

## On Contractible J-Saces

**Narjis A. Dawood**

[narjisabduljabbar@yahoo.com](mailto:narjisabduljabbar@yahoo.com)

Dept. of Mathematics / College of Education for Pure Science/ Ibn Al –  
Haitham- University of Baghdad

**Suaad G. Gasim**

[Suaad.gedaan@yahoo.com](mailto:Suaad.gedaan@yahoo.com)

Dept. of Mathematics / College of Education for Pure Science/ Ibn Al –  
Haitham- University of Baghdad

### Abstract

Jordan curve theorem is one of the classical theorems of mathematics, it states the following : If  $C$  is a graph of a simple closed curve in the complex plane the complement of  $C$  is the union of two regions,  $C$  being the common boundary of the two regions. One of the region is bounded and the other is unbounded. We introduced in this paper one of Jordan's theorem generalizations. A new type of space is discussed with some properties and new examples. This new space called Contractible J-space.

**Key words :** Contractible J- space, compact space and contractible map.

For more information about the Conference please visit the websites:

<http://www.ihsciconf.org/conf/>  
[www.ihsciconf.org](http://www.ihsciconf.org)

## 1. Introduction

Recall the Jordan curve theorem which states that, if  $C$  is a simple closed curve in the plane  $\mathbb{R}^2$ , then  $\mathbb{R}^2 \setminus C$  is disconnected and consists of two components with  $C$  as their common boundary, exactly one of these components is bounded (see, [1]).

Many generalizations of Jordan curve theorem are discussed by many researchers, for example not limited, we recall some of these generalizations. In 1967, Kopperman, Khalimsky and Meyer stated a generalization in  $\mathbb{Z}^2$  equipped with the Khalimsky topology, (see[2]). In 1991, Kong, Kopperman and Meyer introduced the following result: If  $\Gamma$  is an  $n$ -connected closed curve in  $\mathbb{Z}^2$ , then  $\mathbb{Z}^2 \setminus \Gamma$  has two and only two  $\bar{n}$ -connectivity components ( $n + \bar{n} = 12, n = 4, 8$ ). This result is a kind of generalization of the classical Jordan curve theorem in  $\mathbb{R}^2$ , (see [3]). In 1999, E.Micael introduced and studied  $J$ -spaces and strong  $J$ -spaces which are considered to be generalizations of properties of Jordan curve theorem, (see [4]).

In 2007, Y.Nanjing introduced the concept of  $LJ$ -spaces exploited the common generalization of Lindelöf spaces and  $J$ -spaces, (see [5]). In 2007, A.Kornitowicz worked hard to mark crucial points in the proof of Jordan curve theorem, (see[6]). In 2008, E.Bouassida introduced a new proof of the Khalimsky's Jordan curve theorem using the specificity of the Khalimsky's plane as an Alexandroff topological space and the specific properties of connectivity on these spaces, (see [7]). In this paper we introduced another generalization of Jordan curve theorem by using the concept of contractible space.

Suitability with our work, we assumed all functions are continuous and all spaces are  $T_2$ , in spite of most of our results are useful wanting that presumption.

## 2. Preliminaries

In this section, we give some important definitions and properties that we need in our work.

**Definition (2.1)** [1]: Two continuous maps  $f_0, f_1: X \rightarrow Y$  are said to be homotopic if there is a continuous map  $F: X \times I \rightarrow Y$  ( $I$  is the closed interval  $[0,1]$ ), such that  $F(x, 0) = f_0(x)$  and  $F(x, 1) = f_1(x)$ . This homotopic denoted by  $f_0 \cong f_1$ .

**Definition (2.2)** [1]: Two spaces  $X$  and  $Y$  are of the same homotopic type if there exist continuous maps  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  such that  $gf \cong I: X \rightarrow X$  and  $fg \cong I: Y \rightarrow Y$ . The maps  $f$  and  $g$  are then called homotopy equivalences, we also say that  $X$  and  $Y$  are homotopy equivalent.

**Definition (2.3)** [8]: If  $Y$  is a subspace of a topological space  $X$ , a retraction from  $X$  to  $Y$  is a continuous mapping  $r: X \rightarrow Y$  such that  $r(p) = p, \forall p \in Y$ . In this case  $Y$  is called a retract of  $X$ .

**Definition (2.4)** [8]: A subspace  $Y$  of a space  $X$  is called a deformation retract if there is a continuous retract  $r: X \rightarrow Y$  such that the identity map from  $X$  to  $X$  homotopic to the map  $i \circ r$ , where  $i$  is the inclusion of  $Y$  in  $X$ .

**Definition (2.5)** [9]: Let  $X$  be a topological space and  $A$  the subset of  $X \times [0,1]$  given by  $X \times \{1\}$ . By the cone over  $X$ , mean the space  $X \times [0,1]/A$  and denoted by  $TX$ .

**Definition (2.6)** [10]: A space  $X$  is path connected if, for every  $a, b \in X$ , there exists a path in  $X$  from  $a$  to  $b$ .

**Definition (2.7)** [10]: A space  $X$  is simply connected if it is path connected and  $\pi_1(X, x_0) = \{e\}, \forall x_0 \in X$ , where  $\pi_1(X, x_0)$  is the fundamental group of a space  $X$  at the basepoint  $x_0$ .

For more information about the Conference please visit the websites:

<http://www.ihsciconf.org/conf/>  
[www.ihsciconf.org](http://www.ihsciconf.org)

**Examples (2.8)** [11]:

1. The unite sphere  $S^{n-1}$  in  $\mathbb{R}^n$  is path connected  $\forall n > 1$ .
2. The unite ball  $B^n$  in  $\mathbb{R}^n$  is path connected.
3. Every open ball and every closed ball in  $\mathbb{R}^n$  is path connected.

**Definition (2.9)** [12]: A function from  $X$  to  $Y$  is said to be null- homotopic if it is homotopic to some constant function.

**Definition (2.10)** [11]: A space  $X$  is called contractible space if the identity function  $i_X: X \rightarrow X$  is null- homotopic.

**Examples (2.11):**

1. The Euclidean space  $\mathbb{R}^n$  is contractible [13].
2. A discrete space with more than one point is not contractible [14].

In the following we give some results about trivial spaces:

**Remarks (2.12):**

1. Any subspace (with more than one element) of a discrete space is not contractible since every subspace of a discrete space is also discrete.
2. A subset  $Y$  of  $\mathbb{R}$  is contractible if  $Y$  is not discrete space (with more than one element).
3. An indiscrete space is a contractible space; this follows from the fact says that any function with indiscrete codomain is continuous.
4. Any subspace of an indiscrete space is contractible since every subspace of an indiscrete space is also indiscrete.

**Definition (2.13)** [15]: A subset  $Y$  of  $\mathbb{R}^n$  is said to be convex if for every pair of points in  $X$ , the line segment connecting the points is also in  $X$ .

**Propositions (2.14)** [15]:

1. Every convex subset of  $\mathbb{R}^n$  is contractible.
2. Any open ball in  $\mathbb{R}^n$  is contractible.

**Theorem (2.15):**

The following conditions are equivalent for any space  $X$

1.  $X$  is a contractible space.
2.  $X$  is a homotopy equivalent to a point [16].
3. There exists a point  $x_0 \in X$  such that  $\{x_0\}$  is a deformation retract of  $X$  [17].
4.  $X$  is a retract of any cone over it [16].
5. Every map  $f: X \rightarrow Y$ , for arbitrary  $Y$ , is null-homotopic [18].
6. Every map  $f: Y \rightarrow X$ , for arbitrary  $Y$ , is null-homotopic [18].

**Proposition (2.16)** [10]: Every contractible space is path connected space.

**Proposition (2.17)** [10]: Every contractible space is simply connected space.

**Remark (2.18)** [19]: The convers of Propositions (14.1) and (15.1) is not true in general. For example,  $S^n$  is path connected for every integer  $n \geq 1$ , and simply connected for every integer  $n \geq 2$ . Yet these spheres are not contractible.

**Remark (2.19)** [1]: The continuous image of a contractible space need not be contractible. For example:

$f: [a, b] \rightarrow S^1$  is continuous and onto, since  $S^1$  is a quotient space for  $[a, b]$  by the relation  $x \sim y$  if  $x = a$  and  $y = b$ . Note that  $[a, b]$  is contractible, but  $S^1$  is not.

**Theorem (2.20)** [20]: If  $X$  is a contractible space, then  $\pi_1(X, x_0) = \{e\}$  for all  $x_0 \in X$ .

For more information about the Conference please visit the websites:

<http://www.ihsciconf.org/conf/>  
[www.ihsciconf.org](http://www.ihsciconf.org)

**Proposition (2.21)** [18]: A retract of a contractible space is contractible.

**Proposition (2.22)** [10]: If  $X$  is a contractible and  $Y$  is path connected, then any two continuous maps from  $X$  onto  $Y$  are homotopic (and each is null-homotopic).

**Proposition (2.23)** [16]: Two homeomorphic spaces are homotopy equivalent. Thus the classification of spaces up to homotopy equivalence is coarser than the homeomorphism classification.

**Remarks (2.24)** [21]:

1. Homotopy relation on the collection of all topological spaces is an equivalence relation.
2. Homotopy relation is an equivalence relation on the collection of all maps from  $X$  to  $Y$ .

**Definition (2.25)** [22]: A closed continuous function with compact preimages of points is called perfect.

**Theorem (2.26)** [22]: If a function  $f: X \rightarrow Y$  is a perfect function, then for any compact subset  $F$  of  $Y$ , the preimage  $f^{-1}(F)$  is a compact subset of  $X$ .

### 3. Contractible J- Space

**Definition (3.1)**: By a contractible J- space we mean the space satisfies the property which provides; for every proper closed subsets  $E, F$  of  $X$  with  $E \cup F = X$  and  $E \cap F$  compact, either  $E$  or  $F$  is contractible.

**Remark (3.2)**: If the closed cover in definition (3.1) is not proper, then every contractible J- space must be contractible.

**Remark (3.3)**: If a topological space  $X$  has no proper closed cover, then  $X$  is a contractible J- space. It follows from this fact that every indiscrete space is a contractible J- space.

**Remark (3.4)**: A space  $X$  is a contractible J- space if, whenever every subspace of it is contractible.

**Example (3.5)**: The usual space  $\mathbb{R}$  is a contractible J- space, for if  $\{E, F\}$  is a closed cover of  $\mathbb{R}$  with  $E \cap F$  compact, then  $E$  and  $F$  can not be both discrete, thus  $E$  or  $F$  is contractible ( see Remark (2.12), no2).

**Example (3.6)**: A discrete space with more than two points is not contractible J- space, follows from Example (2.11), no2.

**Remark (3.7)**: A subspace  $Y$  of  $\mathbb{R}$  is a contractible J- space if it is not discrete space (with more than two elements) follows from (Remark (2.12), no2).

**Example (3.8)**: Let  $X$  be a non empty set and  $A$  is a proper subset of  $X$ . Define a topology on  $X$  by  $\tau = \{X, \emptyset, A\}$ , then  $(X, \tau)$  is a contractible J- space since  $X$  has no proper closed cover.

**Remark (3.9)**: If  $X$  is a contractible space, then it is not contractible J- space in general.

**For instance**: Let  $X$  be a subspace of Euclidian space  $\mathbb{R}^2$  such that  $X = \{(x, y) \in \mathbb{R}^2, (x - 1)^2 + y^2 \leq 1\} \cup \{(x, y) \in \mathbb{R}^2, (x + 1)^2 + y^2 \leq 1\}$ . Let  $E, F$  be two subsets of  $X$  such that

$$E = \{(x, y) \in \mathbb{R}^2, (x - 1)^2 + y^2 \leq 1\} \cup \{(x, y) \in \mathbb{R}^2, (x + 1)^2 + y^2 = 1\},$$

$F = \{(x, y) \in \mathbb{R}^2, (x - 1)^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2, (x + 1)^2 + y^2 \leq 1\}$  then  $E$  and  $F$  are closed subsets of  $X$  with  $A \cup B = X$  and  $E \cap F = \{(x, y) \in \mathbb{R}^2, (x - 1)^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2, (x + 1)^2 + y^2 = 1\}$  which is compact subset of  $X$ , but neither  $E$  nor  $F$  is contractible. Hence  $X$  is not contractible J- space, but  $X$  is closed ball in  $\mathbb{R}^2$ .

For more information about the Conference please visit the websites:

<http://www.ihsciconf.org/conf/>

[www.ihsciconf.org](http://www.ihsciconf.org)

**Remark (3.10):** A contractible J- space need not be contractible.

**For example:** The unite circle  $S^1$  as a subspace of  $\mathbb{R}^2$  is not contractible, but it is contractible J- space since every proper subset of  $S^1$  is contractible.

**Theorem (3.11):** For any space X, these following conditions are valent:

1. X is a contractible J- space.
2. For every proper closed cover  $\{E, F\}$  with  $E \cap F$  compact E or F is homotopy equivalent to a point.
3. For every proper closed cover  $\{E, F\}$  with  $E \cap F$  compact, there exists  $x_0 \in E$  ( or  $x_0 \in F$ ) such that  $\{x_0\}$  is a deformation retract of E (or of F).
4. For every proper closed cover  $\{E, F\}$  with  $E \cap F$  compact, E or F is a retract of any cone over it.
5. For every proper closed cover  $\{E, F\}$  with  $E \cap F$  compact, every map f from E (or F) to an arbitrary space Y, is null- homotopic.
6. For every proper closed cover  $\{E, F\}$  with  $E \cap F$  compact, every map f from an arbitrary space Y to E (or F) is null- homotopic.

**Proof:** Follows from theorem (2.15) and definition (3.1).

**Proposition (3.12):** If X is a contractible J- space, then for every proper closed cover  $\{E, F\}$  with  $E \cap F$  compact, E or F is path connected.

**Proof:** Follows from proposition (2.16) and definition (3.1).

**Remark (3.13):** If for every proper closed cover  $\{E, F\}$  of a space X with  $E \cap F$  compact, E or F is path connected, then X need not be contractible J- space.

**For example:** Let us take the example of remark (3.9), as we saw in this example X is not contractible J- space, but for every proper closed cover  $\{E, F\}$  of X with  $E \cap F$  compact, E or F is path connected.

**Proposition (3.14):** If X is a contractible J- space, then for every proper closed cover  $\{E, F\}$  with  $E \cap F$  compact, E or F is simply connected.

**Proof:** Follows from proposition (2.17) and definition (3.1).

**Remark (3.15):** The opposite direction of proposition (3.14) is not true in general.

**For example:** Let X be a subspace of  $\mathbb{R}^3$  such that  $X = E \cup F$ , where  $E = \{(x, y, z) \in \mathbb{R}^3, (x - 1)^2 + y^2 + z^2 = 1\}$  and  $F = \{(x, y, z) \in \mathbb{R}^3, (x - 3)^2 + y^2 + z^2 = 1\}$ , then  $\{E, F\}$  is a closed cover of X with  $E \cap F = \{(2, 0, 0)\}$  which is compact, but neither E nor F is contractible. Hence X is not contractible J- space, but E and F are simply connected since both of them are homotopic equivalent to  $S^2$ .

**Remark (3.16):** The property of being contractible J- space is not a weak hereditary property, and thus not hereditary property.

**For example:** The usual space  $\mathbb{R}$  is a contractible J- space, but the natural numbers  $\mathbb{N}$  as a subspace of  $\mathbb{R}$  is not contractible J- space since the induced topology of the usual topology with respect to  $\mathbb{N}$  is the discrete topology.

**Proposition (3.17):** If A is a subset of a contractible J- space with compact boundary, then  $cl(A)$  or  $cl(X \setminus A)$  is contractible.

**Proof:** Consider the closed cover  $\{cl(A), cl(X \setminus A)\}$  of X, such that  $cl(A) \cap cl(X \setminus A) = \partial A$  which is compact, it follows by definition of contractible J- space that  $cl(A)$  or  $cl(X \setminus A)$  is contractible.

**Remark (3.18):** If X and Y are two contractible J- spaces, then  $X \times Y$  need not be so.

**For example:** Let  $X = \{1, 2\}$  and  $\tau = D =$  the discrete topology, then X is contractible J- space since  $\{\{1\}, \{2\}\}$  is the only proper closed cover of X with  $\{1\} \cap \{2\} = \emptyset$  which is

For more information about the Conference please visit the websites:

<http://www.ihsciconf.org/conf/>  
[www.ihsciconf.org](http://www.ihsciconf.org)

compact and  $\{1\}$  and  $\{2\}$  are contractible. But  $X \times X = \{(1,1), (1,2), (2,1), (2,2)\}$ , is not contractible J- space since it has more than two elements and by example (3.6).

**Definition (3.19):** A continuous function  $f: X \rightarrow Y$  is said to be contractible if  $f(A)$  is a contractible subspace of  $Y$  for each contractible subspace  $A$  of  $X$ .

**Remarks (3.20):**

- The identity function on any topological space is a contractible function.
- Any constant function is a contractible function.
- Any function defined from any topological space to an indiscrete space is contractible function.

**Example (3.21):** A function  $f: (\mathbb{R}, \tau) \rightarrow \mathbb{R}$  such that  $f(x) = x, \forall x \in \mathbb{R}$ , is not contractible function since  $\mathbb{N}$  is a contractible subspace of  $\mathbb{R}$  with the indiscrete topology, but  $f(\mathbb{N}) = \mathbb{N}$  is not contractible subset of  $\mathbb{R}$  with the usual topology.

**Remark (3.22):** A continuous function need not be contractible function.

**For example:** Let  $f: [a, b] \rightarrow S^1$  such that  $f(x) = e^{2ix}, \forall x \in [a, b]$ , clear that  $f$  is continuous onto function, but not contractible function since  $[a, b]$  is a contractible set while  $S^1$  is not.

**Remark (3.23):** A contractible function is not necessary to be continuous function.

**For example:** Let  $X = \{1,2,3\}$ , and  $\tau = \{X, \emptyset, \{1\}\}$ , and let  $f: X \rightarrow X$  such that  $f(2) = f(3) = 1$  and  $f(1) = 2$ , then  $f$  is a contractible function since every subset of  $X$  is contractible, and thus  $f(A) \subseteq X$  is contractible for each contractible  $A \subseteq X$ . But  $f$  is not continuous function since  $\{1\} \in \tau$  while  $f^{-1}(\{1\}) = \{2,3\} \notin \tau$ .

**Proposition (3.24):** The property of being contractible J- space is preserved by the perfect and contractible function from  $X$  onto  $Y$ .

**Proof:** Let  $E, F$  be closed subset of  $Y$  with  $E \cup F = Y$  and  $E \cap F$  compact, then  $f^{-1}(E), f^{-1}(F)$  are closed subsets of  $X$  since  $f$  is continuous, and  $f^{-1}(E) \cap f^{-1}(F) = f^{-1}(E \cap F)$  which is compact since  $f$  is perfect, and  $f^{-1}(E) \cup f^{-1}(F) = X$ , but  $X$  is contractible J-space, so  $f^{-1}(E)$  or  $f^{-1}(F)$  is contractible, it follows by definition of contractible function that  $f(f^{-1}(E))$  or  $f(f^{-1}(F))$  is contractible, but  $f$  is surjective, so  $E$  or  $F$  is contractible. Hence  $Y$  is contractible J- space.

**Proposition (3.25):** Every homeomorphism function is a contractible function.

**Proof:**  $f: X \rightarrow Y$  be a homeomorphism function, and let  $A$  be a contractible subset of  $X$ , we have to show that  $f(A)$  is contractible subset of  $Y$ . Note that  $A$  and  $f(A)$  are homeomorphic spaces, it follows by proposition (2.23) that  $A$  and  $f(A)$  are homotopy equivalent. Since  $A$  is contractible, it follows by theorem (2.15) and remark (2.24) that  $f(A)$  contractible.

**Corollary (3.26):** If the topological spaces  $X$  and  $Y$  are homeomorphic spaces and one of them is contractible J- space, then so is the other.

**Proof:** Follows from propositions (3.24) and (3.25).

## References

- [1] C. Kosniowski, A First cours in Algebraic Topology, Cambridge University Press, First Published, 1980.
- [2] E. D. Khalimsky; R. Kopperman and P. R. Meyer, Computer Graphics and Connected Topologies on Finite Closed Sets, Topology Appl. 36,1-7. 1967
- [3] T. Y. Kong; R. Kopperman and P. R. Meyer, A Topological Approach to Digital Topology, American Math. Monthly, 98, 901-917. 1991
- [4] E. Michael; J- spaces, Topology and its Application 102,315-339. 2000
- [5] Y. Nanjing; LJ- spaces, Czechoslovak Math. Journal, 57(132) 1223-1237. 2007.

For more information about the Conference please visit the websites:

<http://www.ihsciconf.org/conf/>  
[www.ihsciconf.org](http://www.ihsciconf.org)

- [6] A. Kornitowicz, A proof of the Jordan Curve Theorem Via the Brouwer Fixed Point Theorem, 6(1)33-40. 2007.
- [7] E. Bouassida, The Jordan Curve Theorem in the Khalimsky Plane, Applied General Topology, 9(2) 253- 262. 2008.
- [8] W. Fulton, A first Cours: Algebraic Topology, Springer Science & Business Media, 1997.
- [9] B. Mendelson, Introduction to Topology, Third Edition, Courier Corporation, 1990.
- [10] J. J. Rotman, An Introduction to Algebraic Topology, Springer Science & Business Media, 2013.
- [11] James R. Munkres, Topology A First Course, Prentice- Hall, 1974.
- [12] E. H. Spanier, Algebraic Topology, Springer Science & Business Media, 1994.
- [13] A.R. Shastri, Basic Algebraic Topology, CRC Press, 2016.
- [14] S. T. Bahadur, Elements of Topology, CRC Press, Taylor & Francis Group, 2015.
- [15] W. F. Basener, Topology and its Applications, John Wiley & Sons, 2006.
- [16] C. R. F. Maunder, Algebraic Topology, Courier Corporation, 1996.
- [17] W. S.Massey, A Basic Course in Algebraic Topology, Springer Science & Business Media, 1991.
- [18] A.Hatcher, Algebraic Topology, Cambridge University Press, 2002.
- [19] A. Ouahab; L. Go'miewicz and S. Djebali, Solution Sets for Differential Equations and Inclusions, Walter De Gruyter, 2012.
- [20] M. S. Nakahara, Geometry Topology and Physics, Second Edition, Taylor and Francis Group, New York London, 2003.
- [21] R. Anant Shastri., Algebraic Topology, CRC Press, 2013.
- [22] M. Tkachenko and Arhangel's Kii A., Topological Groups and Related Structures, An Introduction to Topological Algebra, Springer Science & Business Media, 2008.

For more information about the Conference please visit the websites:

<http://www.ihsciconf.org/conf/>  
[www.ihsciconf.org](http://www.ihsciconf.org)