ c_{24}\ \ c_{31}\ c_{32}\ \ c_{33}\ \ c_{34}\ \ ]^T$

$F=[u_1(0)\ u_2(0)\ u_3(0)\ f_1(z^*{}_0)\ f_2(z^*{}_0)\ f_3(z^*{}_0)\ \dots\ \ f_1(z^*{}_5)\ f_2(z^*{}_5)\ f_3(z^*{}_5)]^T$

Finally , Gauss elimination method may be used to solve system (21) to find $c_{11}=-0.067$, $c_{12}=-0.221$, $c_{13}=0.314$, $c_{14}=-0.023$, $c_{21}=-0.036$, $c_{22}=0.743$, $c_{23}=0.205$, $c_{24}=-0.913$, $c_{31}=-0.05$, $c_{32}=0.952$, $c_{33}=0.04$, $c_{34}=-1.031$

So the approximate solutions:

$$S_1(z^*)=-0.067e^{3z^*}-0.221e^{-3z^*}+0.314e^{2z^*}-0.023e^{-2z^*}$$

$$S_2(z^*)=-0.036e^{3z^*}+0.743e^{-3z^*}+0.205e^{2z^*}-0.913e^{-2z^*}$$

$$S_3(z^*)=-0.05e^{3z^*}+0.952e^{-3z^*}+0.04e^{2z^*}-1.013e^{-2z^*}.$$

Table (1), presents a comparison between the exact and numerical solution and gives the least square error

Table (1): Numerical results of the illustrative example above:

w	$\lvert s_1 - y_1 \rvert$	$\lvert s_2 - y_2 \rvert$	$\lvert s_3 - y_3 \rvert$
0	3×10^{-3}	9×10^{-3}	1×10^{-3}
0.1	0.011	8.218×10^{-3}	0.037
0.2	9.646×10^{-3}	0.014	0.044
0.3	4.877×10^{-3}	0.021	0.035
0.4	5.265×10^{-4}	0.023	0.019
0.5	4.506×10^{-3}	0.021	3.865×10^{-3}
0.6	6.268×10^{-3}	0.011	5.624×10^{-3}
0.7	6.535×10^{-3}	5.162×10^{-3}	8.192×10^{-3}
0.8	7.993×10^{-3}	0.028	5.257×10^{-3}
0.9	0.016	0.058	1.732×10^{-3}
1	0.04	0.094	6.85×10^{-3}
LSE	25×10^{-8}	2.665×10^{-5}	1×10^{-6}

To show the implementation of the method figure (1) is given the approximate solution of the system of fractional integro-differential equation.

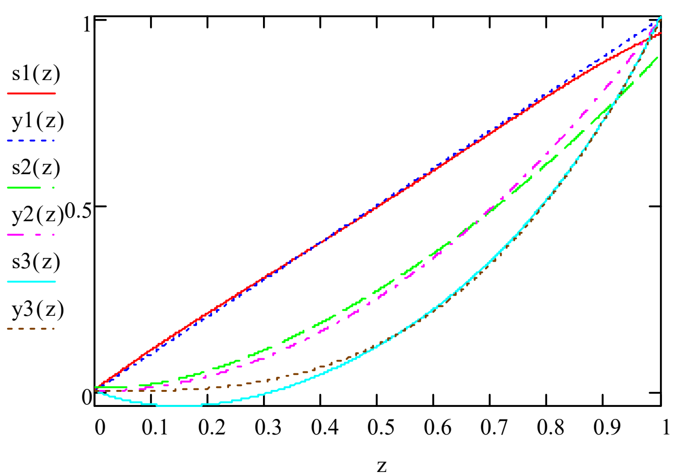

Figure (1): Exact and approximate solution of the illustrative example

Conclusions

In this paper, the application of generalized spline functions investigated to obtain approximate solution of system of linear fractional integro-differential equations and we give illustrative example with different α to show this approximation. As a comparison with the exact solution, table (1) and figure (1) showed the result.

References

[1]. JD. Munkhammar,"Fractional calculus and the Taylor-Riemann series", Undergraduate math Journal ,2005.

[2]. A. Arikoglu, and I. Ozko,"Solution of fractional Integro-differential equation by using fractional differential transform method ",Choas , solution and Fractals ,.5521-529, 2009.

[3]. I. Podlobny ,"Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications", Network; Academic press; 1999.

[4]. J.H. Ahlberge ; E.N. Nilson, and J.L. walsh ,"The theory of splines and their application ", Academic press , New york ,1967.

[5].Fadhel,S.Fadhel;Suha.N.Al-Rawi,and Nabaa N.Hassan ,"Generalized Spline Approximation Method for Solving Ordinary and partial Differential Equations", Eng.&Tech .journal ,2010.

[6]. I. podlubuy ,"Fractional Differential Equation ",Academic press, san Diago, .24-32, 1999.

[7].Mariya kamenova Ishtera ,"properties and applications of the Caputo fractional operator",Thesis Department of Mathmetics Unversity Karlsruhe(TH), 2005.

[8].M.Asgari, "Numerical Solution for solving a system of Fractional Integro-differential Equations", IAENG International Journal of Applied Mathematics , 2015.