T-Abso and T-Abso Quasi Primary Fuzzy Submodules

Wafaa H. Hanoon
Department of Computer Science, College of Education for Girls, University of Kufa, Iraq.
wafaah.hannon@uokufa.edu.iq
Hatem Y. Khalaf
Department of Mathematics, College of Education for Pure Science Ibn- Haitham
University of Baghdad, Baghdad, Iraq.
dr.hatamyahya@yahoo.com

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Abstract
Let $\acute{M}$ be a unitary R-module and R is a commutative ring with identity. Our aim in this paper to study the concepts T-ABSO fuzzy ideals, T-ABSO fuzzy submodules and T-ABSO quasi primary fuzzy submodules, also we discuss these concepts in the class of multiplication fuzzy modules and relationships between these concepts. Many new basic properties and characterizations on these concepts are given.

Keywords: T-ABSO fuzzy ideal, T-ABSO fuzzy submodule, Quasi- prime fuzzy submodule, T-ABSO primary fuzzy submodule, T-ABSO quasi primary fuzzy submodule, Multiplication fuzzy module.

1. Introduction
In this paper all ring is commutative with identity and all modules are unitary. Deniz S. et al in [1] presented the concept of 2-absorbing fuzzy ideal which is a generalization of prime fuzzy ideal. Prime submodule which play an important turn in the module theory over a commutative ring. A prime submodule N of an R-module $\acute{M}$, $N\neq\acute{M}$, with property $a\in R$, $x\in\acute{M}$, $ax\in N$ implies that $\in N$ or $a\in(N:\acute{M})$ [2]. This concept was generalized to concept of prime fuzzy submodule which was presented by Rabi [3]. In 1999, Abdul-Razakm, presented and studied quasi-prime submodule let $N<\acute{M}$, N be called a quasi-prime if for a, $b\in R$, $m\in\acute{M}$, $abm\in N$, implies either $am\in N$ or $bm\in N$ [4]. In 2001, Hatam generalized it to fuzzy quasi-prime submodules [5]. Darani, et al in [6] presented the definition of 2-absorbing submodule. Let $N<\acute{M}$, N be called 2-absorbing submodule of $\acute{M}$ if whenever r, $b\in R$, $x\in\acute{M}$ and $rbx\in N$, then $rx\in N$ or $bx\in N$ or rb $\in(N:\acute{M})$. Hatam and wafaa expanded this concept that is**:** if X be a fuzzy module of an R-module $\acute{M}$. A proper fuzzy submodule A of X is called T-ABSO fuzzy submodule if whenever a_s , b_l be fuzzy singletons of R, and $x_v\subseteq X$, $\forall\, s,l,v\in L$, such that $a_s b_l x_v\subseteq A$, then either $a_s b_l\subseteq(A:_R X)$ or $a_s x_v\subseteq A$ or $b_l x_v\subseteq A$ [7]. McCasland and Moore presented the concept of $\acute{M}$-radical of N such: Let N be a proper module of a nonzero R-module $\acute{M}$, then the $\acute{M}$-radical of N, denoted by $\acute{M}$-rad N is defined to be the intersection of all prime module including N, see [8]. Mostafanasab et al, were presented the connotation of 2-absorbing primary submodule. So, A proper submodule N of an R-module $\acute{M}$ is called 2-absorbing primary submodule of $\acute{M}$ if whenever a, $b\in$ R and $m\in\acute{M}$ and $abm\in N$, then $am\in\acute{M}$-rad N or $bm\in\acute{M}$-rad N or $ab\in(N:_R \acute{M})$,

[9]. Rabi and Hassan in 2008 were presented the concept of quasi primary fuzzy submodule. A proper fuzzy submodule A of fuzzy module X is said to be quasi primary fuzzy submodule if $(A{:}B)$ is a primary fuzzy ideal of R for each fuzzy submodule B of X such that $A \subset B$ [10]. Suat K. et al, studied and presented the connotation of 2-absorbing quasi primary submodule, i.e., A proper submodule N of $\acute{M}$ is said to be 2-absorbing quasi primary submodule if the condition $abq \in N$ implies either $ab \in \sqrt{N{:}_R \acute{M}}$ or $aq \in \acute{M}$-rad(N) or $bq \in \acute{M}$-rad(N) for every a, $b \in$ R and $q \in \acute{M}$ [11]. This paper is composed of two sections.

In section (1) we present the definition of T-ABSO fuzzy ideals and we give some characterizations of this definition for ideals. Also many properties and outcomes of this concept are given. In section (2) we present the definition of T-ABSO fuzzy submodules, many basic properties and outcomes are studied. In section (3) we present the concept of T-ABSO quasi primary fuzzy submodules and we study the relationships this concept with among T-ABSO fuzzy submodules and T-ABSO primary fuzzy submodules. Several important results have been demonstrated. Note that we denote to fuzzy module, submodule.

2. T-ABSO F. Ideals

In this section, we introduce the concepts of T-ABSO and T-ABSO primary ideals. Some concepts and propositions which are needed in the next section.

Definition 1. [1]

Let $\hat{H}$ be a non-constant F. ideal of R. Then $\hat{H}$ is called T-ABSO F. ideal if for any F. points a_s, b_l, r_k of R, $a_s b_l\, r_k \in \hat{H}$ implies that either $a_s b_l \in \hat{H}$ or $a_s\, r_k \in \hat{H}$ or $b_l\, r_k \in \hat{H}$.
The following proposition characterize T-ABSO F. ideal in terms of its level ideal.

Lemma 2. [1]

Let A be F. ideal of R. If A is T-ABSO F. ideal, then A_v is T-ABSO ideal of R, $\forall\, v \in$ L,

Recall that Let $\hat{H}$ be any F. ideal of R. Then the radical F. of $\hat{H}$, denoted by $\sqrt{\hat{H}}$, is defined by: $\sqrt{\hat{H}} = \cap\{\underset{\smile}{U}{:}\underset{\smile}{U}$ is a prime F. ideal of R containing $\hat{H}\}$ [12].

Now, we give these propositions which are used in the next section.

Proposition 3.

Suppose that R be a ring and $\hat{H}$ is T-ABSO F. ideal of R. Then $\sqrt{\hat{H}}$ is T-ABSO F. ideal of R and $a_v^2 \subseteq \hat{H}$ for each F. singleton $a_v \subseteq \sqrt{\hat{H}}$, $\forall\, v \in$ L.

Proof. Let $\hat{H}$ be T-ABSO F. ideal and $a_v \subseteq \sqrt{\hat{H}}$, hence $a \in \sqrt{\hat{H}_v}$. Then $a^2 \in \hat{H}_v$.
So that $\hat{H}(a^2) \geq v$. Thus $(a_v)^2 \subseteq \hat{H}$. Since $(a_v)^2 = a_v^2$, so that $a_v^2 \subseteq \hat{H}$. Now, let a_s, b_l, r_k be F. singletons of R such that $a_s b_l r_k \subseteq \sqrt{\hat{H}}$. Then $(a_s\, b_l, r_k)^2 = a_s^2\, b_l^2\, r_k^2 \subseteq \hat{H}$. Since $\hat{H}$ is T-ABSO F. ideal, then either $a_s^2\, b_l^2 \subseteq \hat{H}$ or $a_s^2\, r_k^2 \subseteq \hat{H}$ or $b_l^2\, r_k^2 \subseteq \hat{H}$, since $(a_s b_l)^2 = a_s^2\, b_l^2$, $(a_s r_k)^2 = a_s^2\, r_k^2$, $(b_l r_k)^2 = b_l^2\, r_k^2$ hence either $(a_s b_l)^2 \subseteq \hat{H}$ or $(a_s r_k)^2 \subseteq \hat{H}$ or $(b_l r_k)^2 \subseteq \hat{H}$. So that either $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_k \subseteq \sqrt{\hat{H}}$ or $b_l r_k \subseteq \sqrt{\hat{H}}$. Thus $\sqrt{\hat{H}}$is T-ABSO F. ideal of R.

Lemma 4.

Let Ĥ⊆ Ƥ be F. ideal of a ring R, where Ƥ is a prime F. ideal. Then the following expressions are equivalent:

1- Ƥ is a minimal prime F. ideal of Ĥ;

2- For each F. singleton a_v⊆ Ƥ, there exists F. singleton b_l of R\Ƥ and a non-negative integer n such that $b_l a_v^n$ ⊆ Ĥ , ∀ v, l ∈ L.

Proof. (1) ⇒ (2) Let Ƥ be a minimal prime F. ideal of Ĥ and a_v ⊆ Ƥ, suppose that for every F. singleton b_l of R\Ƥ, $b_l a_v^n$ ⊈ Ĥ, ∀n∈ N. Inparticular, a_v^n ⊈ Ĥ, ∀n ∈N.

Let A={1, a_v, a_v^2 ,...} and B={K: K is F. ideal of R such that K∩A=∅, Ĥ⊆ K⊆ Ƥ}. Then B≠∅, since Ĥ⊆B, it is obvious B is partially ordered by inclusion. By [13], B has a maximal F. ideal say Ụ. Then Ụ is a prime F. ideal by [12], such that Ĥ⊆ Ụ⊆ Ƥ. Since Ƥ is a minimal prime F. ideal of Ĥ, so Ụ=Ƥ this is a contradiction to a_v⊆ Ƥ=Ụ, hence $b_l a_v^n$ ⊆ Ĥ.

(2) ⇒(1) Suppose that for each F. singleton a_v⊆ Ƥ, there exists F. singleton b_l of R\ Ƥ and n∈N such that $b_l a_v^n$ ⊆ Ĥ. Let K be a prime F. ideal of R such that Ĥ⊆ K⊆ Ƥ.

We claim that Ƥ⊆ K. Since a_v⊆ Ƥ , then there exists F. singleton b_l⊆ R\ Ƥ and n∈N such that $b_l a_v^n$ ⊆ Ĥ⊆ K. Since K is a prime F. ideal, then either b_l⊆ K or a_v^n ⊆ K. Hence a_v⊆ K as b_l⊆ R\ Ƥ. So that Ƥ⊆ K, then Ƥ=K; that is Ƥ is a minimal prime F. ideal of Ĥ.

Proposition 5.

Suppose that Ĥ is T-ABSO F. ideal of a ring R. Then there are at most two prime F ideals of R that are minimal over Ĥ.

Proof**.** Assume that K={$Ƥ_i$: $Ƥ_i$ is a prime F. ideal of R which is minimal over Ĥ}. Let K have at least three prime F. ideals. Let $Ƥ_1$, $Ƥ_2$ ∈ K be two different prime F. ideals. Then there exists F. singleton $a_s \subseteq Ƥ_1 \backslash Ƥ_2$ and there exists F singleton $b_l \subseteq Ƥ_2 \backslash Ƥ_1$.

We show that $a_s b_l$ ⊆ Ĥ. By lemma (4), there exist F. singletons $x_v \not\subseteq Ƥ_1$ and $y_h \not\subseteq Ƥ_2$, such that $x_v a_s^n$ ⊆ Ĥ and $y_h b_l^m$ ⊆ Ĥ for some n, m ≥1. Since Ĥ is T-ABSO F. ideal of R, we have $x_v a_s$ ⊆ Ĥ and $y_h b_l$ ⊆ Ĥ. Since $a_s , b_l \not\subseteq Ƥ_1 \cap Ƥ_2$ and $x_v a_s$, $y_h b_l$ ⊆ Ĥ ⊆ $Ƥ_1 \cap Ƥ_2$, we get $x_v \subseteq Ƥ_2 \backslash Ƥ_1$ and $y_h \subseteq Ƥ_1 \backslash Ƥ_2$, thus $x_v , y_h \not\subseteq Ƥ_1 \cap Ƥ_2$. Since $x_v a_s$ ⊆ Ĥ and $y_h b_l$ ⊆ Ĥ, have $(x_v + y_h) a_s b_l$ ⊆ Ĥ. Observe that $(x_v + y_h) \not\subseteq Ƥ_1$ and $(x_v + y_h) \not\subseteq Ƥ_2$. Since $(x_v + y_h) a_s \not\subseteq Ƥ_2$ and $(x_v + y_h) b_l \not\subseteq Ƥ_1$, we conclude that neither $(x_v + y_h) a_s$ ⊆ Ĥ nor $(x_v + y_h) b_l$ ⊆ Ĥ and hence $a_s b_l$ ⊆ Ĥ. Now, suppose there exists $Ƥ_3$∈ K such that $Ƥ_3$ is neither $Ƥ_1$ nor $Ƥ_2$.Then we can choose $r_k \subseteq Ƥ_1 \backslash (Ƥ_2 \cup Ƥ_3)$, $c_n \subseteq Ƥ_2 \backslash (Ƥ_1 \cup Ƥ_3)$ and $d_m \subseteq Ƥ_3 \backslash (Ƥ_1 \cup Ƥ_2)$. By the same way we show that $r_k c_n$ ⊆ Ĥ. Since Ĥ⊆$Ƥ_1 \cap Ƥ_2 \cap Ƥ_3$ and $r_k c_n$ ⊆ Ĥ, we get either $r_k \subseteq Ƥ_3$ or $c_n \subseteq Ƥ_3$ this is a discrepancy. Hence K has at most two prime F. ideals of R.

Proposition 6

Let Ĥ be T-ABSO F. ideal of R. Then one of the following expressions must hold

1- $\sqrt{Ĥ}$ = Ƥ is a prime F. ideal of R such that $Ƥ^2$ ⊆ Ĥ

2- $\sqrt{Ĥ}$= $Ƥ_1 \cap Ƥ_2$, $Ƥ_1 Ƥ_2$ ⊆ Ĥ, and $(\sqrt{Ĥ})^2$ ⊆ Ĥ where $Ƥ_1, Ƥ_2$ are the only distinct prime F. ideals of R that are minimal over Ĥ.

Proof. By proposition (5), we get either $\sqrt{\hat{H}}$=Ƥ is a prime F. ideal of R or $\sqrt{\hat{H}}$ = $Ƥ_1 \cap Ƥ_2$, where $Ƥ_1, Ƥ_2$ are the only distinct prime F. ideals of R that are minimal over Ĥ. Assume that $\sqrt{\hat{H}}$ = Ƥ is prime F. ideal of R. Let F. singletons $a_s, b_l \subseteq Ƥ$. By proposition (3), we have $a_s^2, b_l^2 \subseteq \hat{H}$. So that $a_s(a_s + b_l)b_l \subseteq \hat{H}$. Since Ĥ is T-ABSO F. ideal, we get $a_s(a_s + b_l) = a_s^2 + a_s b_l \subseteq \hat{H}$ or $(a_s + b_l)b_l = a_s b_l + b_l^2 \subseteq \hat{H}$ or $a_s b_l \subseteq \hat{H}$. From each case implies that $a_v b_l \subseteq \hat{H}$, and so $Ƥ^2 \subseteq \hat{H}$. Suppose that $\sqrt{\hat{H}}$ = $Ƥ_1 \cap Ƥ_2$, where $Ƥ_1, Ƥ_2$ are the only distinct prime F. ideals of R that are minimal over Ĥ. Let F singletons $a_s, b_l \subseteq \sqrt{\hat{H}}$. By the same way of above, we have $a_s b_l \subseteq \hat{H}$ and hence $(\sqrt{\hat{H}})^2 \subseteq \hat{H}$. Now, we show that, $Ƥ_1 Ƥ_2 \subseteq \hat{H}$. By proposition (3), we have $x_v^2 \subseteq \hat{H}$ for each F singleton $x_v \subseteq \sqrt{\hat{H}}$. Let be F singleton $y_h \subseteq Ƥ_1 \backslash Ƥ_2$ and $r_k \subseteq Ƥ_2 \backslash Ƥ_1$. By the proof of proposition (5), we have $y_h r_k \subseteq \hat{H}$. Let F singletons $c_n \subseteq \sqrt{\hat{H}}$ and $d_m \subseteq Ƥ_2 \backslash Ƥ_1$, choose F singleton $f_u \subseteq Ƥ_1 \backslash Ƥ_2$. Then $f_u d_m \subseteq \hat{H}$ by the proof of proposition (5) and $(c_n + f_u) \subseteq Ƥ_1 \backslash Ƥ_2$. Thus $c_n d_m + f_u d_m = (c_n + f_u)d_m \subseteq \hat{H}$. So that $c_n d_m \subseteq \hat{H}$. By the same method we show that if $c_n \subseteq \sqrt{\hat{H}}$ and $d_m \subseteq Ƥ_1 \backslash Ƥ_2$ .Thus $Ƥ_1 Ƥ_2 \subseteq \hat{H}$.

Proposition 7

Let Ĥ be T-ABSO F. ideal of R such that $\sqrt{\hat{H}}$ = Ƥ is a prime F. ideal, of Rand suppose that Ĥ≠Ƥ. For each F. singleton $a_v \subseteq P \backslash \hat{H}$, let $A_{a_v} = \{b_l \subseteq R: b_l a_v \subseteq \hat{H}\}$, ∀ *v, l* ∈L. Then A_{a_v} is a prime F. ideal of R included Ƥ. Futhermore, either $A_{b_l} \subseteq A_{a_v}$ or $A_{a_v} \subseteq A_{b_l}$ for each F. singletons $a_v, b_l \subseteq Ƥ \backslash \hat{H}$.

Proof. Let $a_v \subseteq Ƥ \backslash \hat{H}$. Since $Ƥ^2 \subseteq \hat{H}$ by proposition (6), we have Ƥ⊆ A_{a_v}. Assume that Ƥ≠A_{a_v} and $b_l r_k \subseteq A_{a_v}$ for some F. singleton b_l, r_k of R. Since Ƥ⊆ A_{a_v}, we may suppose that $b_l \nsubseteq Ƥ$ and $r_k \nsubseteq Ƥ$, hence $b_l r_k \nsubseteq \hat{H}$. Since $b_l r_k \subseteq A_{a_v}$ we have $b_l r_k a_v \subseteq \hat{H}$. Since Ĥ is T-ABSO F. ideal of R and $b_l r_k \nsubseteq \hat{H}$, we have either $b_l a_v \subseteq \hat{H}$ or $r_k a_v \subseteq \hat{H}$, thus either $b_l \subseteq A_{a_v}$ or $r_k \subseteq A_{a_v}$. Hence A_{a_v} is a prime F. ideal of R included Ƥ. Now, let $a_v, b_l \subseteq Ƥ \backslash \hat{H}$ for F. singletons a_v, b_l of R and assume that F singleton $r_k \subseteq A_{a_v} \backslash A_{b_l}$. Since Ƥ ⊆ A_{b_l}, so $r_k \subseteq A_{a_v} \backslash Ƥ$. We show that $A_{b_l} \subseteq A_{a_v}$. Let F singleton x_s of R such that $x_s \subseteq A_{b_l}$. Since Ƥ⊆ A_{a_v} , we may suppose that $x_s \subseteq A_{b_l}$ Ƥ. Since $r_k \nsubseteq Ƥ$ and $x_s \nsubseteq Ƥ$, we have $r_k x_s \nsubseteq \hat{H}$. Since $r_k(a_v + b_l)x_s \subseteq \hat{H}$ and $r_k x_s, r_k b_l \nsubseteq \hat{H}$, we have $(a_v + b_l)x_s \subseteq \hat{H}$. Hence $a_v x_s \subseteq \hat{H}$ since $(a_v + b_l)x_s \subseteq \hat{H}$ and $x_s b_l \subseteq \hat{H}$. Hence $x_s \subseteq A_{a_v}$. So that $A_{b_l} \subseteq A_{a_v}$.

Proposition 8. Assume that Ĥ is F. ideal of R such that Ĥ≠$\sqrt{\hat{H}}$ and $\sqrt{\hat{H}}$ is a prime F. ideal of R. Then the following expressions are equivalent:

1- Ĥ is T-ABSO F. ideal of R;

2- $A_{a_v} = \{b_l \subseteq R: b_l a_v \subseteq \hat{H}\}$, ∀ *v,l* ∈L, is a prime F. ideal of R for each F. singleton $a_v \subseteq \sqrt{\hat{H}} \backslash \hat{H}$.

Proof. (1) ⇒ (2) This is obvious by proposition (7).

(2)⇒(1) Assume that $a_v b_l r_k \subseteq \hat{H}$ for F. singletons a_v, b_l, r_k of R.

Since $\sqrt{\hat{H}}$ is a prime F. ideal of R, we may suppose that $a_v \subseteq \sqrt{\hat{H}}$.

If $a_v \subseteq \hat{H}$, then $a_v b_l \subseteq \hat{H}$. Thus suppose that $a_v \subseteq \sqrt{\hat{H}})\backslash\hat{H}$. Hence $b_l r_k \subseteq A_{a_v}$. But A_{a_v} is a prime F. ideal of R , then by proposition (7), either $b_l a_v \subseteq \hat{H}$ or $r_k a_v \subseteq \hat{H}$. Thus $\hat{H}$ is T-ABSO F. ideal of R.

Proposition 9.

Assume that $\hat{H}$ is a non-constant proper F. ideal of a ring R. Then the following expressions are equivalent:

1- $\hat{H}$ is T-ABSO F. ideal of R;

2- If $\underset{\wedge}{U}KT \subseteq \hat{H}$ for F. ideals $\underset{\wedge}{U}, K, T$ of R, $\underset{\wedge}{U}K \subseteq \hat{H}$ or $KT \subseteq \hat{H}$ or $\underset{\wedge}{U}T \subseteq \hat{H}$.

Proof. (1)⇒(2) Assume that $\underset{\wedge}{U}KT \subseteq \hat{H}$ for F. ideals $\underset{\wedge}{U}$, K, T of R. By proposition (5), we have $\sqrt{\hat{H}}$ is a prime F. ideal of R or $\sqrt{\hat{H}} = Ƥ_1 \cap Ƥ_2$ where $Ƥ_1, Ƥ_2$ are non-constant distinct prime F. ideals of R that are minimal over $\hat{H}$. If $\hat{H}=\sqrt{\hat{H}}$, then it is readily showed that, $\underset{\wedge}{U}K \subseteq \hat{H}$ or $KT \subseteq \hat{H}$ or $\underset{\wedge}{U}T \subseteq \hat{H}$. Thus suppose that $\hat{H} \neq \sqrt{\hat{H}}$. We see the following:

(1) Assume that $\sqrt{\hat{H}}$ is a prime F. ideal of R. Then we perhaps suppose that $\underset{\wedge}{U} \subseteq \sqrt{\hat{H}}$ and$\underset{\wedge}{U} \nsubseteq \hat{H}$. Let F. singleton a_v of R such that $a_v \subseteq \underset{\wedge}{U}\backslash\hat{H}$. Since $a_v\, KT \subseteq \hat{H}$, we have $KT \subseteq A_{a_v}$ where $A_{a_v} = \{b_l \subseteq R: b_l a_v \subseteq \hat{H}\}$. Since A_{a_v} is a prime F. ideal of R by proposition (8), we have either $K \subseteq A_{a_v}$ or $T \subseteq A_{a_v}$. If $K \subseteq A_{x_s}$ and $T \subseteq A_{x_s}$ for each F. singleton $x_s \subseteq \underset{\wedge}{U}\backslash\hat{H}$, then $\underset{\wedge}{U}K \subseteq \hat{H}$ (and $\underset{\wedge}{U}T \subseteq \hat{H}$) and we are finished. Hence suppose that $K \subseteq A_{r_k}$and $T \nsubseteq A_{r_k}$ for some F. singleton $r_k \subseteq \underset{\wedge}{U}\backslash\hat{H}$. Since $\{A_{w_h}: w_h \subseteq \underset{\wedge}{U}\backslash\hat{H}\}$, is a set of prime F. ideals of R that are linearly ordered by proposition (7), since $K \subseteq A_{r_k}$ and $T \nsubseteq A_{r_k}$, we have $K \subseteq A_{z_n}$ for some F. singleton $z_n \subseteq \underset{\wedge}{U}\backslash\hat{H}$. Thus$\underset{\wedge}{U}K \subseteq \hat{H}$.

(2) Assume that $\sqrt{\hat{H}} = Ƥ_1 \cap Ƥ_2$ where $Ƥ_1, Ƥ_2$ are non-constant distinct prime F. ideals of R that are minimal over $\hat{H}$. We suppose that $\underset{\wedge}{U} \subseteq Ƥ$. If either $K \subseteq Ƥ_2$ or $T \subseteq Ƥ_2$, then either $\underset{\wedge}{U}K \subseteq \hat{H}$ or $\underset{\wedge}{U}T \subseteq \hat{H}$ because $Ƥ_1 Ƥ_2 \subseteq \hat{H}$ by proposition (6). Hence suppose that $\underset{\wedge}{U} \subseteq \sqrt{\hat{H}}$ and$\underset{\wedge}{U} \nsubseteq \hat{H}$. By the same way in (1) and by proposition (7), we are finished from this proof.

(2) ⇒ (1) it is trivial. Now, we give the concept of T-ABSO quasi primary F. ideal as follows:

Definition 10.

A proper F. ideal $\hat{H}$ of R is called T-ABSO quasi primary F. ideal of R if $\sqrt{\hat{H}}$ is T-ABSO F. ideal of R.

Proposition 11.

A proper F. ideal $\hat{H}$ of R is T-ABSO quasi primary F. of R iff whenever for each F. singleton a_s, b_l, r_h of R, ∀*s, l, h* ∈ L, such that $a_s b_l r_h \subseteq \hat{H}$, then $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_h \subseteq \sqrt{\hat{H}}$ or $b_l r_h \subseteq \sqrt{\hat{H}}$.

Proof. (⇐) Suppose that $\hat{H}$ is a proper F. ideal of R and whenever for each F. singleton a_s, b_l, r_h of R, such that $a_s b_l r_h \subseteq \hat{H}$, then $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_h \subseteq \sqrt{\hat{H}}$ or $b_l r_h \subseteq \sqrt{\hat{H}}$. Let $a_s b_l r_h \subseteq \sqrt{\hat{H}}$, $a_s r_h \nsubseteq \sqrt{\hat{H}}$ and $b_l r_h \nsubseteq \sqrt{\hat{H}}$. Since $a_s b_l r_h \subseteq \sqrt{\hat{H}}$, then there exists $n \in Z^+$ such tha $(a_s b_l r_h)^n = a_s^n b_l^n r_h^n \subseteq \hat{H}$. Since $a_s^n r_h^n \nsubseteq \hat{H}$ and $b_l^n r_h^n \nsubseteq \hat{H}$, then we have $a_s^n b_l^n = (a_s b_l)^n \subseteq \hat{H}$. So that $a_s b_l \subseteq \sqrt{\hat{H}}$. Thus $\sqrt{\hat{H}}$ is T-ABSO F. ideal of R and so that $\hat{H}$ is T-ABSO quasi primary F. of R.
(⇒) Let $\hat{H}$ be T-ABSO quasi primary F. ideal of R and for each F. singleton a_s, b_l, r_h of R, such that $a_s b_l r_h \subseteq \hat{H}$. Since $\hat{H} \subseteq \sqrt{\hat{H}}$ and $\sqrt{\hat{H}}$ is T-ABSO F. ideal of R. So that $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_h \subseteq \sqrt{\hat{H}}$ or $b_l r_h \subseteq \sqrt{\hat{H}}$. The proposition specificities T-ABSO quasi primary F. ideal in terms of its level ideal is given as follow

Proposition 12.

A F. ideal $\hat{H}$ of R is T-ABSO quasi primary F. iff the level ideal $\hat{H}_v$ is T-ABSO quasi primary ideal of R, ∀ $v \in$ L.

Proof. (⟹) Let $abr \in \hat{H}_v$ for each $a, b, r \in$ R then $\hat{H}(abr) \geq v$ hence $(abr)_v \subseteq \hat{H}$. So that $a_s b_l r_k \subseteq \hat{H}$ where $v = \min\{s, l, k\}$. Since $\hat{H}$ is T-ABSO quasi primary F., then either $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_k \subseteq \sqrt{\hat{H}}$ or $b_l r_k \subseteq \sqrt{\hat{H}}$ hence either $(ab)_v \subseteq \sqrt{\hat{H}}$ or $(ar)_v \subseteq \sqrt{\hat{H}}$ or $(br)_v \subseteq \sqrt{\hat{H}}$ and so $ab \in \sqrt{\hat{H}_v}$ or $ar \in \sqrt{\hat{H}_v}$ or $br \in \sqrt{\hat{H}_v}$. Thus $\hat{H}_v$ is T-ABSO quasi primary ideal of R.
(⟸) Let $a_s b_l r_k \subseteq \hat{H}$ for F. singletons a_s, b_l, r_k of R, ∀ $s, l, k \in$ L. Hence $(abr)_v \subseteq A$, where $v = \min\{s, l, k\}$, so that $\hat{H}(abr) \geq v$ and $abr \in \hat{H}_v$. But $\hat{H}_v$ is T-ABSO quasi primary ideal then either $ab \in \sqrt{\hat{H}_v}$ or $ar \in \sqrt{\hat{H}_v}$ or $br \in \sqrt{\hat{H}_v}$, hence either $(ab)_v \subseteq \sqrt{\hat{H}}$ or $(ar)_v \subseteq \sqrt{\hat{H}}$ or $(br)_v \subseteq \sqrt{\hat{H}}$. So that either $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_k \subseteq \sqrt{\hat{H}}$ or $b_l r_k \subseteq \sqrt{\hat{H}}$. Thus $\hat{H}$ is T-ABSO quasi primary F. ideal of R. The following theorem gives a characterization of T-ABSO quasi primary F. ideal.

Theorem 13.

Let $\hat{H}$ be a proper F. ideal of R. Then $\hat{H}$ is T-ABSO quasi primary F. ideal iff whenever $\underset{\wedge}{U}KT \subseteq \hat{H}$ for some F. ideals $\underset{\wedge}{U}$, K, T of R, then $\underset{\wedge}{U}K \subseteq \sqrt{\hat{H}}$ or $\underset{\wedge}{U}T \subseteq \sqrt{\hat{H}}$ or $KT \subseteq \sqrt{\hat{H}}$.**Proof.**
(⇐) Assume that $\underset{\wedge}{U}KT \subseteq \hat{H}$ for some F. ideals $\underset{\wedge}{U}$, K, T of R, then $\underset{\wedge}{U}K \subseteq \sqrt{\hat{H}}$ or $\underset{\wedge}{U}T \subseteq \sqrt{\hat{H}}$ or $KT \subseteq \sqrt{\hat{H}}$ and let $a_s b_l r_k \subseteq \hat{H}$ for F. singleton a_s, b_l, r_k of R . Hence $< a_s >< b_l >< r_k > \subseteq \hat{H}$ and so that $< a_s >< b_l > \subseteq \sqrt{\hat{H}}$ or $< a_s >< r_k > \subseteq \sqrt{\hat{H}}$ or $< b_l >< r_k > \subseteq \sqrt{\hat{H}}$. Then $a_s b_l \subseteq \sqrt{\hat{H}}$ or $a_s r_k \subseteq \sqrt{\hat{H}}$ or $b_l r_k \subseteq \sqrt{\hat{H}}$. By proposition (11), then $\hat{H}$ is T-ABSO quasi primary F. ideal of R.
(⇒) Assume that $\hat{H}$ is T-ABSO quasi primary F. ideal of R and $\underset{\wedge}{U}KT \subseteq \hat{H}$ for some F. ideals $\underset{\wedge}{U}, K, T$ of R, then $\underset{\wedge}{U}KT \subseteq \sqrt{\hat{H}}$. Since $\sqrt{\hat{H}}$ is T-ABSO F. ideal of R, then $\underset{\wedge}{U}K \subseteq \sqrt{\hat{H}}$ or $\underset{\wedge}{U}T \subseteq \sqrt{\hat{H}}$ or $KT \subseteq \sqrt{\hat{H}}$ by proposition (9).

3. T-ABSO F. Subm.

In this section we present the concept of T-ABSO F. subm. and we introduce many basic properties and results about this concept.

Definition 14.

Let X be F. M. of an R-M. $\acute{M}$. A proper F. subm. A of X is called T-ABSO F. subm. if whenever a_s , b_l be F. singletons of R, and $x_v \subseteq X$, $\forall\, s, l, v \in L$ such that $a_s b_l x_v \subseteq A$, then either $a_s b_l \subseteq (A{:}_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$, see [7].

The proposition specificities T-ABSO F. subm. in terms of its level subm. is given as follow:

Proposition 15.

Let A be T-ABSO F. subm. of F. M. X of an R-M. $\acute{M}$., iff the level subm. A_v is T-ABSO subm. of X_v, for all $v \in$ L, see[7].

Remarks and Examples

1. The intersection of two distinct prime F. subms. of F. M. X of an R-M, $\acute{M}$ is T-ABSO F. subm.

Proof. Let A and B be two distinct prime F. subms. of X. Suppose that F. singletons $a_s, b_l$of R, $x_v \subseteq X$ such that $a_s b_l x_v \subseteq A \cap B$, but $a_s x_v \nsubseteq A \cap B$ and $b_l x_v \nsubseteq A \cap B$. Then $a_s x_v \nsubseteq A$, $b_l x_v \nsubseteq A$, $a_s x_v \nsubseteq B$ and $b_l x_v \nsubseteq B$ these are impossible since A and B are prime F. subms. So suppose that $a_s x_v \nsubseteq A$ and $b_l x_v \nsubseteq B$. Since $a_s b_l x_v \subseteq A$ and $a_s b_l x_v \subseteq B$, then $b_l \subseteq (A{:}_R X)$ and $a_s \subseteq (B{:}_R X)$. So that $a_s b_l \subseteq (A{:}_R X) \cap (B{:}_R X) = (A \cap B{:}_R X)$. Thus $A \cap B$ is T-ABSO F. subm. of X. (2). Every prime F. subm. is T-ABSO F. subm. **Proof.** Let A be a prime F. subm. of F. M. X of an R-M. $\acute{M}$. Let$a_s b_l x_k \subseteq A$ for F. singletons a_s, b_l of R and $x_k \subseteq X$. Then $(abx)_v \subseteq A$ where $v =$ min{s, l, k }. Since A is a proper subm. of X then A_v is a proper subm. of X_v, hence A_v is prime subm. of X_v. So that A_v is T-ABSO subm. (see [14]), hence $ab \in (A{:}_R X)_v$, then either $(ab)_v \subseteq (A{:}_R X)$ or $(ax)_v \subseteq A$ or $(bx)_v \subseteq A$. So either $ab \in (A_v{:}_R X_v)$ or $ax \in A_v$ or $bx \in A_v$.
Since $(A_v{:}_R X_v) = (A{:}_R X)_v$ by [5]. So that Then either $a_s b_l \subseteq (A{:}_R X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Thus A is T-ABSO F. subm. of X. However, the converse incorrect in general, for example:

Let $X: Z_{24} \to L$ such that $X(y) = \begin{cases} 1 & if\ y \in Z_{24} \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z_{24} as Z-M.

Let $A: Z_{24} \to L$ such that $A(y) = \begin{cases} v & if\ y \in (\bar{6}) \\ 0 & o.w. \end{cases} \forall\, v \in L$

It is obvious that A is F. subm. of X. Now $A_v = (\bar{6})$ is not prime subm. of Z_{24} ,since $2.\bar{3} \in (\bar{6})$ but $\bar{3} \notin (\bar{6})$ and $2 \notin ((\bar{6}){:}_z Z_{24}) = 6Z$. But $(\bar{6}) = (\bar{2}) \cap (\bar{3})$ is T-ABSO subm.of Z_{24} as Z-M. by [14]. So A_v is T-ABSO subm., but not prime subm.,implies that A is T-ABSO F. subm., but not prime F. subm. (3) It obvious every quasi-prime F. subm. is T-ABSO F. subm. However T-ABSO F. subm. may not be quasi-prime F.
subm. for example:

Let X:Z→L such that $X(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z-M. Z.

Let A: Z→L such that $A(y) = \begin{cases} v & if\ y \in 6Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A is F. subm. of X.

A_v=6Z is T-ABSO subm. of Z, since if x, y, z∈ Z and xyz∈ 6Z= A_v then at least one of x, y and z is even or one of them is 6. Then either xy∈ A_v or xz∈ A_v or yz∈ A_v. But 6Z= A_v is not quasi-prime, since 2.3.1∈6Z, but 2.1∉6Z and 3.1∉6Z. So that A is T-ABSO F. subm., but A is not quasi-prime F. subm. (4) Let A, B be two F. subms. of F. M. X of an R-M. $\acute{M}$, and B⊂A. If A is T-ABSO F. subm. of X, then it is not necessary that B is a T-ABSO F. subm., for example:

Let X: $Z_{24} \rightarrow L$ such that $X(y) = \begin{cases} 1 & if\ y \in\ Z_{24} \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z-M. Z_{24} .

Let A: $Z_{24} \rightarrow L$ such that $A(y) = \begin{cases} v & if\ y \in (\bar{2}) \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

And B: $Z_{24} \rightarrow L$ such that $B(y) = \begin{cases} v & if\ y \in (\overline{12}) \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A and B are F. subms. of X.

Now, A_v =$(\bar{2})$ and B_v =$(\overline{12})$ where $B_v \subset A_v$, since A_v =$(\bar{2})$ is maximal subm. of Z_{24} as Z-M., then A_v is prime subm. by[15]. Implies that A_v is T-ABSO subm. by [14]. But $2.2.\bar{3} \in B_v$, $2.\bar{3} \notin B_v$ and 2.2=4$\notin(B_v{:_z}\,Z_{24}) = 12Z$. Thus B_v is not T-A BSO subm. Of Z_{24} as Z-M. hence B is not T-ABSO F. subm. (5) Let A and B be F. subms. of F. M. X of an R-M . $\acute{M}$ and A⊂B. If A is T-ABSO F . subm. of X, then A is T-ABSO F. subm. of B. **Proof.** If B =X, then don´t need to prove. Let $a_s b_l x_k \subseteq A$ for F. singletons a_s, b_l of R and $x_k \subseteq B$, implies $(abx)_v \subseteq A$ hence $v = \min\{s, l, k\}$ abx∈A_v, where a,b∈R, x∈B_v. Since A⊂B implies where $A_v \subset B_v$. Since A is T-ABS O F. subm. of X, then A_v is T-ABSO subm. Of X_v. Hence A_v is T-ABSO subm. Of B_v by [14], so that either $ab \in (A_v{:_R}\,B_v) \rightarrow ab \in (A{:_R}\,B)_v$ or $ax \in A_v$ or $bx \in A_v$, then $(ab)_v \subseteq (A{:_R}\,B)$ or $(ax)_v \subseteq A$ or $(bx)_v \subseteq A$, implies either $a_s b_l \subseteq (A{:_R}\,B)$ of T-A or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Thus A is T-ABSO F. subm. of B. (6) The sum BSO F. subm. is not necessary T-ABSO F. subm., for example:

Let X: Z→L such that $X(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z-M. Z.

Let A: Z→L such that $A(y) = \begin{cases} v & if\ y \in 2Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A is F. subm. of X.

Let B: Z→L such that $B(y) = \begin{cases} v & if\ y \in 3Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that B is F. subm. of X. Now, A_v=2Z , , B_v=3Z where A_v and B_v be T- ABSO subms. of Z-M. Z, but $A_v + B_v = Z = X_v$ is not T-ABSO subm., implies that A+B=X is not T-ABSO F. subm. (7) Let A and B be two F. subms. of F. M. X of an R-M. $\acute{M}$. If A is T-ABSO F. subm. then it is not necessary that B is T-ABSO F. subm., for example:

Let X: Z→L such that $X(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z-M. Z.

Let A: Z→L such that $A(y) = \begin{cases} v & if\ y \in 12Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

Let B: Z→L such that $B(y) = \begin{cases} v & if\ y \in 10Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A and B are F. subms. of X . Now, A_v=2Z , B_v=20Z where A_v is T- ABSO subm. of Z as Z-M., but 2Z≅20Z and B_v=20Z is not T-ABSO subm. of Z as Z-M. since 2.2.5∈B_v=20Z, but 2.5∉ B_v=20Z and 2.2∉ B_v=20Z. Thus A≅B where A is T- ABSO F. subm. of X and B is not T-ABSO F. subm. of X. (8) The intersection of two T-ABSO F. subms. need not be T-ABSO F. subm., for example:

Let X: Z→L such that $X(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z-M. Z.

Let A: Z→L such that $A(y) = \begin{cases} v & if\ y \in 12Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

Let B: Z→L such that $B(y) = \begin{cases} v & if\ y \in 10Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A and B are F. subms. of X . A_v=12Z , B_v=10Z are T-ABSO subms. in the Z as Z-M. But $A_v \cap B_v$=12Z∩10Z=120Z which is not T-ABSO since 2.6.10 ∈120Z, but 2.10∉120Z and 6.10∉120Z and 2.6∉120Z. Hence A and B subms., but A∩B is not T- ABSO F. subm (9) Let A be T-ABSO are two T-ABSO F. subm. of F. M. X of an R-M. $\acute{M}$. Then for each B⊆X, either B⊆A or B∩A is T-ABSO F. subm. of B.

Proof. Assume that B⊈A then B∩A⊊B Let a_s, b_l be F. singletons of R and $x_k \subseteq B$, such that $a_s b_l x_k \subseteq B \cap A$, implies $a_s b_l x_k \subseteq A$. Since A is T-ABSO F. subm., thus either $a_s b_l \subseteq (A{:_R}\, X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Then either $a_s b_l \subseteq (B \cap A{:_R}\, B)$ or $a_s x_k \subseteq B \cap A$ or $b_l x_k \subseteq B \cap A$. Thus B∩A is T-ABSO F. subm. of B.

Proposition 17.

Let $f\colon \acute{M}_1 \longrightarrow \acute{M}_2$ be an epimorphism, where X_1, X_2 are F. M. of R- M. $\acute{M}_1$ and $\acute{M}_2$ resp. If B is T-ABSO F. subm. of X_2, then $f^{-1}(B)$ is T-ABSO F. subm. of X_1. **Proof.** Since B is F. subm. of X_2, then $f^{-1}(B)$ is F. subm. of X_1, since f is epimorphism. Let $a_s b_l x_k \subseteq f^{-1}(B)$ for F. singletons a_s , b_l of R and $x_k \subseteq X_1$. Then $a_s b_l f(x_k) \subseteq B$ and since B is TABSO F. subm., then either $a_s f(x_k) \subseteq B$ or $b_l f(x_k) \subseteq B$ or $a_s b_l \subseteq (B{:_R}\, X_2)$. Hence either $a_s x_k \subseteq f^{-1}(B)$ or $b_l f(x_k) \subseteq f^{-1}(B)$ or $a_s b_l X_2 \subseteq B$. But $f(X_1) \subseteq X_2$, so that $a_s b_l f(X_1) \subseteq B$, hence $a_s b_l X_1 \subseteq f^{-1}(B)$, implies $a_s b_l \subseteq (f^{-1}(B){:_R}\, X_1$ Thus $f^{-1}(B)$ is T-ABSO F. subm. of X_1

Proposition 18.

Let $f\colon \acute{M}_1 \longrightarrow \acute{M}_2$ be an epimorphism, and X_1, X_2 are F. M. of R-M. $\acute{M}_1$ and $\acute{M}_2$ resp. Let $A \subseteq X_1$ such that F-$ker\ f \subseteq A$. Then A is T-ABSO F. subm. of X_1 iff $f(A)$ is T-ABSO F. subm. of X_2 .

Proof.(⇒) Let a_s, b_l be F. singletons of R and $y_h \subseteq X_2$ where $y_h = f(x_k)$ for some F. singleton $x_k \subseteq X_1$, such that $a_s b_l y_h \subseteq f(A)$. Hence $a_s b_l f(x_k) \subseteq f(A)$ $a_s b_l f(x_k) \subseteq f(A)$ since f is onto. Then $a_s b_l f(x_k) = f(z_n)$ for some F. singleton $z_n \subseteq A$. So that $f(a_s b_l x_k) =$

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$f(z_n)$, hence $f(a_s b_l x_k) - f(z_n) = 0_1$; that is $f(a_s b_l x_k - z_n) = 0_1$, implies $a_s b_l x_k - z_n \subseteq F - kerf \subseteq A$.
So that $a_s b_l x_k \subseteq A$. Since A is T-ABSO F. subm., then either $a_s b_l \subseteq (A:_R X_1)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Hence either $a_s b_l X_1 \subseteq A \rightarrow f(a_s b_l X_1) \subseteq f(A)$ or $f(a_s x_k) \subseteq f(A)$ or $f(b_l x_k) \subseteq f(A)$, implies either $a_s b_l f(X_1) \subseteq f(A) \rightarrow a_s b_l X_2 \subseteq f(A)$ or $a_s f(x_k) \subseteq f(A)$ or $b_l f(x_k) \subseteq f(A)$. Then either $a_s b_l \subseteq (f(A):_R X_2)$ or $a_s y_h \subseteq f(A)$ or $b_l y_h \subseteq f(A)$. Thus $f(A)$ is T-ABSO F. subm. of X_2.
($\Leftarrow$) Let $a_s b_l x_k \subseteq A$ for F. singletons a_s, b_l of R and $x_k \subseteq X_1$. Hence $f(a_s b_l x_k) \subseteq f(A)$, implies $a_s b_l f(x_k) \subseteq f(A)$. But $f(A)$ is T-ABSO F. subm., then either $a_s b_l \subseteq (f(A):_R X_2)$ or $a_s f(x_k) \subseteq f(A)$ or $b_l f(x_k) \subseteq f(A)$.
If $a_s b_l \subseteq (f(A): X_2)$, then $a_s b_l X_2 \subseteq f(A)$, implies $a_s b_l f(X_1) \subseteq f(A)$ since f is onto. Hence $f(a_s b_l X_1) \subseteq f(A)$, so that $a_s b_l X_1 \subseteq A$; that is $a_s b_l \subseteq (A:_R X_1)$. If $a_s f(x_k) \subseteq f(A)$ then $f(a_s x_k) = f(z_n)$ for some F. singleton $z_n \subseteq A$,$\forall$ $n \in$ L. Hence $f(a_s x_k) - f(z_n) = 0_1$, implies $a_s x_k - z_n \subseteq F - kerf \subseteq A$. So that $a_s x_k \subseteq A$. If $b_l f(x_k) \subseteq f(A)$, then by the same way above, we have $b_l x_k \subseteq A$. Therefore, A is T-ABSO F. subm. of X_1.

Proposition 19. Let A be a proper F. subm. of F. M. X of an R-M $\acute{M}$. Then A is T-ABSO F. subm. of X iff $a_s b_l B \subseteq A$ for F. singletons a_s, b_l of R and B is F . subm. of X implies $a_s b_l \subseteq (A:_R X)$ or $a_s B \subseteq A$ or $b_l B \subseteq A$.
Proof. ($\Rightarrow$) Let A be T-ABSO F. subm. and $a_s b_l B \subseteq A$. Assume that $a_s b_l \nsubseteq (A: X)$, $a_s B \nsubseteq A$ and $b_l B \nsubseteq A$. Then there exist F. singletons $x_v, y_k \subseteq B$, such that $a_s x_v \nsubseteq A$ and $b_l y_k \nsubseteq A$. Since $a_s b_l x_v \subseteq A$ and $a_s b_l \nsubseteq (A:_R X)$, $a_s x_v \nsubseteq A$, we have $b_l x_v \subseteq A$. Also since $a_s b_l y_k \subseteq A$ and $a_s b_l \nsubseteq (A:_R X)$, $b_l y_k \nsubseteq A$, we have $a_s y_k \subseteq A$. Now, since $a_s b_l (x_v + y_k) \subseteq A$ and $a_s b_l \nsubseteq (A:_R X)$, we have $a_s (x_v + y_k) \subseteq A$ or $b_l (x_v + y_k) \subseteq A$. If $a_s (x_v + y_k) \subseteq A$, then $(a_s x_v + a_s y_k) \subseteq A$ and since $a_s y_k \subseteq A$, we get $a_s x_v \subseteq A$, this is a discrepancy. If $b_l (x_v + y_k) \subseteq A$, then $(b_l x_v + b_l y_k) \subseteq A$ and since $b_l x_v \subseteq A$, we get $b_l y_k \subseteq A$ this is a discrepancy. Thus either $a_s b_l \subseteq (A:_R X)$ or $a_s B \subseteq A$ or $b_l B \subseteq A$.
($\Leftarrow$) It is obvious. The next theorem gives a characterization of T-ABSO F. subm.

Theorem 20.

Let A be a proper F. subm. of F. M. X of an R-M. $\acute{M}$. Then the following expressions are equivalent:

1- A is T-ABSO F. subm. of X;
2- If $\hat{H}\underset{\frown}{U}B \subseteq A$, for some F. ideals $\hat{H}$, $\underset{\frown}{U}$ of R and F. subm. B of X, then either $\hat{H}B \subseteq A$ or $\underset{\frown}{U}B \subseteq A$ or $\hat{H}\underset{\frown}{U} \subseteq (A:_R X)$.

Proof. (1)$\Rightarrow$(2) Suppose that A is T-ABSO F. subm. of X and $\hat{H}\underset{\frown}{U}B \subseteq A$ for some F. ideals $\hat{H}$, $\underset{\frown}{U}$ of R and some F. subm. B of X. Let $\hat{H}\underset{\frown}{U} \nsubseteq (A:_R X)$, to prove $\hat{H}B \subseteq A$ or $\underset{\frown}{U}B \subseteq A$. Assume that $\hat{H}B \nsubseteq A$ and $\underset{\frown}{U}B \nsubseteq A$, then there exist F. singletons $a_s \subseteq \hat{H}$ and $b_l \subseteq \underset{\frown}{U}$, such that $a_s B \nsubseteq A$ and $b_l B \nsubseteq A$. But $a_s b_l B \subseteq A$ and neither $a_s B \nsubseteq A$ nor $b_l B \nsubseteq A$ and A is T-ABSO F. subm., so that $a_s b_l \subseteq (A:_R X)$. Since $\hat{H}\underset{\frown}{U} \nsubseteq (A:_R X)$, then there exist F. singletons $x_v \subseteq \hat{H}$ and $y_k \subseteq \underset{\frown}{U}$, such that $x_v y_k \nsubseteq (A:_R X)$. But $x_v y_k B \subseteq A$, so that $x_v B \subseteq A$ or $y_k B \subseteq A$ by proposition (19).

Now we have the following:
(1) If $x_v B \subseteq A$ and $y_k B \nsubseteq A$, since $a_s y_k B \subseteq A$ and $y_k B \nsubseteq A$, $a_s B \nsubseteq A$, so that $a_s y_k \subseteq (A:_R X)$ by proposition (19). Since $x_v B \subseteq A$ and $a_s B \nsubseteq A$, hence $(a_s + x_v)B \nsubseteq A$. On the other hand, $(a_s + x_v)y_k B \subseteq A$ and neither $(a_s + x_v)B \subseteq A$ nor $y_k B \subseteq A$, we get $(a_s + x_v)y_k \subseteq (A:_R X)$ by proposition (19). But $(a_s + x_v)y_k = (a_s y_k + x_v y_k) \subseteq (A:_R X)$ and $a_s y_k \subseteq (A:_R X)$, we get $x_v y_k \subseteq (A:_R X)$ this is a discrepancy.
(2) If $y_k B \subseteq A$ and $x_v B \nsubseteq A$. By the same way of (1), we get a discrepancy.
(3) If $x_v B \subseteq A$ and $y_k B \subseteq A$. Since $y_k B \subseteq A$ and $b_l B \nsubseteq A$, we have $(b_l + y_k)B \nsubseteq A$. But $a_s(b_l + y_k)B \subseteq A$ and neither $a_s B \subseteq A$ nor $(b_l + y_k)B \subseteq A$. Thus $a_s(b_l + y_k) \subseteq (A:_R X)$ by proposition (19). Since $a_s b_l \subseteq (A:_R X)$ and $(a_s b_l + a_s y_k) \subseteq (A:_R X)$, we get $a_s y_k \subseteq (A:_R X)$. Since $(a_s + x_v)b_l B \subseteq A$ and neither $b_l B \subseteq A$ nor $(a_s + x_v)B \subseteq A$, we have$(a_s + x_v)b_l \subseteq (A:_R X)$ by proposition (19). But $(a_s + x_v)b_l = (a_s b_l + x_v b_l) \subseteq (A:_R X)$ and since $a_s b_l \subseteq (A:_R X)$, we have $x_v b_l \subseteq (A:_R X)$. Now, since $(a_s + x_v)(b_l + y_k)B \subseteq A$ and neither $(a_s + x_v)B \subseteq A$ nor $(b_l + y_k)B \subseteq A$, we get $(a_s + x_v)(b_l + y_k) \subseteq (A:_R X)$ by proposition (19), where$(a_s + x_v)(b_l + y_k) = (a_s b_l + a_s y_k + x_v b_l + x_v y_k) \subseteq (A:_R X)$. But $(a_s b_l + a_s y_k + x_v b_l) \subseteq (A:_R X)$, so that $x_v y_k \subseteq (A:_R X)$ this is a discrepancy. Thus $\hat{H}B \subseteq A$ or $\check{U}B \subseteq A$
(2) ⇒ (1) It is obvious.

Theorem 21.

If A is T-ABSO F. subm. of F. M. X of an R-M. $\acute{M}$, then $(A:_R X)$ is T-ABSO F. ideal of R.

Proof. Let $a_s b_l r_k \subseteq (A:_R X)$ for F. singletons a_s, b_l, r_k of R.
If $a_s r_k \nsubseteq (A:_R X)$ and $b_l r_k \nsubseteq (A:_R X)$, then there exist F. singletons $x_v, y_h \subseteq X \setminus A$, such that $a_s r_k x_v \nsubseteq A$ and $b_l r_k y_h \nsubseteq A$. Since $a_s b_l (r_k(x_v + y_h)) \subseteq A$ and A is T-ABSO F. subm., then either $a_s b_l \subseteq (A:_R X)$ or $a_s r_k(x_v + y_h) \subseteq A$ or $b_l r_k(x_v + y_h) \subseteq A$. If $a_s r_k(x_v + y_h) \subseteq A$ and since $a_s r_k x_v \nsubseteq A$, then we have $a_s r_k y_h \nsubseteq A$. So that $a_s b_l(r_k y_h) \subseteq A$ and $b_l r_k y_h \nsubseteq A$, hence $a_s b_l \subseteq (A:_R X)$. By the same method if $b_l r_k(x_v + y_h) \subseteq A$, we get $a_s b_l \subseteq (A:_R X)$.
Thus $(A:_R X)$ is T-ABSO F. ideal of R.

Theorem 22.

Let X be multiplication F. M. of an R-M. $\acute{M}$, and A is a proper F. subm. of X. If $(A:_R X)$ is T-ABSO F. ideal of R, then A is T-ABSO F. subm. of X.

Proof. Let $a_s b_l x_v \subseteq A$ for F. singletons a_s, b_l of R and $x_v \subseteq X$, then $a_s b_l < x_v > \subseteq A$. But $< x_v > = \hat{H}X$ for some F. ideal $\hat{H}$ of R since X is multiplication F. M., so that $a_s b_l \hat{H}X \subseteq A$. Thus $a_s b_l \hat{H} \subseteq (A:_R X)$, so we have that $< a_s >< b_l > \hat{H} \subseteq (A:_R X)$. Since $(A:_R X)$ is T-ABSO F. ideal of R, we get either $< a_s > \hat{H} \subseteq (A:_R X)$ or $< b_l > \hat{H} \subseteq (A:_R X)$ or $< a_s >< b_l > \subseteq (A:_R X)$ by
Proposition (9).
1) If $< a_s > \hat{H} \subseteq (A:_R X)$, then $< a_s > \hat{H}X \subseteq A$ and so $< a_s >< x_v > \subseteq A$. Hence $a_s x_v \subseteq A$
2) If $< b_l > \hat{H} \subseteq (A:_R X)$, then by the same method $b_l x_v \subseteq A$.
3) If $< a_s >< b_l > \subseteq (A:_R X)$, then $a_s b_l \subseteq (A:_R X)$.

By combining theorem (21) and theorem (22), we have the following corollary:

Corollary 23.

Let A be a proper F. subm. of a multiplication F. M. X of an R-M. $\acute{M}$. Then A is T-ABSO F. subm. of X iff $(A:_R X)$ is T-ABSO F. ideal of R.

Remark 24.

The condition X is multiplication F. M. can't be deleted from theorem (22). See the following example:

Let $X: Z_{p^\infty} \to L$ such that $X(y) = \begin{cases} 1 & if\ y \in Z_{p^\infty} \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z-M. Z_{p^∞} .

Let $A: Z_{p^\infty} \to L$ such that $A(y) = \begin{cases} v & if\ y \in (0) \\ 0 & o.w. \end{cases} \quad \forall\, v \in L$

It is obvious that A is F. subm. of X.

Now, $A_v = (0)$ is not T-ABSO subm. of $X_v = Z_{p^\infty}$, since $p^2 < \frac{1}{p^2} + Z >= (0)$ but $p < \frac{1}{p^2} + Z >\neq (0)$ and $p^2 \notin \left((0):_Z Z_{p^\infty}\right) = 0$. Note (0) is a prime ideal in Z , so that $\left((0):_Z Z_{p^\infty}\right) = 0$ is T-ABSO ideal in Z ; that is $(A_v:_R X_v)$ is T-ABSO ideal in Z, then $(A:_R X)$ is T-ABSO F. ideal in Z. Thus A is not T-ABSO F. subm. of X, but $(A:_R X)$ is T-ABSO F. ideal in Z.

Now, we gave the following theorem is a characterization of T-ABSO F. subm.

Theoerm 25.

Let A be a proper F. subm. of a multiplication F. M. X of $\acute{M}$. Then A is T-ABSO F. subm. of X iff $A_1A_2A_3 \subseteq A$ implies that $A_1A_2 \subseteq A$ or $A_1A_3 \subseteq A$ or $A_2A_3 \subseteq A$, where A_1, A_2, A_3 are F. subm. of X.

Proof. (⇒)Since X is a multiplication F., then $A_1 = \hat{H}X$, $A_2 = \underset{\wedge}{U}X$ and $A_3 = KX$ for some F. ideals $\hat{H}, \underset{\wedge}{U}$ and K of R . So that the product of A_1, A_2 and A_3 as follows: $A_1A_2A_3 = \hat{H}\underset{\wedge}{U}KX \subseteq A$. by [16]. Hence $\hat{H}\underset{\wedge}{U}K \subseteq (A:_R X)$. Since A is T-ABSO F. subm. of X, then$(A:_R X)$ is T-ABSO F. ideal by theorem (21). So by proposition (9), either $\hat{H}\underset{\wedge}{U} \subseteq (A:_R X)$ or $\hat{H}K \subseteq (A:_R X)$ or $\underset{\wedge}{U}K \subseteq (A:_R X)$. Hence either $\hat{H}\underset{\wedge}{U}X \subseteq A$ or $\hat{H}KX \subseteq A$ or $\underset{\wedge}{U}KX \subseteq A$, then $A_1A_2 \subseteq A$ or $A_1A_3 \subseteq A$ or $A_2A_3 \subseteq A$.

(⇐) Let $\hat{H}\underset{\wedge}{U}B \subseteq A$ for some F. ideals $\hat{H}, \underset{\wedge}{U}$ of R and B is F. subm. of X.

Since X is a multiplication F. M., then $B = EX$ for some F. ideal E of R. Then $\hat{H}\underset{\wedge}{U}EX \subseteq A$. Let $A_1 = \hat{H}X$ and $A_2 = \underset{\wedge}{U}X$, so that $A_1A_2B = \hat{H}\underset{\wedge}{U}EX \subseteq A$. So by hypotheses either $A_1B \subseteq A$ or $A_2B \subseteq A$ or $A_1A_2 \subseteq A$, hence $\hat{H}EX \subseteq A$ or $\underset{\wedge}{U}EX \subseteq A$ or $\hat{H}\underset{\wedge}{U}X \subseteq A$. Thus $\hat{H}B \subseteq A$ or $\underset{\wedge}{U}B \subseteq A$ or $\hat{H}\underset{\wedge}{U} \subseteq (A:_R X)$. Therefore, A is T-ABSO F. subm. of X by theorem (20). Now, the definitions of finitely generated F. M. see [17, Definition (2.11)] and faithful F. M. see [3, Definition (3.2.6)]. We give the following proposition.

Proposition 26.

Let X be a finitely generated multiplication F. M. of an R-M. $\acute{M}$. If $\hat{H}$ is T-ABSO F. ideal of R such that $F\text{-}annX \subseteq \hat{H}$, then $\hat{H}X$ is T-ABSO F. subm. of X.

Proof. Let $a_s b_l x_v \subseteq \hat{H}X$, where a_s , b_l be F. singletons of R and $x_v \subseteq X$, hence $a_s b_l < x_v > \subseteq \hat{H}X$. But X is a multiplication F. M., then $< x_v >= \underset{\wedge}{U}X$ for some F. ideal $\underset{\wedge}{U}$ of R. Thus

$a_s b_l \underset{\wedge}{U} X \subseteq \hat{H}X$. So that $a_s b_l \underset{\wedge}{U} \subseteq \hat{H} + F - annX = \hat{H}$ since $F\text{-}annX \subseteq \hat{H}$. But $\hat{H}$ is T-ABSO F. ideal of R, so that either $a_s b_l \subseteq \hat{H}$ or $a_s \underset{\wedge}{U} \subseteq \hat{H}$ $b_l \underset{\wedge}{U} \subseteq \hat{H}$. Then we have $a_s b_l X \subseteq \hat{H}X$ or $a_s \underset{\wedge}{U} X \subseteq \hat{H}X$ or $b_l \underset{\wedge}{U} X \subseteq \hat{H}X$, so that $a_s b_l \subseteq (\hat{H}X:_R X)$ or $a_s < x_v > \subseteq \hat{H}X$ or $b_l < x_v > \subseteq \hat{H}X$, hence $a_s b_l \subseteq (\hat{H}X:_R X)$ or $a_s x_v \subseteq \hat{H}X$ or $b_l x_v \subseteq \hat{H}X$. So that $\hat{H}X$ is T-ABSO F. subm. of X.

Corollary 27.

Let X be a faithful finitely generated multiplication F. M. of $\acute{M}$. If $\hat{H}$ is T-ABSO F. ideal of R, then $\hat{H}X$ is T-ABSO F. subm. of X.

Proof. By proposition (26), it follows immediately.

Corollary 28.

Suppose that X be a faithful finitely generated multiplication F. M. of $\acute{M}$. Then every proper F. subm. of X is T-ABSO iff every proper F. ideal of R is T-ABSO.

Proof. (⇐) By corollary (27), it follows immediately.

(⟹) Let $\hat{H}$ be a proper F. ideal of R. Then A=$\hat{H}X$ is a proper subm. of X. Since A is T-ABSO F. subm., so that $(A:_R X)$ is T-ABSO F. ideal by theorem (21). But X is a multiplication F. M., hence A=$(A:_R X)\ X$ by [5]. Thus $\hat{H}X$= $(A:_R X)\ X$. Since X is a faithful finitely generated multiplication F. M., then X_v is a faithful finitely generated multiplication M. by [16, 17], implies that $X_v = \acute{M}$ is cancellation R-M. by [18]. Hence X is a cancellation F. M. by [8]. Therefore $\hat{H}$= $(A:_R X)$; that is $\hat{H}$ is T-ABSO F. ideal of R.

Recall that Let X be F. M. of an R-M. $\acute{M}$, and let A be F. subm. of X. A is called a pure F. subm., if for each F. ideal $\hat{H}$ of R such that $\hat{H}A$=$\hat{H}X \cap A$, see [19].

Proposition 29.

Let A be a proper pure F. subm. of F. M. X of $\acute{M}$. If 0_1 is T-ABSO F. subm. of X, then A is T-ABSO F. subm. of X.

Proof. Let $a_s b_l x_v \subseteq A$ where a_s, b_l F. singletons of R and $x_v \subseteq X$.

Put $\hat{H} =< a_s b_l >$, hence $a_s b_l x_v \subseteq \hat{H}X \cap A$, but $\hat{H}X \cap A$=$\hat{H}A$. So $a_s b_l x_v = a_s b_l y_h$,

for some F. singleton $y_h \subseteq A$, then $a_s b_l (x_v - y_h) \subseteq 0_1$, but 0_1 is T-ABSO F. subm., hence $a_s (x_v - y_h) \subseteq 0_1$ or $b_l (x_v - y_h) \subseteq 0_1$ or $a_s b_l \subseteq F - annX \subseteq (A:_R X)$.

So we have $a_s x_v = a_s y_h \subseteq A$ or $b_l x_v = b_l y_h \subseteq A$ or $a_s b_l \subseteq (A:_R X)$.

Therefore A is T-ABSO F. subm. of X.

Now, we give the concept of a cancellative F. M. as follows:

Definition 30. A F. M. X of $\acute{M}$ is called a cancellative F. if whenever $a_s x_v = a_s y_k$ for F. singletons a_s of R and $x_v, y_k \subseteq X$, $\forall s,v,k \in$ L, then $x_v = y_k$

Proposition 31.

Let X be a cancellative F. M. of $\acute{M}$, and A be a proper F. subm. of X. Then A is a pure F. subm. of X iff A is T-ABSO F. subm.of X with $(A:_R X) = 0_1$.

Proof. (⇒) Assume that A is a pure F. subm. of X and $a_s b_l x_v \subseteq A$ such that $a_s b_l \nsubseteq (A:_R X)$ for F. singletons a_s, b_l of R and $x_v \subseteq X$. Then $a_s b_l x_v \subseteq a_s b_l X \cap A = a_s b_l A$, hence $a_s b_l x_v = a_s b_l y_k$ for some F. singleton $y_k \subseteq A$. Since X is a cancellative F. M., then $b_l x_v = b_l y_k \subseteq A$. Thus A is T-ABSO F. subm. of X.
Now, assume that F. singleton $r_h \subseteq (A:_R X)$ with $r_h \neq 0_1$. Since $A \neq X$ there exists F. singleton $x_v \subseteq X \backslash A$ such that $r_h x_v \subseteq r_h X \cap A = r_h A$, so there exists F. singleton $y_k \subseteq A$, such that $r_h x_v = r_h y_k$, hence $x_v = y_k$ this is a contradication. So that $(A:_R X) = 0_1$.
(⇐) Suppose that A is T-ABSO F. subm. of X. Let $a_s b_l x_v \subseteq a_s b_l X \cap A$ for F. singletons a_s, b_l of R and $x_v \subseteq X$. We may suppose that $a_s b_l \neq 0_1$. Since A is T-ABSO F. subm. of X, then either $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. If $b_l x_v \subseteq A$ and b_l be F. singleton of R, $a_s b_l x_v \subseteq a_s b_l A$. Thus $a_s b_l X \cap A \subseteq a_s b_l A$. By the same method to prove the case if $a_s x_v \subseteq A$; that is $a_s b_l A \subseteq a_s b_l X \cap A$. Thus $a_s b_l X \cap A = a_s b_l A$. So that A is a pure F. subm.

4. T-ABSO Quasi Primary F. Subm.

In this section we present the concept of T-ABSO quasi primary F. subm. and study the relationships this concept among T-ABSO F. subm. and T-ABSO primary F. subm. Many basic properties and outcomes are given. Now, we give the following definition:

Definition 32.

Let A be a proper F. subm. of non-empty F. M. X of an R-M. $\acute{M}$. Then the X-F. radical of A, denoted by X-R(A) is defined to the intersection of all prime F. subm. including A. We give the pursue lemma which are needed in the next proposition.

Lemma 33.

Let X be a multiplication F. M. of $\acute{M}$, let A be a proper F. subm. of X. Then the following expressions are equivalent:

1- A is a prime F. subm. of X.
2- $(A:_R X)$ be a prime F. ideal of R .
3- A=ĤX for some a prime F. ideal Ĥ of R with *F-annX*⊆Ĥ.

Proof. (1)→(2) It follows by [20, proposition (2.5)].
(2)→(3) Since X is a multiplication F. M., so that $A = (A:_R X)X$ by[5].
Put Ĥ=$(A:_R X)$ be a prime F. ideal of R. Now, since *F-annX*=$(0_1:_R X)$ and $(0_1:_R X) \subseteq (A:_R X) =$ Ĥ. So that *F-annX* ⊆ Ĥ .
(3)→(1) Let $a_s x_v \subseteq A$ for F. singleton a_s of R and $x_v \subseteq X$, and $x_v \nsubseteq A$ to prove $a_s \subseteq (A:_R X)$. By(3), A=ĤX for some a prime F. ideal Ĥ of R with *F-annX* ⊆Ĥ, so that *F-annX* is a prime F. ideal of R, but *F-annX*=$(0_1:_R X)$, hence $(0_1:_R X)$ is a prime F. ideal of R. Let $a_s b_l \subseteq (0_1:_R X)$, for F. singleton b_l of R, and $b_l \nsubseteq (0_1:_R X)$, then $a_s \subseteq (0_1:_R X)$. Since $(0_1:_R X) \subseteq (A:_R X)$, so that $a_s \subseteq (A:_R X)$. Thus A is a prime F. subm. of X.

Lemma 34.

Let X be a finitely generated multiplication F. M. of $\acute{M}$ and let A be F. subm. of X. Then $X - R(A) = \sqrt{A:_R X} \, . X$.

Proof. If X-R(A)=X , then the result is directly.
So that X-R(A)≠X, if B is any prime F. subm. of X which contains A, we get $(A:_R X) \subseteq (B:_R X)$. We prove that $(B:_R X)$ is a prime F. ideal. Assume that $a_s b_l \subseteq (B:_R X)$ for F. singleton a_s, b_l of R, so that $a_s b_l X \subseteq B$, then either $b_l X \subseteq B$ or $b_l x_v \subseteq X/B$ for some F. singleton $x_v \subseteq X$. But B is a prime F. subm. and $a_s(b_l x_v) \subseteq B$, then either $(b_l x_v) \subseteq B$ or $a_s \subseteq (B:_R X)$. Thus $a_s \subseteq (B:_R X)$ or $b_l \subseteq (B:_R X)$. So that $(B:_R X)$ is a prime F. ideal. Hence $\sqrt{A:_R X} \subseteq (B:_R X)$ by [13], then $\sqrt{A:_R X}\,.X \subseteq (B:_R X)X$. Since B is an arbitary prime F. subm. containing A , we get $\sqrt{A:_R X}\,.X \subseteq X - R(A)$ (1).
Now, since X is a multiplication F. M., hence $X - R(A) = (X - R(A):_R X)X$.
We must prove that $(X - R(A):_R X) \subseteq \sqrt{A:_R X}$. Let K be any prime F. ideal such that $(A:_R X) \subset K$. Since K is a prime F. ideal containing *F-annX*=$(0_1:_R X)$, then KX is a prime F. subm. of X containing A=$(A:_R X)X$ by lemma (33). Thus $(X - R(A):_R X)X = X - R(A) \subseteq KX$, hence $(X - R(A):_R X) \subseteq K$, then $(X - R(A):_R X) \subseteq \sqrt{A:_R X}$ by [13], hence $X - R(A) = (X - R(A):_R X)X \subseteq \sqrt{A:_R X}\,.X$. So that $-R(A) \subseteq \sqrt{A:_R X}\,.X$ (2) .
From (1) and (2), we get $-R(A) = \sqrt{A:_R X}\,.X$.

Before the next proposition we give these lemmas and definition which are needed in the proof of the next proposition. We give this definition as follows:

Definition 35.

Let X be F. M. of an R-M. $\acute{M}$. If Ƥ is a maximal F. ideal of R then we define $F - G_{Ƥ}(X) = \{x_v \subseteq X : (1_v - a_s)x_v = 0_1$ for some F. sigleton $a_s \subseteq Ƥ, \forall v, s \in L\}$.

It is obvious $F - G_{Ƥ}(X)$ is F. subm. of X . X is calld Ƥ-cyclic F. M. if there exist F. singleton $b_l \subseteq Ƥ$ *and* $x_v \subseteq X$ such that $(1_v - b_l)X \subseteq < x_v >$, $\forall l, v \in L$.

Lemma 36.

Let R be a commutative ring with unity. Then F. M. X of an R-M. $\acute{M}$ is a multiplication F. M. iff for every maximal F. ideal Ƥ of R either $X = F - G_{Ƥ}(X)$ or X is Ƥ-cyclic F. M.

Proof. (⇒) Assume that X is a multiplication F. M. Let Ƥ be maximal F. ideal of R. Suppose that X=ƤX , let F. singleton $x_v \subseteq X$, then $< x_v >= \hat{H}X$ for some F. ideal Ĥ of R. Hence $< x_v >= \hat{H}X = \hat{H}ƤX = Ƥ\hat{H}X = Ƥ < x_v >$, then $x_v = a_s x_v$ for some F. sigleton $a_s \subseteq Ƥ$. Thus $(1_v - a_s)x_v = 0_1$, so that $x_v \subseteq F - G_{Ƥ}(X)$. It follows that $X = F - G_{Ƥ}(X)$
Now, suppose that X≠ƤX, then there exists F. sigleton $x_v \subseteq X$, $x_v \nsubseteq ƤX$. So that there exists an ideal Ụ of R such that $< x_v >=$ ỤX. It is obvious that Ụ⊈ Ƥ and so $(1_v - b_l) \subseteq$ Ụ for some F. singleton $b_l \subseteq Ƥ$. Hence $(1_v - b_l)X \subseteq < x_v >$. Thus X is Ƥ-cyclic F. M.(⇐) Suppose that for each maximal F. ideal Ƥ of R either $X = F - G_{Ƥ}(X)$ or X is Ƥ-cyclic F. M. Let A be F. subm. of X and $\hat{H} = (A:_R X)$. It is obvious that ĤX⊆A. Suppose that F. singleton $y_k \subseteq A$ and $K = \{r_h \subseteq R: r_h y_k \subseteq \hat{H}X\}$. Assume that K≠R, then there exists a maximal F. ideal E of R such that K⊆ E by [13, proposition(1.3.2.4)]. If X=$F - G_E(X)$ then $(1_v - a_s)y_k = 0_1$ for some F. singleton $a_s \subseteq E$, and $(1_v - a_s) \subseteq K \subseteq E$ this is a discrepancy. Thus by

hypothesis there exist F. singletons $b_l \subseteq E$, $z_n \subseteq X$ such that $(1_v - b_l)X \subseteq < z_n >$. It follows that $(1_v - b_l)A$ is F. subm. of $< z_n >$ and so tha $(1_v - b_l)A = D\, z_n$ where D is F. ideal $\{r_h \subseteq R: r_h z_n \subseteq (1_v - b_l)A\}$ of R. Note that $(1_v - b_l)D\, X = D\,(1_v - b_l)X \subseteq D\, z_n \subseteq A$. So that $(1_v - b_l)D \subseteq \hat{\text{H}}$. Thus for F. singleton $y_k \subseteq A$, $(1_v - b_l)^2 y_k \subseteq (1_v - b_l)^2 A = (1_v - b_l)D\, z_n \subseteq \hat{\text{H}}X$.
So that $(1_v - b_l)^2 \subseteq K \subseteq E$ this is a discrepancy. Thus K=R and $y_k \subseteq \hat{\text{H}}X$. Therefore A=$\hat{\text{H}}X$ and X is a multiplication F. M.

Lemma 37.

Let X be a multiplication F. M. of an R-M. $\acute{M}$, then
$\cap_{i\in\Lambda}(\hat{\text{H}}_i X) = (\cap_{i\in\Lambda}(\hat{\text{H}}_i + F - annX))X$ for any non-empty collection of F. ideals $\hat{\text{H}}_i (i \in \Lambda)$ of R.

Proof. Assume that X is a multiplication F. M. Let $\hat{\text{H}}_i (i \in \Lambda)$ be any non-empty collection of F. ideals of R, let $\underset{\wedge}{\text{U}} = \cap_{i\in\Lambda}(\hat{\text{H}}_i + F - annX)$, then $\underset{\wedge}{\text{U}}X$=$(\cap_{i\in\Lambda}(\hat{\text{H}}_i + F - annX))X$. It is obvious that $\underset{\wedge}{\text{U}}X \subseteq \cap_{i\in\Lambda}(\hat{\text{H}}_i X)$. Now, let be F. singleton $x_v \subseteq \cap_{i\in\Lambda}(\hat{\text{H}}_i X)$ and let $G = \{a_s \subseteq R: a_s x_v \subseteq \underset{\wedge}{\text{U}}X\}, \forall s, v \in L$ Suppose that G≠R, then there exists a maximal F. ideal Ƥ of R such that $G \subseteq$ Ƥ, it is obvious that $x_v \nsubseteq F - G_{Ƥ}(X)$ and hence X is Ƥ-cyclic F. M. by lemma (36). Then there exist F. singletons $a_s \subseteq$ Ƥ and $y_k \subseteq X$ such that $(1_v - a_s)X \subseteq < y_k >$. Hence $(1_v - a_s)x_v \subseteq \cap_{i\in\Lambda}(\hat{\text{H}}_i y_k)$. for each $i \in \Lambda$ there exists F. singleton $b_{l_i} \subseteq \hat{\text{H}}_i$, $\forall l_i \in L$, such that $(1_v - a_s)x_v = b_{l_i} y_k$. Choose $j \in \Lambda$, for each $i \in \Lambda$, $b_{l_j} y_k = b_{l_i} y_k$, so that $(b_{l_j} - b_{l_i})y_k = 0_1$, implies that: $(1_v - a_s)(b_{l_j} - b_{l_i})X = (b_{l_j} - b_{l_i})(1_v - a_s)X \subseteq (b_{l_j} - b_{l_i}) < y_k > = 0_1$, $(1_v - a_s)(b_{l_j} - b_{l_i}) = 0_1$. Thus $(1_v - a_s)b_{l_j} = (1_v - a_s)b_{l_i} \subseteq \hat{\text{H}}_i (i \in \Lambda)$, then $(1_v - a_s)b_{l_j} \subseteq \underset{\wedge}{\text{U}}$. Hence$(1_v - a_s)^2 x_v = (1_v - a_s)b_{l_j} y_k \subseteq \underset{\wedge}{\text{U}}X$.

It follows that $(1_v - a_s)^2 \subseteq G \subseteq$ Ƥ this is a discrepancy. Thus G=R and $x_v \subseteq \underset{\wedge}{\text{U}}X$, so that $\cap_{i\in\Lambda}(\hat{\text{H}}_i X) \subseteq \underset{\wedge}{\text{U}}X$ implies that $\cap_{i\in\Lambda}(\hat{\text{H}}_i X) = \underset{\wedge}{\text{U}}X$ That is $\cap_{i\in\Lambda}(\hat{\text{H}}_i X) = (\cap_{i\in\Lambda}(\hat{\text{H}}_i + F - annX))X$. Now, we give the proposition as follows:

Proposition 38.

Let X be a multiplication finitely generated F. M. of an R-M. $\acute{M}$ and A be T-ABSO F. subm. of X. Then one of the following satisfy:

1- X-R(A)=Ƥ is a prime F. subm. of X such that $Ƥ^2 \subseteq A$.
2- X-R(A)=$Ƥ_1 \cap Ƥ_2$, $Ƥ_1 Ƥ_2 \subseteq A$ and $(X - R(A))^2 \subseteq A$ where $Ƥ_1, Ƥ_2$ are the only distinct minimal prime F. subms. of A.

Proof. By theorem (21), $(A:_R X)$ is T-ABSO F. ideal of R. So that either $R((A:_R X)) = \underset{\wedge}{\text{U}}$ is a prime F. ideal of R such that $\underset{\wedge}{\text{U}}^2 \subseteq (A:_R X)$ or $R((A:_R X)) = \underset{\wedge}{\text{U}}_1 \cap \underset{\wedge}{\text{U}}_2$, $\underset{\wedge}{\text{U}}_1 \underset{\wedge}{\text{U}}_2 \subseteq (A:_R X)$ and $R((A:_R X))^2 \subseteq (A:_R X)$ where $\underset{\wedge}{\text{U}}_1, \underset{\wedge}{\text{U}}_2$ are the only distinct minimal prime F. ideals of $(A:_R X)$ by proposition (6), where $R((A:_R X)) = \sqrt{A:_R X}$. if the first case satisfies, then since X is F. multiplication, we have X-R(A)=R($(A:_R X)$)X=$\underset{\wedge}{\text{U}}X$ is a prime F. subm. of X. Put $\underset{\wedge}{\text{U}}X$=Ƥ by lemma (33) and lemma (34), and $(\underset{\wedge}{\text{U}}X)^2 = \underset{\wedge}{\text{U}}^2 X \subseteq (A:_R X)X = A$. Now, suppose that the latter case satisfies, then by lemma(33), $\underset{\wedge}{\text{U}}_1 X$ and $\underset{\wedge}{\text{U}}_2 X$ are the only distinct minimal prime F. subms.

of A and X- R(A) = $R((A:_R X))X$ = $(\underset{\smile}{U}_1 \cap \underset{\smile}{U}_2)X$ = $\underset{\smile}{U}_1X \cap \underset{\smile}{U}_2X$ by lemma (37). Moreover $(\underset{\smile}{U}_1X)(\underset{\smile}{U}_2X) = (\underset{\smile}{U}_1\underset{\smile}{U}_2)X \subseteq (A:_R X)X = A$ and $(X - R(A))^2 = (\mathrm{R}((A:_R X))\mathrm{X})^2 = (\mathrm{R}((A:_R X)))^2X \subseteq (A:_R X)X = A$.

We give the definition of T-ABSO primary F. subm. as follows:

Definition 39. Let A be a proper F. subm. of F. M. X of $\acute{M}$, A is called T-ABSO primary F. subm. of X if whenever F. singletons a_s, b_l of R and $x_v \subseteq X$ such that $a_s b_l x_v \subseteq A$, then either $a_s x_v \subseteq X - \mathrm{R}(A)$ or $b_l x_v \subseteq X - \mathrm{R}(A)$ or $a_s b_l \subseteq (A:_R X)$.

The following proposition characterize T-ABSO primary F. subm. in terms of its level subm.

Proposition 40.

Let A be T-ABSO primary F. subm. of F. M. X of Ḿ. for all v∈ L, iff the level subm. A_v is T-ABSO primary subm. of X_v.

Proof. (⇒) Let $abx \in A_v$ for any $a,b \in$R and $x \subseteq X_v$, then $A(abx) \geq v$, so $(abx)_v \subseteq A$ implies that $a_s b_l x_k \subseteq A$ where v =min$\{s, l, k\}$. Since A be T-ABSO primary F. subm., so either $a_s x_k \subseteq X - R(A)$ or $b_l x_k \subseteq X - R(A)$ or $a_s b_l \subseteq (A:_R X)$.

If $a_s x_k \subseteq X - R(A)$, then $(ax)_v \subseteq X - R(A)$, so $ax \in X_v - R(A_v)$.

If $b_l x_k \subseteq X - R(A)$, then $(bx)_v \subseteq X - R(A)$, so $bx \in X_v - R(A_v)$.

If $a_s b_l \subseteq (A:_R X)$ then $(ab)_v \subseteq (A:_R X)$, so $ab \in (A:_R X)_v = (A_v:_R X_v)$.

Hence $ab \in (A_v:_R X_v)$. Thus A_v is T-ABSO primary subm. of X_v.

(⟸)Let $a_s b_l x_k \subseteq A$ for F. singletons a_s, b_l of R and $x_k \subseteq X$,∀s, l, k ∈L,
hence $(abx)_v \subseteq A$ where $v = \min\{s, l, k\}$ so that $A(abx) \geq v$, implies $abx \in A_v$, but A_v is T-ABSO primary subm. of X_v so either $ax \in X_v - \mathrm{R}(A_v)$ or $bx \in X_v - \mathrm{R}(A_v)$ or $ab \in (A_v:_R X_v)$. Since $(A_v:_R X_v) = (A:_R X)_v$, hence $ab \in (A:_R X)_v$. Then either $(ax)_v \subseteq X - R(A)$ or $(bx)_v \subseteq X - \mathrm{R}(A)$ or $(ab)_v \subseteq (A:_R X)$, implies either $a_s x_k \subseteq X - R(A)$ or $b_l x_k \subseteq X - R(A)$ or $a_s b_l \subseteq (A:_R X)$. Thus A be T-ABSO primary F. subm. of X.

Remark 41.

Every T-ABSO F. subm. is T-ABSO primary F. subm., but the converse in general incorrect**,** for example:

Let X: $Z \to L$ such that $X(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is obvious that X is F. M. of Z-M. Z.

Let A: $Z \to L$ such that $A(y) = \begin{cases} v & if\ y \in 12Z \\ 0 & o.w. \end{cases}$ $\forall\ v \in L$

It is obvious that A is F. subm. of X.

Now, $A_v = 12Z$ and $X_v = Z$ as Z-M. Note that $A_v = 12Z$ is not T-ABSO subm. since $2.2.3 \in 12Z = A_v$ but $2.2 \notin 12Z = A_v$ and $2.3 \notin 12Z = A_v$.

But $X_v - \mathrm{R}(A_v) = Z - \mathrm{R}(12Z) = 2Z \cap 3Z = 6Z$ where $2Z$ and $3Z$ are prime subms. of X_v containing A_v. So that A_v is T-ABSO primary subm. of X_v since 2.3=6∈6Z. Thus A is not T-ABSO F. subm., but it is T-ABSO primary F. subm. of X . We give the concept of T-ABSO quasi primary F. subm. as follows:

Definition 42. A proper F. subm. A of F. M. X of $\acute{M}$ is called T-ABSO quasi primary F. subm. If $a_s b_l x_v \subseteq A$ implies either $a_s b_l \subseteq \sqrt{A:_R X}$ or $a_s x_v \subseteq X - R(A)$ or $b_l x_v \subseteq X - R(A)$ for each F. singleton a_s, b_l of R and $x_v \subseteq X$, $\forall s,l,v \in$ L.

The following proposition characterize T-ABSO quasi primary F. subm. in terms of its level subm.

Proposition 43.

Let A be T-ABSO quasi primary F. subm. of F. M. X of $\acute{M}$ iff the level subm. A_v is T-ABSO quasi primary subm. of X_v $\forall$ $v \in$L.

Proof. ($\Rightarrow$) Let $abx \in A_v$ for any $a, b \in$ R and $x \in X_v$, then $A(abx) \geq v$, so $(abx)_v \subseteq A$ implies that $a_s b_l x_k \subseteq A$ where $v = \min\{s, l, k\}$. Since A be a T-ABSO quasi primary F. subm., so either $a_s b_l \subseteq \sqrt{A:_R X}$ or $a_s x_k \subseteq X - R(A)$ or $b_l x_k \subseteq X - R(A)$.
If $a_s b_l \subseteq \sqrt{A:_R X}$ then $(ab)_v \subseteq \sqrt{A:_R X}$, so ab
$\in \left(\sqrt{A:_R X}\right)_v = \sqrt{A_v:_R X_v}$. Henc e$ab \in \sqrt{A_v:_R X_v}$.
If $a_s x_k \subseteq X -$ R(A), then $(ax)_v \subseteq X - R(A)$, so $ax \in X_v - R(A_v)$.
If $b_l x_k \subseteq X -$ R(A), then $(bx)_v \subseteq X - R(A)$, so $bx \in X_v - R(A_v)$.
Thus A_v is a T-ABSO quasi primary subm. of X_v.
($\Leftarrow$) Let $a_s b_l x_k \subseteq A$ for F. singletons a_s, b_l of R and $x_k \subseteq X$, hence $(abx)_v \subseteq A$ where $v = \min\{s, l, k\}$ so that $A(abx) \geq v$, implies $abx \in A_v$, but A_v is T-ABSO quasi primary subm. of X_v, so either $ab \in \sqrt{A_v:_R X_v}$ or $ax \in X_v - R(A_v)$ or $bx \in X_v - R(A_v)$. Since $\sqrt{A_v:_R X_v} = \left(\sqrt{A:_R X}\right)_v$, hence $ab \in \left(\sqrt{A:_R X}\right)_v$. Then either $(ab)_v \subseteq \sqrt{A:_R X}$ or $(ax)_v \subseteq X - R(A)$ or $(bx)_v \subseteq X -$ R(A), implies either $a_s b_l \subseteq \sqrt{A:_R X}$ $a_s x_k \subseteq X - R(A)$ or $b_l x_k \subseteq X - R(A)$ where $v = \min\{s, l, k\}$. Thus A be T-ABSO quasi primary F. subm. of X.

Theorem 44.

Let A be a proper F. subm. of F. M. X of $\acute{M}$. Then the following expressions are equivalent:

1- A is T-ABSO quasi primary F. subm. of X;

2- For every F. singleton a_s, b_l of R, $\forall s$, $(A:_X a_s^n b_l^n) = X$ for some $n \in Z^+$or $(A:_X a_s b_l) \subseteq (X - R(A):_X a_s) \cup (X - R(A):_X b_l)$.

3- For every F. singleton a_s, b_l of R, $\forall s,l \in$ L, $(A:_X a_s^n b_l^n) = X$ for some $n \in Z^+$ or $(A:_X a_s b_l) \subseteq (X - R(A):_X a_s)$ or $(A:_X a_s b_l) \subseteq (X - R(A):_X b_l)$.

Proof. (1)→(2) Assume that A is T-ABSO quasi primary F. subm. of X, let F. singleton a_s, b_l of R.
If $a_s b_l \subseteq \sqrt{A:_R X}$, then $(a_s b_l)^n = a_s^n b_l^n \subseteq (A:_R X)$ for some $n \in Z^+$, hence $(A:_X a_s^n b_l^n) = X$. Now, suppose that $a_s b_l \nsubseteq \sqrt{A:_R X}$. Let $x_v \subseteq (A:_X a_s b_l)$, then $a_s b_l x_v \subseteq A$. Since A is T-ABSO quasi primary F. subm., then $a_s x_v \subseteq X - R(A)$ or $b_l x_v \subseteq X - R(A)$. So that $(A:_X a_s b_l) \subseteq (X - R(A):_X a_s) \cup (X - R(A):_X b_l)$.
(2)→(3) By (2), we have $(A:_X a_s b_l) \subseteq (X - R(A):_X a_s) \cup (X - R(A):_X b_l)$.
So that $(A:_X a_s b_l) \subseteq (X - R(A):_X a_s)$ or $(A:_X a_s b_l) \subseteq (X - R(A):_X b_l)$.

(3)→(1) Let $a_s b_l x_v \subseteq A$ and $a_s b_l \nsubseteq \sqrt{A:_R X}$ for F. singletons a_s, b_l of R and $x_v \subseteq X$, hence $(a_s b_l)^n = a_s^n b_l^n \nsubseteq (A:_R X)$ for some $n\in Z^+$, then $(A:_X a_s^n b_l^n) \neq X$. By (3), we have that $x_v \subseteq (A:_X a_s b_l) \subseteq (X - R(A):_X a_s)$ or $x_v \subseteq (A:_X a_s b_l) \subseteq (X - R(A):_X b_l)$. Thus $x_v a_s \subseteq X - R(A)$ or $x_v b_l \subseteq X - R(A)$. So that A is T-ABSO quasi primary F. subm. of X.

Lemma 45.

Let X be F. M. of $\acute{M}$. Suppose that A is T-ABSO quasi primary F. subm. of X and $a_s b_l B \subseteq A$ for F. singleton a_s, b_l of R, $\forall\, s, l \in$ L , and F. subm. B of X . If $a_s b_l \nsubseteq \sqrt{A:_R X}$, then $a_s B \subseteq X - R(A)$ or $b_l B \subseteq X - R(A)$.

Proof. Since $B \subseteq (A:_X a_s b_l)$ and $(A:_X a_s^n b_l^n) \neq X$ for some $n\in Z^+$, by theorem (44), we get $B \subseteq (A:_X a_s b_l) \subseteq (X - R(A):_X a_s)$ or $B \subseteq (A:_X a_s b_l) \subseteq (X - R(A):_X b_l)$.
Then $a_s B \subseteq X - R(A)$ or $b_l B \subseteq X - R(A)$.

Theorem 46.

Let A be a proper F. subm. of F. M. X of $\acute{M}$, then the following expressions are equivalent:
1- A is T-ABSO quasi primary F. subm. of X;
2- For F. singleton a_s of R, $\forall s \in$L, F. ideal Ĥ of R and F. subm. B of X with $a_s \hat{H} B \subseteq A$, then either

$a_s \hat{H} \subseteq \sqrt{A:_R X}$ or $a_s B \subseteq X - R(A)$ or $\hat{H} B \subseteq X - R(A)$;

3- For F. ideals Ĥ, Ụ of R, and F. subm. B of X with $\hat{H}\underset{\wedge}{U} B \subseteq A$, then eithe $\hat{H}\underset{\wedge}{U} \subseteq \sqrt{A:_R X}$ or $\hat{H} B \subseteq X - R(A)$ or $\underset{\wedge}{U} B \subseteq X - R(A)$.

Proof. (1)→(2) Assume that $a_s \hat{H} B \subseteq A$ with $a_s \hat{H} \nsubseteq \sqrt{A:_R X}$ and $\hat{H} B \nsubseteq X - R(A)$. Then there exist F. singletons $b_l, r_k \subseteq \hat{H}$, such that $a_s b_l \nsubseteq \sqrt{A:_R X}$ and $r_k B \nsubseteq X - R(A)$. Now, we prove that $a_s B \subseteq X - R(A)$. Suppose that $a_s B \nsubseteq X - R(A)$. Since $a_s b_l B \subseteq A$, by lemma (45), we have $b_l B \subseteq X - R(A)$, hence $(b_l + r_k) B \nsubseteq X - R(A)$. By using lemma (45), we have $a_s(b_l + r_k) = a_s b_l + a_s r_k \subseteq \sqrt{A:_R X}$, because $a_s(b_l + r_k) B \subseteq A$. Since $a_s b_l + a_s r_k \subseteq \sqrt{A:_R X}$ and $a_s b_l \nsubseteq \sqrt{A:_R X}$, we have $a_s r_k \nsubseteq \sqrt{A:_R X}$. Since $a_s r_k B \subseteq A$, by lemma (45), we have $r_k B \subseteq X - R(A)$ or $a_s B \subseteq X - R(A)$ this is a discrepancy. So that $a_s B \subseteq X - R(A)$.
(2)→(3) Suppose that $\hat{H}\underset{\wedge}{U} B \subseteq A$ with $\hat{H}\underset{\wedge}{U} \nsubseteq \sqrt{A:_R X}$ for F. ideals Ĥ, Ụ of R and F. subm. B of X. Hence $a_s \underset{\wedge}{U} \nsubseteq \sqrt{A:_R X}$ for some F. singleton $a_s \subseteq \hat{H}$. Now, we prove that $\hat{H} B \subseteq X - R(A)$ or $\underset{\wedge}{U} B \subseteq X - R(A)$. Assume that $\hat{H} B \nsubseteq X - R(A)$ and $\underset{\wedge}{U} B \nsubseteq X - R(A)$. Since $a_s \underset{\wedge}{U} B \subseteq A$, by (2), we have $a_s B \subseteq X - R(A)$, then there exists $y_h \subseteq \hat{H}$ such that $y_h B \nsubseteq X - R(A)$ since the assumption $\hat{H} B \nsubseteq X - R(A)$. Since $y_h \underset{\wedge}{U} B \subseteq A$, we have $y_h \underset{\wedge}{U} \subseteq \sqrt{A:_R X}$, hence $(a_s + y_h)\underset{\wedge}{U} \nsubseteq \sqrt{A:_R X}$. Since $(a_s + y_h)\underset{\wedge}{U} B \subseteq A$, we get $(a_s + y_h) B \subseteq X - R(A)$ and so $y_h B \subseteq X - R(A)$ this is a discrepancy. Thus $\hat{H} B \subseteq X - R(A)$.
(3)→(1) Let $a_s b_l x_v \subseteq A$, for F. singletons a_s, b_l of R and $x_v \subseteq X$. Put $\hat{H} = < a_s >$, $\underset{\wedge}{U} = < b_l >$ and $B = < x_v >$, then $\hat{H}\underset{\wedge}{U} B \subseteq A$. By (3), we have either $\hat{H}\underset{\wedge}{U} \subseteq \sqrt{A:_R X}$ or $\hat{H} B \subseteq X - R(A)$ or $\underset{\wedge}{U} B \subseteq X - R(A)$; that is either $< a_s >< b_l > \subseteq \sqrt{A:_R X}$ or $< a_s >< x_v > \subseteq X -$

$R(A)$ or $< b_l >< x_v >\subseteq X - R(A)$. Hence either $a_s b_l \subseteq \sqrt{A:_R X}$ or $a_s x_v \subseteq X - R(A)$ or $b_l x_v \subseteq X - R(A)$. Thus A is T-ABSO quasi primary F. subm. of X.

Theorem 47.

Let X be F. M. of $\acute{M}$, and A be F. subm. of X. Then the following are satisfied:

1- If is a multiplication F. M. and $(A:_R X)$ is T-ABSO quasi primary F. ideal of R, then A is T- ABSO quasi primary F. subm. of X.

2- If X is a finitely generated multiplication F. M. and A is T-ABSO quasi primary F. subm. of X, then $(A:_R X)$ is T-ABSO quasi primary F. ideal of R.

Proof. (1) Assume that X is a multiplication F. M., $(A:_R X)$ is T-ABSO quasi primary F. ideal of R and $\hat{H}\underset{\wedge}{U}B\subseteq A$ for F. ideals $\hat{H}$, $\underset{\wedge}{U}$ of R and F. subm. B of X. Since X is a multiplication F. M., we have $B=KX$ for some F. ideal K of R. So that $\hat{H}\underset{\wedge}{U}B=\hat{H}\underset{\wedge}{U}KX\subseteq A$, then $\hat{H}\underset{\wedge}{U}K \subseteq (A:_R X)$. Since $(A:_R X)$ is T-ABSO quasi primary F. ideal of R, so by theorem (13), we have $\hat{H}\underset{\wedge}{U} \subseteq \sqrt{A:_R X}$ or $\hat{H}K \subseteq \sqrt{A:_R X} \subseteq (X - R(A):_R X)$ or $\underset{\wedge}{U}K \subseteq \sqrt{A:_R X} \subseteq (X - R(A):_R X)$. Hence $\hat{H}\underset{\wedge}{U} \subseteq \sqrt{A:_R X}$ or $\hat{H}B \subseteq X - R(A)$ or $\underset{\wedge}{U}B \subseteq X - R(A)$. Then A is T-ABSO quasi primary F. subm. of X by theorem (46).

(2) Assume that A is T-ABSO quasi primary F. subm. of a finitely generated multiplication F. M. X. Let F. singletons a_s, b_l, r_k of R , such that $a_s b_l r_k \subseteq (A:_R X)$ with $a_s b_l \nsubseteq \sqrt{A:_R X}$. Hence $a_s b_l (r_k x_v) \subseteq A$ for evey F. singleton $x_v \subseteq X$. Since A is T-ABSO quasi primary F. subm. of X and $a_s b_l \nsubseteq \sqrt{A:_R X}$. Then we have $a_s r_k x_v \subseteq X - R(A)$ or $b_l r_k x_v \subseteq X - R(A)$ for all $x_v \subseteq X$. Hence we have $(X - R(A):_X a_s r_k) \cup (X - R(A):_X b_l r_k) = X$, so that $(X - R(A):_X a_s r_k) = X$ or $(X - R(A):_X b_l r_k) = X$. Then we have $a_s r_k \subseteq (X - R(A):_R X) = \sqrt{A:_R X}$ or $b_l r_k \subseteq (X - R(A):_R X) = \sqrt{A:_R X}$. Thus $(A:_R X)$ is T-ABSO quasi primary F. ideal of R.

Theorem 48.

Let X be a finitely generated multiplication F. M. of $\acute{M}$. For any F. subm. A of X, the following expressions are equivalent:

1- A is T-ABSO quasi primary F. subm. of X;

2- X-R(A) is T-ABSO F. subm. of X.

Proof. (1)→(2) Assume that A is T-ABSO quasi primary F. subm. of X. By theorm (47) and proposition (6), then we have $\sqrt{A:_R X} = \underset{\wedge}{U}$ is a prime F. ideal of R or $\sqrt{A:_R X} = \underset{\wedge}{U}_1 \cap \underset{\wedge}{U}_2$ where $\underset{\wedge}{U}_1$, $\underset{\wedge}{U}_2$ are distinct prime F. ideals minimal over $(A:_R X)$. If $\sqrt{A:_R X} = \underset{\wedge}{U}$, hence X-R(A)=$\underset{\wedge}{U}X$ is a prime subm. by lemma (43), so that X-R(A) is T-ABSO F. subm. of X.

Now, if $\sqrt{A:_R X} = \underset{\wedge}{U}_1 \cap \underset{\wedge}{U}_2$ where $\underset{\wedge}{U}_1$, $\underset{\wedge}{U}_2$ are distinct prime F. ideals minimal over $(A:_R X)$, then we have X-R(A)= $(\underset{\wedge}{U}_1 \cap \underset{\wedge}{U}_2)X$. Since F- annX=$(0_1:_R X)$ and $(0_1:_R X) \subseteq (A:_R X)$ and $\underset{\wedge}{U}_1$, $\underset{\wedge}{U}_2$ are distinct prime F. ideals minimal over $(A:_R X)$. So that $F - annX \subseteq \underset{\wedge}{U}_1, \underset{\wedge}{U}_2$.

Then X-R(A)= $((\underset{\wedge}{U}_1 + F - annX) \cap (\underset{\wedge}{U}_2 + F - annX))X = \underset{\wedge}{U}_1 X \cap \underset{\wedge}{U}_2 X$ by lemma (47).

Since $\underset{\smile}{U}_1X$, $\underset{\smile}{U}_2X$ are two distinct prime F. subms., so that X-R(A) is T-ABSO F. subm. of X by remarks and examples(16)part(1).

(2)→(1) Assume that X-R(A) is T-ABSO F. subm. of X. Let $a_s b_l x_v \subseteq A$, for F. singletons a_s, b_l of R and $x_v \subseteq X$. Since $A\subseteq$ X-R(A), then $a_s b_l x_v \subseteq X - R(A)$. But X-R(A) is T-ABSO F. subm. of X, so that $a_s b_l \subseteq (X - R(A):_R X) = \sqrt{A:_R X}$ or $a_s x_v \subseteq X - R(A)$ or $b_l x_v \subseteq X - R(A)$. Thus A is T-ABSO quasi primary F. subm. of X. By combining theorem (47) and theorem (48), we get the following corollary is beneficial to determine T-ABSO quasi primary F. subm. of a finitely generated multiplication F. M.

Corollary 49.

For any F. subm. A of a finitely generated multiplication F. M. X of $\acute{M}$. Then the following expressions are equivalent:

1- A is T-ABSO quasi primary F. subm. of X;
2- X-R(A) is T-ABSO F. subm. of X;
3- X-R(A) is T-ABSO primary F. subm. of X;
4- X-R(A) is T-ABSO quasi primary F. subm. of X;
5- $\sqrt{A:_R X}$ is T-ABSO F. ideal of R;
6- $\sqrt{A:_R X}$ is T-ABSO primary F. ideal of R,
7- $\sqrt{A:_R X}$ is T-ABSO quasi primary F. ideal of R;
8- $(A:_R X)$ is T-ABSO quasi primary F. ideal of R.

4. Conclusions

Through our research we concluded to the concepts (prime and quasi-prime) F. subm. lead to the concept T-ABSO F. subm. we reached the concept T-ABSO F. subm.one of the most important conclusions is the theorem (20), and explan the relationship if A is T-ABSO F. subm. with $(A:_R X)$ is T-ABSO F. ideal under the class of a multiplication F. M. in corollary (23). Also we concluded the relationship $X - R(A)$ with $\sqrt{A:_R X}$ under the class of a multiplication F. M. in lemma (45), and explan the relationships A is T-ABSO quasi primary F. subm.with $(A:_R X)$ is T-ABSO quasi primary F. ideal and A is T-ABSO quasi primary F. subm.with $X - R(A)$ is T-ABSO F. subm. under the class of a multiplication F. M. as in theorem (47), and theorem (48).

References

1. Deniz, S.; Gürsel, Y.; Serkan, O.; Bayram, A. E.; Bijan, D. On 2-Absorbing Primary Fuzzy Ideals of Commutative Rings. *Mathematical Problems in Engineering*. **2017,** *2017,* 1-7.
2. Lu. C. P. Prime Submodules of Modules. *Commutative Mathematics*. **1981,** *33,* 61-69.
3. Rabi. H. J. Prime Fuzzy Submodules and Prime Fuzzy Modules. M.Sc. Thesis. University of Baghdad. **2001.**
4. Abdul-Razak, M. H. Quasi-Prime Modules and Quasi-Prime Submodules. M.Sc. Thesis. College of Education Ibn Al-Haitham. University of Baghdad. **1999.**
5. Hatam, Y. K. Fuzzy Quasi-Prime Modules and Fuzzy Quasi-Prime Submodules. M.Sc. Thesis. University of Baghdad. **2001.**

6. Darani, A.Y.; Soheilnia, F. 2-Absorbing and Weakly 2-Absorbing Submodules. *Thai Journal of Mathematics.* **2011,** *9, 3,* 577-584.
7. Khalaf, H. Y.; Hannon, W. H. Small and Classical T-ABSO Fuzzy Submodules. *Global Journal of Pure and Applied Mathematics*. **2018,** *14, 3,* 443–457.
8. McCasland, R. L.; Moore, M. E. On Radical of Submodules of Finitely Generated Modules. *Canad. Math. Bull.* **1986,** *29, 1,* 37-39.
9. Mostafanasab, H.; Yetkin, E.; Tekir, U.; Darani, A. Y. On 2- absorbing Primary Submodules of Modules over Commutative Rings. *An. St. Univ. Ovidius Constanta.* **2016,** *24, 1,* 335-351.
10. Rabi, H. J.; Hassan, K. H. On Quasi Prime Fuzzy Submodules and Quasi Primary Fuzzy Submodules. *J.Thi-Qar Sci.* **2008**, *1, 1,* 37-42.
11. Suat, K.; Rabia, N.U.; Unsal, T. On 2-Absorbing Quasi Primary Submodules. *Faculty of ciences and Mathematics. University of Niˇs, Serbia*. **2017**. 2943–2950.
12. Malik, D. S.; Mordeson, J. N. Fuzzy Maximal, Radical, and Primary Ideals of a Ring. *Information sciences*. **1991,** *53*, 237-250.
13. Hussein, W. R. Some Results of Fuzzy Rings. M.Sc. Thesis. University of Baghdad. **1999**.
14. Abdulrahman, A. H. 2-Absorbing Submodules (Modules) and Some of Their Generalizations. M.Sc. Thesis. University of Baghdad. **2015**.
15. Robert, W. *Foundations of Module and Ring Theory*. Gordon and Breach Science Publishers, **1991.**
16. Rashed, H. G. Fully Cancellation Fuzzy Modules and Some Generalizations. M.Sc. Thesis. University of Baghdad. **2017**.
17. Hadi, I. M. A.; Hamil, M. A. Cancellation and Weakly Cancellation Fuzzy Modules. *Journal of Basrah Reserchs (Sciences)*. **2011.** *37, 4.*
18. El-Bast, Z. A.; Smith, P. F. Multiplication Modules.*Comm.In Algebra*. **1988**, *16*, 755-779. 19. Maysoun, A. H. F-Regular Fuzzy Modules. M.Sc. Thesis. University of Baghdad. **2002**.
19. Abou-Draeb, A. T.; Al-Zreegawi, B.; Al-Duja, Y. Kh. Prime Fuzzy Submodules and Strongly Prime Fuzzy Submodules Over Near-ring. *Journal of Babylon Univ. Pure and Applied Science*. **2014**, *22, 3,* 984-996.