Strongly $\mathcal{K}$-nonsingular Modules

Tha'ar Younis Ghawi
thar.younis@qu.edu.iq
Department of Mathematics, College of Education, AL-Qadisiyah University
AL-Qadisiyah, Iraq.

Article history: Received 12 August 2018, Accepted 26 September 2018, Publish January 2019

Abstract

A submodule N of a module M is said to be s-essential if it has nonzero intersection with any nonzero small submodule in M. In this article, we introduce and study a class of modules in which all its nonzero endomorphisms have non-s-essential kernels, named, strongly $\mathcal{K}$-nonsingular. We investigate some properties of strongly $\mathcal{K}$-nonsigular modules. Direct summand, direct sums and some connections of such modules are discussed.

Keywords: Modules; S-essential submodules; nonsingular modules; Strongly $\mathcal{K}$-nonsigular modules.

1. Introduction

A proper submodule N of a module M is said to be small if for any submodule K of M with $N + K = M$ implies $K = M$[1]. A nonzero module M is called Hollow if all its proper submodules are small [2]. The dual concept of small submodule is an essential submodule, where a nonzero submodule N of a module M is called essential if for any submodule K of M with $N \cap K = 0$ implies $K = 0$. A nonzero R-module M is said to be uniform if all its nonzero submodules are essential [3]. As mixing of concepts small and essential submodules, we introduced the following class of submodules. A submodule N of M is said to be s-essential if for any small K in M with $N \cap K = 0$ implies $K = 0$ [4]. It is clear essential submdules implies s-essential. Roman C.S. in [5], recall that an R-module M is called $\mathcal{K}$-nonsigular if for any endomorphism φ of M which has essential kernel, $\varphi = 0$. $\mathcal{K}$-a nonsingular module is studied in detail by [6]. In this research, we introduced concept of strongly $\mathcal{K}$-nonsigular modules which is stronger than $\mathcal{K}$-nonsigular modules. An R-module M is said to be strongly $\mathcal{K}$-nonsigular if for each endomorphism of M which has s-essential kernel, is zero. In section 2, we give some characterizations and properties of this concept. In section 3, we proved a strongly $\mathcal{K}$-nonsigular module is inherited by direct summands. Also, we give a condition for finite direct sums of strongly $\mathcal{K}$-nonsigular modules to be strongly $\mathcal{K}$-nonsigular. Several connections between strongly $\mathcal{K}$-nonsigular and other classes, also some examples are proved in section 4. Throughout this work, all rings are associative with identity and all modules are unitary right R-modules. For a right R-module M, the notations$N \subseteq M$, $N \leq M$, $N \ll M$, $N \trianglelefteq M$, $N \trianglelefteq^s M$ or $N \leq^{\oplus} M$ denotes that N is a subset, a submodule, a small submodule, an essential submodule, a s-essential submodule, or direct summand of M, respectively. Also, for $N \leq M$, we denote the endomorphism ring of M by $End_R(M)$, $r_R(N) = \{r \in R | Nr = 0\}$ and $[N:_R M] = \{r \in R | Mr \subseteq N\}$.

Starting, we will state some properties of s-essential submodules in [4, Prop. 2.7] which needed in this work.

Proposition 1: Let M be a module. Then;

(1) Assume N, K, L are submodules of M with $K \leq N$.

(i) If $K \trianglelefteq^s M$, then $K \trianglelefteq^s N$ and $N \trianglelefteq^s M$.

(ii) $N \trianglelefteq^s M$ and $L \trianglelefteq^s M$ if and only if $N \cap L \trianglelefteq^s M$.

(2) If $\varphi: M \to \grave{M}$ is a homomorphism with $K \trianglelefteq^s \grave{M}$, then $\varphi^{-1}(K) \trianglelefteq^s M$.

(3) If $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \trianglelefteq^s M_1 \oplus M_2$ if and only if $K_i \trianglelefteq^s M_i$ for $i = 1,2$.

2. Strongly $\mathcal{K}$-nonsigular Modules

In this section, we introduce the class of strongly $\mathcal{K}$-nonsigular modules as a stronger class of $\mathcal{K}$-nonsigular modules. Several various properties are proved.

Definition 2. An R-module M is said to be strongly $\mathcal{K}$-nonsigular if for all $\varphi \in End_R(M)$ with $ker\varphi$ is s-essential in M, implies $\varphi = 0$. Also, a ring R is strongly $\mathcal{K}$-nonsigular if it is a strongly $\mathcal{K}$-nonsigular R-module.

for $N \leq M$, if $Hom_R\left(\frac{M}{N}, M\right) = 0$ then N is called quasi-invertible [7].

Firstly, we are now in a position to give a characterization the notion of strongly $\mathcal{K}$-nonsigular modules.

Theorem 3. A module M is strongly $\mathcal{K}$-nonsigular if and only if all its s-essential submodules are quasi-invertible.

Proof. Assume M is a strongly $\mathcal{K}$-nonsigular R-module. Let $N \trianglelefteq^s M$ and N is not quasi-invertible, *i.e.* $Hom_R\left(\frac{M}{N}, M\right) \neq 0$, so there exists $(0 \neq)\varphi: \frac{M}{N} \to M$. Consider $\psi = \varphi \circ \pi \in End_R(M)$, where π is a natural epimorphism map. It is clear that $N \subseteq ker\psi$, but $N \trianglelefteq^s M$, this implies $ker\psi \trianglelefteq^s M$, and hence $\psi = 0$, as M is strongly $\mathcal{K}$-nonsigular, thus $\varphi = 0$, a contradiction. Therefore $N \trianglelefteq^s M$ and N is quasi-invertible. Conversely, let $(0 \neq)f \in End_R(M)$. If $kerf \trianglelefteq^s M$, so by hypothesis $kerf$ is quasi-invertible. But, we can define a homomorphism $h: \frac{M}{kerf} \to M$ by $h(m + Kerf) = f(m)$ for all $m \in M$. So $h \neq 0$ and hence $Hom_R\left(\frac{M}{kerf}, M\right) \neq 0$ which is a contradiction with $kerf$ is quasi-invertible. Therefore $kerf \ntrianglelefteq^s M$ and M is a strongly $\mathcal{K}$-nonsigular R-module. ■

Corollary 4. Let M be a strongly $\mathcal{K}$-nonsigular module. If $N \trianglelefteq^s M$, then $r_R(N) = r_R(M)$.

Proof. Assume $N \trianglelefteq^s M$, then by previous Theorem, N is a quasi-invertible submodule, and so $r_R(N) = r_R(M)$ by [7, Prop. 1.1.4]. ■

Proposition 5. Let M be an R-module, $R^* = R/A$ and $A \subseteq r_R(M)$. Then M is a strongly $\mathcal{K}$-nonsingular R-module if and only if M is a strongly $\mathcal{K}$-nonsingular R^*-module.

Proof. Assume $\pi: R \to R^*$ is a natural epimorphism, so by [8, Ex. P.51] $Hom_R\left(\frac{M}{N}, M\right) = Hom_{R^*}\left(\frac{M}{N}, M\right)$ for each submodule N of M. So, the result is follow. ■

Proposition 6. Let M be a strongly $\mathcal{K}$-nonsingular module with M/X is a projective module for all $X \trianglelefteq^s M$. Then M/A is a strongly $\mathcal{K}$-nonsingular module, for all $A \trianglelefteq^s M$.

Proof. For $B/A \trianglelefteq^s M/A$, to prove that $Hom_R\left(\frac{M/A}{B/A}, \frac{M}{A}\right) = 0$, that is; $Hom_R\left(\frac{M}{B}, \frac{M}{A}\right) = 0$. If false, so there is a nonzero homomorphism $\varphi: \frac{M}{B} \to \frac{M}{A}$. Note that $B \trianglelefteq^s M$ (in fact, $A \subseteq B \subseteq M$ with $A \trianglelefteq^s M$), so by hypothesis M/B is projective, hence there is a homomorphism $\psi: \frac{M}{B} \to M$ such that $\varphi = \pi \circ \psi$. It is clear $\psi \neq 0$, this implies $Hom_R\left(\frac{M}{B}, M\right) \neq 0$ with $B \trianglelefteq^s M$, is a contradiction with M is strongly $\mathcal{K}$-nonsigular. Thus $\varphi = 0$ and M/A is a strongly $\mathcal{K}$-nonsigular R-module. ■

Definition 7. Let M be a module, define the s-$\mathcal{K}$-nonsigular submodule of M by $Z_s^{\mathcal{K}}(M) = \sum_{\varphi \in S} Im\varphi$, where $S = End_R(M)$ and $ker\varphi \trianglelefteq^s M$.

Now, we will give another characterization for a strongly $\mathcal{K}$-nonsigular module as follows.

Proposition 8. Let M be a module. Then M is strongly $\mathcal{K}$-nonsigular if and only if $Z_s^{\mathcal{K}}(M) = 0$.

Proof. If M is a strongly $\mathcal{K}$-nonsigular module, then for all $\varphi \in End_R(M)$ with $ker\varphi \trianglelefteq^s M$, implies $Im\varphi = 0$, and hence $Z_s^{\mathcal{K}}(M) = \sum_{\varphi \in S} Im\varphi = 0$, where $S = End_R(M)$ and $ker\varphi \trianglelefteq^s M$. Conversely, assume $Z_s^{\mathcal{K}}(M) = 0$. Let $\psi \in End_R(M)$ such that $ker\psi \trianglelefteq^s M$, then $Im\psi \subseteq Z_s^{\mathcal{K}}(M)$ and so $\psi = 0$. Hence M is a strongly $\mathcal{K}$-nonsigular module. ■

Let M be a module, recall that a submodule N is supplement of $K \leq M$ if, N is a minimal in the set of submodules $L \leq M$ with $K + L = M$ (Equivalently, N is supplement of $K \leq M$ if and only if $K + N = M$ and $K \cap N \ll N$) [9]. We say that a submodule N of a module M is a supplement if it is a supplement for some submodule L of M.

The transitive property of s-essential submodules need not be hold, see [4, Ex. 2.8]. So, we will give a condition for which the transitive property is hold of s-essential submodules.

Lemma 9. Let M be a module, and let N is a supplement submodule in M with $K \subseteq N \subseteq M$. If $K \trianglelefteq^s N$ and $N \trianglelefteq^s M$, then $K \trianglelefteq^s M$.

Proof. Assume $L \ll M$ with $K \cap L = 0$. If $L \subseteq N$, but N is a supplement in M, then by [10, Prop. 20.2] $L \ll N$, and hence $L = 0$, since $K \trianglelefteq^s N$. Now, if $L \nsubseteq N$. We have $L \cap N \subseteq N \subseteq M$, but ($L \ll M$ implies $L \cap N \ll M$), thus again by [10, Prop. 20.2] $L \cap N \ll N$, since N is a supplement in M. But $K \cap (L \cap N) = K \cap L = 0$ and $K \trianglelefteq^s N$, this implies $L \cap N = 0$, and hence $L = 0$, as $N \trianglelefteq^s M$. ■

Now, we present the following Proposition.

Proposition 10. Let M be a quasi-injective R-module, and let N is a s-essential and supplement submodule in M. If M is a strongly $\mathcal{K}$-nonsingular R-module, then so is N.

Proof. Let $(0 \neq) f: N \to N$ be a homomrphism. Since M is a quasi-injective module, there exists $(0 \neq) \varphi \in End_R(M)$ such that $i \circ f = \varphi \circ i$, where $i: N \to M$ is an inclusion map. As M is strongly $\mathcal{K}$-nonsingular, we get $ker\varphi \ntrianglelefteq^s M$. Clearly, $kerf \subseteq ker\varphi$ then $kerf \ntrianglelefteq^s M$. If $kerf \trianglelefteq^s N$, and since N(supplement)$\trianglelefteq^s M$, so by previous Lemma, $kerf \trianglelefteq^s M$, is a contradiction. Therefore $kerf \ntrianglelefteq^s N$, and N is a strongly $\mathcal{K}$-nonsingular module. ∎

A quasi-injective module $\bar{\bar{M}}$ is called quasi-injective hull of a module M if, there exists a monomorphism $\varphi: M \to \bar{\bar{M}}$ with $Im\varphi \trianglelefteq \bar{\bar{M}}$ [11].

Corollary 11. Let $\bar{\bar{M}}$ be a strongly $\mathcal{K}$-nonsingular module. If M is a supplement in $\bar{\bar{M}}$, then M is strongly $\mathcal{K}$-nonsingular.

Next, we will study the behavior of s-essential submodule and strongly $\mathcal{K}$-nonsingular module under localization. Firstly, we have the following Lemma.

Lemma 12. Let M be a module, $N \leq K \leq M$ and let S is a multiplicative closed subset of R, provided $S^{-1}L_1 = S^{-1}L_2$ *iff* $L_1 = L_2$ for all $L_1, L_2 \leq M$. Then the following hold.
(i) $N \ll K$ in M as R-module if and only if $S^{-1}N \ll S^{-1}K$ in $S^{-1}M$ as $S^{-1}R$-module.
(ii) $N \trianglelefteq^s K$ in M as R-module if and only if $S^{-1}N \trianglelefteq^s S^{-1}K$ in $S^{-1}M$ as $S^{-1}R$-module.

Proof. (i) Assume $N \ll K \leq M$. Let $S^{-1}L \leq S^{-1}K$ with $S^{-1}N + S^{-1}L = S^{-1}K$, where $L \leq K$. But we have $S^{-1}N + S^{-1}L = S^{-1}(N + L)$, so $S^{-1}(N + L) = S^{-1}K$, and hence $N + L = K$ by hypothesis, thus $L = K$, as $N \ll K$. Therefore $S^{-1}L = S^{-1}K$, and so $S^{-1}N \ll S^{-1}K$ in $S^{-1}M$. Conversely, if $N + L = K$ where $L \leq K$. Then $S^{-1}N + S^{-1}L = S^{-1}(N + L) = S^{-1}K$, and hence $S^{-1}L = S^{-1}K$, as $S^{-1}N \ll S^{-1}K$. By hypothesis, $L = K$, and so $N \ll K$ in M.
(ii) If $N \trianglelefteq^s K \leq M$. Let $S^{-1}L \ll S^{-1}K$ such that $S^{-1}N \cap S^{-1}L = S^{-1}0$, where $L \leq K$. By (i), $L \ll K$. But, we have $S^{-1}(N \cap L) = S^{-1}N \cap S^{-1}L = S^{-1}0$, $N \cap L = 0$ by hypothesis. As $N \trianglelefteq^s K$ and $L \ll K$ implies $L = 0$, thus $S^{-1}L = S^{-1}0$. Conversely, suppose $N \cap L = 0$ where $L \ll K$, implies $S^{-1}L \ll S^{-1}K$, by (i). So $S^{-1}N \cap S^{-1}L = S^{-1}(N \cap L) = S^{-1}0$, thus $S^{-1}L = S^{-1}0$, as $S^{-1}N \trianglelefteq^s S^{-1}K$. By hypothesis, $L = 0$. ∎

However, we get the following result.

Proposition 13. Let M be an R-module, and let S is a multiplicative closed subset of R such that $S^{-1}L = S^{-1}K$ *iff* $L = K$ for all $L, K \leq M$. Then M is a strongly $\mathcal{K}$-nonsingular R-module, whenever $S^{-1}M$ is a strongly $\mathcal{K}$-nonsingular $S^{-1}R$-module.

Proof. Assume $(0 \neq) g \in End_R(M)$. We can define an $S^{-1}R$-homomorphism $S^{-1}g: S^{-1}M \to S^{-1}M$ such that $S^{-1}g\left(\frac{m}{s}\right) = \frac{g(m)}{s}$ for each $m \in M$, $s \in S$. It is clear $S^{-1}g \neq 0$, so $ker(S^{-1}g) \ntrianglelefteq^s S^{-1}M$, as $S^{-1}M$ is strongly $\mathcal{K}$-nonsingular. Also, it is easy to see that $ker(S^{-1}g) = S^{-1}(kerg)$, this implies that $S^{-1}(kerg) \ntrianglelefteq^s S^{-1}M$, and hence by Lemma 12 (ii), $kerg \ntrianglelefteq^s M$. ∎

Proposition 14. Let M be an R-module, and let P is a maximal ideal of R. If M_P is a strongly $\mathcal{K}$-nonsingular R_P-module, then M is a strongly $\mathcal{K}$-nonsingular R-module.

Recall that an R-module M is called multiplication if for each submodule N of M, $N = MI$ for some ideal I of R (Equivalently, M a multiplication if and only if $N = M.[N:_R M]$) [12]. If $r_R(M) = 0$, then M is called a faithful R-module. An R-module M is said to be scalar if for any $\varphi \in End_R(M)$, $\varphi(m) = mr$ for some $r \in R$, and for all $m \in M$ [13].

Now, we will studied the strongly $\mathcal{K}$-nonsingular property for rings and modules. But, in a position we need the following Lemma.

Lemma 15. The following holds, for faithful multiplication R-module M.
(i) $N \ll M$ if and only if $I \ll R$, where $N = MI$.
(ii) $N \trianglelefteq^s M$ if and only if $I \trianglelefteq^s R$, where $N = MI$.

Proof. (i) Assume that $N \ll M$. Let J be any ideal of R with $I + J = R$, so $M(I + J) = MR$, that is; $N + MJ = M$, but $N \ll M$ implies $MJ = M$, and so $J = R$, since M is a faithful multiplication R-module. Thus $I \ll R$. Conversely, let $K \leq M$ with $N + K = M$. As M is multiplication, $K = MJ$ for some $J \leq R$. Hence $M(I + J) = N + K = M = MR$, but M is a faithful multiplication R-module, so $I + J = R$, thus $J = R$ (since $I \ll R$). Therefore, $K = MJ = MR = M$, and hence $N \ll M$.
(ii) Let $N \trianglelefteq^s M$. Suppose that $J \ll R$ with $I \cap J = 0$, then $N \cap MJ = MI \cap MJ = M(I \cap J) = 0$, but by (i), $MJ \ll M$, hence $MJ = 0$, implies $J = 0$ (since M is faithful). Thus $I \trianglelefteq^s R$. Conversely, let $K \ll M$ such that $N \cap K = 0$. Since M is multiplication, then there is a small ideal J of R with $K = MJ$, by (i). Hence $M(I \cap J) = MI \cap MJ = N \cap K = 0$, so by faithfulty for M, we get $I \cap J = 0$, then $J = 0$, as $J \ll R$ and $I \trianglelefteq^s R$. Thus $K = MJ = 0$, and so $N \trianglelefteq^s M$. ■

Proposition 16. Let M be a faithful multiplication R-module. If M is a strongly $\mathcal{K}$-nonsingular R-module, then R is strongly $\mathcal{K}$-nonsingular. The converse hold, whenever M is finitely generated.

Proof. Assume that M is a strongly $\mathcal{K}$-nonsingular R-module. Let $(0 \neq)\varphi \in End_R(R)$. For $r \in R$, we know $\varphi(a) = a.\varphi(1)$. We can define $\psi: M \to M$ by $\psi(m) = m.\varphi(1)$ for all $m \in M$. It is easy to see ψ is well-defined and homomorphism. If $\psi = 0$, then $M.\varphi(1) = 0$, hence $\varphi(1) \in r_R(M) = 0$, so $\varphi = 0$ which is a contradiction. Hence $(0 \neq)\psi \in End_R(M)$, and so $ker\psi \ntrianglelefteq^s M$, as M is strongly $\mathcal{K}$-nonsingular. Since M is a multiplication R-module, $ker\psi = M.[ker\psi:_R M]$. But, we have $[ker\psi:_R M] = ker\varphi$, to see this: if $r \in [ker\psi:_R M]$, $Mr \subseteq ker\psi$, so $\psi(Mr) = Mr.\varphi(1) = M.\varphi(r) = 0$, hence $\varphi(r) \in r_R(M) = 0$, thus $r \in ker\varphi$. Now, if $x \in ker\varphi$, $\varphi(x) = x.\varphi(1) = 0$ hence $Mx.\varphi(1) = 0$, so $\psi(Mx) = 0$ implies $Mx \subseteq ker\psi$, thus $x \in [ker\psi:_R M]$. Since $ker\psi \ntrianglelefteq^s M$, so $M.[ker\psi:_R M] \ntrianglelefteq^s M$, so by Lemma 15 (ii), $[ker\psi:_R M] \ntrianglelefteq^s R$, which hence $ker\varphi \ntrianglelefteq^s R$, therefore R is strongly $\mathcal{K}$-nonsingular. Conversely, let $(0 \neq)g \in End_R(M)$. If M is finitely generated multiplication R-module, then M is a scalar R-module, by [14, Th. 2.3]. Hence $g(m) = mr$ for some $r \in R$, and for all $m \in M$. It follows that $h \in End_R(R)$ defined by $h(x) = xr$ for all $x \in R$. Note $h(1) = 1.r = r \neq 0$ (in fact, if $r = 0$ implies $g = 0$), and hence $(0 \neq)h \in End_R(R)$, but R is strongly $\mathcal{K}$-nonsingular, then $kerh \ntrianglelefteq^s R$. On the other hand, we have

$kerh = [kerg:_R M]$ which implies $[kerg:_R M] \ntrianglelefteq^s R$, and hence $M.[kerg:_R M] \ntrianglelefteq^s M$, by Lemma 15 (ii), thus $kerg \ntrianglelefteq^s M$, and M is a strongly $\mathcal{K}$-nonsigular R-module. ∎

Next, proved that the property of strongly $\mathcal{K}$-nonsigular of modules is inherited by isomorphism.

Proposition 17. For two modules M_1 and M_2, if $M_1 \cong M_2$ then M_2 is a strongly $\mathcal{K}$-nonsigular module, whenever M_1 is strongly $\mathcal{K}$-nonsigular.

Proof. Since $M_1 \cong M_2$, there exists an isomorphism $f: M_1 \longrightarrow M_2$. Assume M_1 is a strongly $\mathcal{K}$-nonsigular module. Let $g \in End_R(M_2)$ such that $kerg \trianglelefteq^s M_2$. Consider $\psi = f^{-1} \circ g \circ f \in End_R(M_1)$, where $f^{-1}: M_2 \longrightarrow M_1$ isomorphism. Now, we have $ker\psi = f^{-1}(kerg)$, to see this:
$ker\psi = \{ x \in M_1 | \ f^{-1} \circ g \circ f(x) = 0\} = \{ x \in M_1 | \ g \circ f(x) \in kerf^{-1} = 0\} =$
$\{x \in M_1 | \ f(x) \in kerg\} = \{ x \in M_1 | \, x \in f^{-1}(kerg)\} = f^{-1}(kerg)$. By Proposition 1.1(2), we get $f^{-1}(kerg) \trianglelefteq^s M_1$, (since $kerg \trianglelefteq^s M_2$), this implies $ker\psi \trianglelefteq^s M_1$ and hence $\psi = 0$, as M_1 is strongly $\mathcal{K}$-nonsigular. Thus, $0 = f^{-1} \circ g(Imf) = f^{-1} \circ g(M_2)$, thus $Img \subseteq kerf^{-1} = 0$. Therefore $g = 0$. ∎

Proposition 18. Let M be a faithful scalar R-module. Then R is strongly $\mathcal{K}$-nonsigular if and only if $S = End_R(M)$ is strongly $\mathcal{K}$-nonsigular.

Proof. Since M is a scalar R-module, then by [15, Lemma 3.6.2] $S = End_R(M) \cong R/r_R(M)$, but M is faithful, hence $S = End_R(M) \cong R$. By Proposition 17, the result is follow. ∎

Proposition 19. Let M be a faithful multiplication R-module. If R is strongly $\mathcal{K}$-nonsigular, then $r_R(N) = r_R(M)$ for all $N \trianglelefteq^s M$.

Proof. As M is a faithful multiplication R-module, if $N \trianglelefteq^s M$, there is $I \trianglelefteq^s R$ with $N = MI$, by Lemma 15 (ii). For $r \in r_R(N)$, $Nr = 0$, then $MI.r = 0$, hence $Ir \subseteq r_R(M) = 0$, so $r \in r_R(I)$ implies $r_R(N) = r_R(I)$. Since R is strongly $\mathcal{K}$-nonsigular with $I \trianglelefteq^s R$, then I is a quasi-invertible ideal (by Theorem 2.2), so $r_R(I) = r_R(R) = 0$ by [7, Prop. 1.1.4]. Hence $r_R(N) = 0 = r_R(M)$. ∎

3. Direct Summand and Direct Sums

We start with following result.

Proposition 20. Let M be a strongly $\mathcal{K}$-nonsigular module, and $A \leq M$. If $A \trianglelefteq^s B_i \leq^{\oplus} M$, then $B_1 = B_2$ for $i \in \{1,2\}$.

Proof. Consider $\rho_i: M \longrightarrow B_i$ is the canonical projection map, for $i = 1,2$. We have $\rho_1(A) = A = \rho_2(A)$. Since $(1-\rho_1)\rho_2 \in End_R(M)$, so we have $((1-\rho_1)\rho_2)(A) = (1-\rho_1)(\rho_2(A)) = (1-\rho_1)(\rho_1(A)) = ((1-\rho_1)\rho_1)(A) = 0$ (since ρ_1 is an idempotent), then $A \subseteq ker(1-\rho_1)\rho_2$. Now, $B_2 \leq^{\oplus} M$, so $M = \grave{B}_2 \oplus B_2$ for some $\grave{B}_2 \leq M$. Hence $((1-\rho_1)\rho_2)(\grave{B}_2) = (1-\rho_1)(\rho_2(\grave{B}_2)) = (1-\rho_1)(0) = 0$, thus $\grave{B}_2 \subseteq ker(1-\rho_1)\rho_2$. Therefore $\grave{B}_2 \oplus A \subseteq ker(1-\rho_1)\rho_2$. On the other hand, $\grave{B}_2 \trianglelefteq^s \grave{B}_2$ and $A \trianglelefteq^s B_2$, then $\grave{B}_2 \oplus A \trianglelefteq^s \grave{B}_2 \oplus B_2 = M$ by Proposition 1 (3), and so $ker(1-\rho_1)\rho_2 \trianglelefteq^s M$ which implies $(1-\rho_1)\rho_2 = 0$, as M is strongly $\mathcal{K}$-nonsigular. Hence $\rho_2 = \rho_1\rho_2$, so $B_2 = \rho_2(B_2) = \rho_1\rho_2(B_2) = \rho_1(\rho_2(B_2)) = \rho_1(B_2) \subseteq B_1 \Rightarrow B_2 \subseteq B_1$. Similarly, taking $(1-\rho_2)\rho_1 \in End_R(M)$, and we get $B_1 \subseteq B_2$. ■

Based on our result, we prove that direct summands of a strongly $\mathcal{K}$-nonsigular module inherit the property.

Proposition 21. A direct summand of a strongly $\mathcal{K}$-nonsigular module is strongly $\mathcal{K}$-nonsigular.

Proof. Let M be a strongly $\mathcal{K}$-nonsigular module, and $A \leq^{\oplus} M$, so $M = A \oplus B$ for some $B \leq M$. Assume that $f \in End_R(A)$ such that $kerf \trianglelefteq^s A$. Consider $h = i \circ f \circ \rho \in End_R(M)$, where ρ is the canonical projection map onto A, and i is the inclusion map from A to M. So, we have $Kerh = Kerf \oplus B$, to see this: for $x \in kerh$, $x = a + b$ where $a \in A$ and $b \in B$ with $h(x) = 0$, so $f(a) = i \circ f(a) = i \circ f(\rho(x)) = h(x) = 0$, then $a \in kerf$, and hence $x = a + b \in kerf + B$, that is; $kerh = kerf + B$. On the other hand, $kerf \cap B \subseteq A \cap B = 0$, which implies $kerh = kerf \oplus B$. Since $kerf \trianglelefteq^s A$ and $B \trianglelefteq^s B$, then $kerh = kerf \oplus B \trianglelefteq^s A \oplus B = M$ by Proposition 1.1(3). Thus $h = 0$, as M strongly $\mathcal{K}$-nonsigular. Hence $Imf = f(A) = i \circ f(A) = i \circ f(\rho(M)) = h(M) = 0$. Therfore $f = 0$ and A is strongly $\mathcal{K}$-nonsigular. ■

Definition 22. Let M and N be two R-modules. Then M is called strongly $\mathcal{K}$-nonsigular relative to N if, every $\varphi \in Hom_R(M, N)$ such that $ker\varphi \trianglelefteq^s M$, implies $\varphi = 0$. Obviously, M is strongly $\mathcal{K}$-nonsigular if and only if M is strongly $\mathcal{K}$-nonsigular relative to M.

Proposition 23. If M is a strongly $\mathcal{K}$-nonsigular module. For $N \leq M$, M is strongly $\mathcal{K}$-nonsigular relative to N.

Proof. If $N = M$, clear that M is strongly $\mathcal{K}$-nonsigular relative to N. Assume that $N \neq M$, if $\psi \in Hom_R(M, N)$ with $ker\psi \trianglelefteq^s M$. Consider $h = i \circ \psi$, where i is the inclusion map from N to M. So $h \in End_R(M)$ such that $kerh = ker\psi \trianglelefteq^s M$, then $h = 0$, as M is strongly $\mathcal{K}$-nonsigular, hence $Im\psi = \psi(M) = i(\psi(M)) = h(M) = 0$, thus $\psi = 0$. ■

Lemma 24. For a module M, if $N_i \trianglelefteq^s K_i \leq M$ for $i \in \wedge = \{1,2,\dots,n\}$, then $\bigcap_{i=1}^{n} N_i \trianglelefteq^s \bigcap_{i=1}^{n} K_i$.
Proof. Consider the case when the index set $\wedge = \{1,2\}$. Let $X \ll K_1 \cap K_2$ with $(N_1 \cap N_2) \cap X = 0$, then $N_1 \cap (N_2 \cap X) = 0$. Since $X \ll K_1 \cap K_2 \subseteq K_1$, then $X \ll K_1$ and hence $N_2 \cap X \ll K_1$ implies $N_2 \cap X = 0$, as $N_1 \trianglelefteq^s K_1$. Also, $X \ll K_2$ and $N_2 \trianglelefteq^s K_2$, hence $X = 0$. Thus $N_1 \cap N_2 \trianglelefteq^s K_1 \cap K_2$.■

Theorem 25. Let $M = M_1 \oplus M_2$ be an R-module. Then M is strongly $\mathcal{K}$-nonsigular if and only if M_i is strongly $\mathcal{K}$-nonsigular relative to M_j, for $i, j \in \{1,2\}$.

Proof. Assume $M = M_1 \oplus M_2$ a strongly $\mathcal{K}$-nonsigular module. By Proposition 21, M_i is strongly $\mathcal{K}$-nonsigular, for $i \in \{1,2\}$. Hence M_i is strongly $\mathcal{K}$-nonsigular relative to M_i, for $i \in \{1,2\}$. Now, let $\varphi \in Hom_R(M_1, M_2)$ such that $ker\varphi \trianglelefteq^s M_1$. Consider $\psi = i \circ \varphi \circ \rho \in End_R(M)$, where ρ is the canonical projection map onto M_1, $i: M_2 \longrightarrow M$ is the inclusion map. Clearly, $ker\psi = ker\varphi \oplus M_2$, so $ker\psi = ker\varphi \oplus M_2 \trianglelefteq^s M_1 \oplus M_2 = M$, hence $\psi = 0$ (since M is strongly $\mathcal{K}$-nonsigular). Thus, $\varphi = 0$ and so M_1 is strongly $\mathcal{K}$-nonsigular relative to M_2. M_2 is strongly $\mathcal{K}$-nonsigular relative to M_1, similarly. Conversely, if $f \in End_R(M)$ such that $kerf \trianglelefteq^s M$, so we have $kerf \cap M_1 \trianglelefteq^s M_1$, by Lemma 24. Consider $f|_{M_1}: M_1 \to M$ which defined by $f|_{M_1}(x) = f(x+0)$ for all $x \in M$. We have $ker(f|_{M_1}) = kerf \cap M_1$ as follows: if $a \in kerf \cap M_1$ then $0 = f(a) = f(a+0) = f|_{M_1}(a)$ and $a \in M_1$, thus $a \in ker(f|_{M_1})$. Now, if $x \in ker(f|_{M_1})$ then $0 = f|_{M_1}(x) = f(x+0) = f(x)$, so $x \in kerf \cap M_1$. Consider $g_i = \rho_i \circ f|_{M_1}$, where ρ_i is the canonical projection map onto M_i, for $i \in \{1,2\}$. To prove that $ker(f|_{M_1}) = \bigcap_{i=1}^{2} kerg_i$. If $x \in ker(f|_{M_1})$, $0 = f|_{M_1}(x)$, so $g_i(x) = \rho_i \circ f|_{M_1}(x) = \rho_i\,(f|_{M_1}(x)) = \rho_i(0) = 0$, this implies $x \in \bigcap_{i=1}^{2} kerg_i$. Now, if $x \in \bigcap_{i=1}^{2} kerg_i$, so $g_i(x) = 0 \Rightarrow \rho_i(f|_{M_1}(x)) = 0 \Rightarrow f|_{M_1}(x) \in \bigcap_{i=1}^{2} ker\rho_i = M_2 \cap M_1 = 0 \Rightarrow x \in ker(f|_{M_1})$ for $i \in \{1,2\}$. So $\bigcap_{i=1}^{2} kerg_i = ker(f|_{M_1}) = kerf \cap M_1 \trianglelefteq^s M_1$, hence by Proposition 1, $kerg_1 \trianglelefteq^s M_1$ and $kerg_2 \trianglelefteq^s M_1$. By hypothesis, $g_i = 0 \Rightarrow \rho_i\,(Im\,f|_{M_1}) = 0 \Rightarrow Imf|_{M_1} \subseteq \bigcap_{i=1}^{2} ker\rho_i = 0$ for $i \in \{1,2\}$, implies $f|_{M_1} = 0$. Similarly, we obtain $h_i = \rho_i \circ f|_{M_2} = 0$ for $i \in \{1,2\}$, and hence $f|_{M_2} = 0$. So $f|_{M_i} = 0$ for $i \in \{1,2\}$. Therefore $f = 0$, and $M = M_1 \oplus M_2$ is strongly $\mathcal{K}$-nonsigular. ■

Corollary 26. If $M = \oplus_{i=1}^{n} M_i$. Then M is a strongly $\mathcal{K}$-nonsigular module if and only if M_i is strongly $\mathcal{K}$-nonsigular relative to M_j, for $i, j \in \{1,2,\ldots,n\}$.

Proposition 27. Let $M = M_1 + M_2$ be an R-module, where $M_1, M_2 \leq M$. If $\frac{M}{M_1 \cap M_2}$ is a strongly $\mathcal{K}$-nonsigular R-module, then both of $\frac{M}{M_1}$ and $\frac{M}{M_2}$ is strongly $\mathcal{K}$-nonsigular.

Proof. We have $\frac{M_1}{M_1 \cap M_2} + \frac{M_2}{M_1 \cap M_2} = \frac{M_1 + M_2}{M_1 \cap M_2} = \frac{M}{M_1 \cap M_2}$, also $\frac{M_1}{M_1 \cap M_2} \cap \frac{M_2}{M_1 \cap M_2} = \frac{M_1 \cap M_2}{M_1 \cap M_2} = 0_{\frac{M}{M_1 \cap M_2}}$, thus $\frac{M}{M_1 \cap M_2} = \frac{M_1}{M_1 \cap M_2} \oplus \frac{M_2}{M_1 \cap M_2}$. As $\frac{M}{M_1 \cap M_2}$ is strongly $\mathcal{K}$-nonsigular, so by Proposition 3.2, $\frac{M_i}{M_1 \cap M_2}$ is strongly $\mathcal{K}$-nonsigular for $i = 1,2$. But, we have $\frac{M_2}{M_1 \cap M_2} \cong \frac{M_1 + M_2}{M_1} = \frac{M}{M_1}$ and $\frac{M_1}{M_1 \cap M_2} \cong \frac{M_1 + M_2}{M_2} = \frac{M}{M_2}$, so by Proposition 16, $\frac{M}{M_1}$ and $\frac{M}{M_2}$ are strongly $\mathcal{K}$-nonsigular. ■

4. Connections to other Topics

In this section, we can prove some relations between strongly $\mathcal{K}$-nonsigular modules and other classes of modules, such examples, semisimple, Rickart, quasi-Dedekind and prime modules.

Example 28. Every module has no nonzero small submodule, all its submodules are s-essential, and hence does not strongly $\mathcal{K}$-nonsigular. Notice, every submodule in Z_Z is s-essential, because the zero is the only small submodule of Z_Z, hence Z_Z is not strongly $\mathcal{K}$-nonsigular. In particular, every simple (semisimple) module is not strongly $\mathcal{K}$-nonsigular. But, we know every semisimple module is $\mathcal{K}$-nonsigular.

Remark 29. It is clear that every strongly $\mathcal{K}$-nonsingular module is $\mathcal{K}$-nonsingular, but the converse need not be true, in general, a semisimple module is $\mathcal{K}$-nonsingular but not strongly $\mathcal{K}$-nonsingular.

Lemma 30. Let M be a Hollow (not simple) module, and $A \leq M$. Then A is essential if and only if A is s-essential.

Proof. ⇒) Clear. ⇐) Assume $(0 \neq)A \trianglelefteq^{s} M$ such that $A \cap B = 0$, where $B \leq M$. If $B = M$, then $A = 0$, a contradiction. Thus B is a proper in M, hence $B \ll M$ (since M is Hollow), and so $B = 0$, as $A \trianglelefteq^{s} M$. Therfore $A \trianglelefteq M$. ■

However, we consider the following Proposition by Lemma 30.

Proposition 31. Let M be a Hollow (not simple) module. Then M is strongly $\mathcal{K}$-nonsingular if and only if M is $\mathcal{K}$-nonsingular.

An R-module M is said to be Rickart if $r_M(\varphi) = Ker\varphi$ is a direct summand of M for each $\varphi \in End_R(M)$ [16]. Recall that an R-module M is quasi-Dedekind if, for any $(0 \neq)\varphi \in End_R(M)$, is a monomorphism ($i.e.$ $ker\varphi = 0$) [7].

Obviously, Rickart, quasi-Dedekind modules are $\mathcal{K}$-nonsingular. Note that the Z-module Z_6 is semisimple, so it is Rickart, but not strongly $\mathcal{K}$-nonsingular. Also we know Z_Z is quasi-Dedekind, but it is not strongly $\mathcal{K}$-nonsingular. However, we have the following Corollary which follows by Proposition 4.4.

Corollary 32. For a Hollow (not simple) module M. If M is Rickart (or quasi-Dedekind), then M is strongly $\mathcal{K}$-nonsingular.

Lemma 33. Let M be an R-module. If $S = End_R(M)$ is a regular ring, then M is Rickart.

Proof. Assume $\varphi \in S = End_R(M)$. Since S is a regular ring, so φ a regular element, thus $ker\varphi \leq^{\oplus} M$, by [17, Cor. 3.2]. Hence M is a Rickart module. ■

Corollary 34. If M is a Hollow (not simple) R-module with $S = End_R(M)$ is a regular ring, then M is strongly $\mathcal{K}$-nonsingular.

Proof. It follows directly by Lemma 33 and Corollary 34. ■

Lemma 35. If M is a uniform module has nonzero small submodule, then s-essential submodule implies essential.

Proof. Assume $X \leq M$. Put $X = 0$. Let N be a nonzero small submodule of M, then $X \cap N = 0$ which implies $X \ntrianglelefteq^{s} M$. Hence the result is obtained. ■

Note that Z-module Z is uniform, the zero submodule of Z_Z is s-essential but not essential (in fact, 0 is the only small submodule of Z_Z).

However, we have the following.

Proposition 36. Let M be a uniform module has nonzero small submodule. Then M is strongly $\mathcal{K}$-nonsigular if and only if M is $\mathcal{K}$-nonsigular.

Proof. It follows by Lemma 35. ∎

Recall [18], a module M is called prime if for all nonzero submodule N of M, $r_R(N) = r_R(M)$. Mijbass in [7, Th. 2.3.14], presented the following Theorem.

Theorem 37. A module M is uniform quasi-Dedekind if and only if it is uniform prime.

Proposition 38. Let M be a uniform R-module has nonzero small submodule. Then the following asseretions are equivalent.

(i) M is Rickart.
(ii) M is $\mathcal{K}$-nonsigular.
(iii) M is strongly $\mathcal{K}$-nonsigular.
(iv) M is quasi-Dedekind.
(v) M is prime.
(vi) For $N \trianglelefteq^{s} M$, $r_R(N) = r_R(M)$.

Proof. $(i) \Rightarrow (iv)$ Since M is a uniform R-module, then M is indecomposable. Let $\varphi \in End_R(M)$ with $\varphi \neq 0$, then $ker\varphi \leq^{\oplus} M$, as M is Rickart. So, either $ker\varphi = M$ or $ker\varphi = 0$. If $ker\varphi = M$ then $\varphi = 0$, a contradiction. Hence $ker\varphi = 0$, implies M is quasi-Dedekind.
$(iv) \Rightarrow (i)$ Let $\varphi \in End_R(M)$. If $\varphi = 0$, then $ker\varphi = M \leq^{\oplus} M$. Assume that $\varphi \neq 0$, but M is a quasi-Dedekind module, so $ker\varphi = 0 \leq^{\oplus} M$. Thus M is Rickart.
$(ii) \Leftrightarrow (iii)$ It follows by Proposition 36.
$(ii) \Leftrightarrow (iv)$ Since M is a uniform module, the result is follow.
$(iv) \Leftrightarrow (v)$ It follows by Theorem 37.
$(v) \Leftrightarrow (vi)$ Since M is uniform has nonzero small submodule, then all its nonzero submodules are s-essential, so the result is obtained. ∎

5. Conclusion

The most important results of the article are:

(1) Let M be a faithful multiplication R-module. If M is a strongly $\mathcal{K}$-nonsigular R-module, then R is strongly $\mathcal{K}$-nonsigular. The converse holds, whenever M is finitely generated.
(2) A direct summand of a strongly $\mathcal{K}$-nonsigular module is strongly $\mathcal{K}$-nonsigular.
(3) If $M = \oplus_{i=1}^{n} M_i$. Then M is a strongly $\mathcal{K}$-nonsigular module if and only if M_i is strongly $\mathcal{K}$-nonsingular relative to M_j, for $i, j \in \{1,2, \dots, n\}$.

References

1. Leonard, W.W. Small modules. *Pro. Amer. Math. Soc.* **1966**, *2*, 527-531.
2. Fluery, P. Hollow modules and local endomorphism rings. *Pacific J. Math.* **1974**, *4*, 379-385.

3. Goodearl, K.R. *Ring theory, Nonsingular rings and modules*. Marcel Dekker. Newyork and Basel. **1976**.
4. Zhou, D.X.; Zhang, X.R. Small-essential submodules and morita duality. *Southeast Asian Bulletin of math*. **2011**, *35, 6,* 1051-1062.
5. Roman, C.S. Baer and Quasi-Baer modules. Doctoral Dissertation. The Ohio state Univ. **2004**.
6. Rizvi, S.T; Roman, C.S. On $\mathcal{K}$-nonsigular modules and applications. *comm. In Algebra*. **2007**, 3, 2960-2982.
7. Mijbass, A.S. Quasi-Dedekind modules. Ph.D. Thesis. College of Science. University of Baghdad. Iraq. **1997**.
8. Kasch, F. *Modules and Rings*. Academic press. New York. **1982**.
9. Wisbauer, R. *Foundations of module and ring theory, reading*. Gordon and Breach Science Pub. **1991**.
10. Clark, J.; Lomp C.; Vanaja, N.; Wisbauer, R. *Lifting modules*. Supplements and projectivity in module theory. Frontiers in Mathematics. Birkhauser. Basel. **2006**.
11. Zelmanowitz, J.M. Representation of rings with faithful polyform modules. *comm. In Algebra*. **1986**, *14, 6,* 1141-1169.
12. Branard, B. Multiplication modules. *Journal of Algebra*. **1981**, *3*, 170-178.
13. Shihab, B.N. Scalar reflexive modules. Ph.D. Thesis. University of Baghdad. Iraq. **2004**.
14. Naoum, A.G. On the ring of Endomorphisms of finitely generated multiplication modules. *Periodica Math. Hungarica*. **1990**, *21, 3*, 249-255.
15. Mohamed-Ali, E.A. On Ikeda-Nakayama modules. Ph.D. Thesis. University of Baghdad. Iraq. **2006**.
16. Lee, G.; Rizvi, S.T.; Roman, C.S. Rickart modules. *Commutative In Algebra*. **2010**, *4*, 4005-4027.
17. Ware, R. Endomorphism rings of projective modules. *Trans. Amer. Math. Soc.* **1971**, 233-256.
18. Desale, G.; Nicholson, W.K. Endoprimitive rings. *J. Algebra*. **1981**, 548-560.
19. Hadi, I. M-A and Ghawi, Th.Y. Essentially Quasi-Invertible Submodules and Essentially Quasi-Dedekind Modules. *Ibn AL-Haitham J. For Pure & Appl. Sci.* **2011**, *24, 3*, 102-113.