New Transform Fundamental Properties and Its Applications

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Abstract

In this paper, new transform with fundamental properties are presented. The new transform has many interesting properties and applications which make it rival to other transforms.
Furthermore, we generalize all existing differentiation, integration, and convolution theorems in the existing literature. New results and new shifting theorems are introduced. Finally, comprehensive list of this transforms of functions will be providing.

Keywords: Functions transform, Integral transform, Distribution space.

Introduction

Recently, integral transformations played an important role in many fields of science and engineering [1-4], especially mathematical physics [5], optics [6], engineering mathematics [7, 8], Cryptography [9], image processing [10] and, few others, because they have been successfully used in solving many problems in those fields. The possibility of solving a problem is required transforming the problem from a space to another space where the solving is possible or easy. As well as the possibility of decreasing the independent variables in some problems. Many of these transforms have been introduced which were extensively used and applied on theory and applications, such as Laplace [11, 12], Fourier [2], Sumudu [13, 14], Elzaki [15-17], Aboodh [18], Natural, and ZZ Transforms [19]. Among these the most widely used is Laplace transform.Here, new integral transform is proposed to avoid the complexity of previous transforms.

Definition of New Transform

The transform of a function $f(t)$is defined by

$$\bar{f}(u) = \mathbb{T}\{f(t)\} = \int_0^\infty e^{-t} f\left(\frac{t}{u}\right) dt, \qquad (1)$$

Where u is a real number,for those values of u which the improper integral converges.

Lemma 1

The improper integral $\int_0^\infty e^{-at}\, dt$, converges only for a > 0.

Proof

$$\int_0^\infty e^{-at}\, dt = \lim_{N\to\infty} \int_0^N e^{-at} dt = -\frac{1}{a} \lim_{N\to\infty} (e^{-aN} - 1) = \frac{1}{a}, \quad a > 0$$

In the case that a = 0, the improper integral is diverging, since we are computing the area of a rectangle with sides equal to one and infinity. In the case a < 0, holds

$$\lim_{N\to\infty} -\frac{1}{a}(e^{-aN} - 1) = \infty$$

Therefore, the improper integral converges only for a >0.

Remark:lemma (1) holds when $t\in[0,\infty)$, if $t\in(-\infty,0]$ then the condition of convergence is a<0.

Now, we give illustrated examples

Example 1

$$\mathbb{T}\{t^n\} = \frac{n!}{u^n}\ , \qquad u \neq 0\,, \qquad n = 0,1,2,3, \dots$$

Proof

$$\mathbb{T}\{t^n\} = \int_0^\infty e^{-t} \left(\frac{t}{u}\right)^n dt = u^{-n} \int_0^\infty e^{-t} t^n dt \qquad (2)$$

Using the rule of integral by part n times to get

$$\int_0^\infty e^{-t} t^n dt = -[e^{-t}t^n + ne^{-t}t^{n-1} + \cdots + n!\, e^{-t}]_0^\infty = n! \qquad (3)$$

Substituted the value in (3) into (2) to obtain

$$\mathbb{T}\{t^n\} = \frac{n!}{u^n}\ ,\ \ u \neq 0\,, n = 0,1,2,3, \dots \qquad \square$$

Example 2

$$\mathbb{T}\{e^{at}\} = \frac{u}{u-a} \quad , \quad u \in \begin{cases} R\setminus [0,a], & a \geq 0 \\ R\setminus[a,0], & a < 0 \end{cases}$$

Proof

$$\mathbb{T}\{e^{at}\} = \int_0^\infty e^{-t} e^{\frac{at}{u}} dt = \int_0^\infty e^{-(1-\frac{a}{u})t} dt = \frac{-u}{u-a} e^{-\frac{u-a}{u}t}\Big|_0^\infty = \frac{u}{u-a}$$

By Lemma (1), the condition of convergence is

$$1 - \frac{a}{u} > 0 \quad \rightarrow \quad 1 > \frac{a}{u}$$

If $a \geq 0$ then the condition holds for all $u > a$ or $u < 0$, that is $u \in R\setminus [0,a]$

If $a < 0$ then the condition holds for all $u < a$ or $u > 0$, that is $u \in R\setminus [a,0]$

Example 3

$$1.\ \mathbb{T}\{\sin(at)\} = \frac{au}{a^2+u^2} \quad , u \neq 0$$

$$2.\ \mathbb{T}\{\cos(at)\} = \frac{u^2}{a^2+u^2} \quad , u \neq 0$$

Proof

Using the rule of integral by part with $u \neq 0$, to get

$$\mathbb{T}\{\sin(at)\} = \int_0^\infty e^{-t} \sin\left(\frac{at}{u}\right) dt = \frac{au}{a^2+u^2}$$

Similarly $\mathbb{T}\{\cos(at)\} = \frac{u^2}{a^2+u^2} \quad , u \neq 0$ □

A list of the new transforms for common functionsis presentedin theTable (1). This can be computed in the same manner of previous examples.

Table (1): New transforms for some common functions

$f(t)$	$\bar{f}(u)=\mathbb{T}\{f(t)\}$	$D_{\bar{f}}$
t^n, n=0,1,…	$\frac{n!}{u^n}$	$u\neq 0$
t^a , $a>0$	$\Gamma(a+1)/u^a$	$u\neq 0$
e^{at}	$\frac{u}{u-a}$	$u\in R\backslash[0,a]$ if $a\geq 0$ $u\in R\backslash[a,0]$ if $a<0$
$\sin(at)$	$\frac{au}{a^2+u^2}$	$u\neq 0$
$\cos(at)$	$\frac{u^2}{a^2+u^2}$	$u\neq 0$
$\sinh(at)$	$\frac{-au}{a^2-u^2}$	$\lvert u\rvert>\lvert a\rvert$
$\cosh(at)$	$\frac{-u^2}{a^2-u^2}$	$\lvert u\rvert>\lvert a\rvert$
$u_a(t)=u(t-a)=H(t-a)$	e^{-au}	$u>0$
$\delta(t-a)$	e^{-au}/u	$u>0$
$ln(at)$, a>0	$ln(a/u)-\gamma$	$u>0$

In the next section, we discuss the existence of the new transform

Existence of the New Transform

Does the new transform always exist?It can be shown that

$$\int_0^\infty e^{-t}e^{\left(\frac{t}{u}\right)^2}dt=\infty$$

For every real number u. Hence, the functione^{t^2}does not have a transform.

In this section, we will establish the conditions that ensure the existence of the new transform of a function. We first review some relevant definitions from calculus,and for simplicity, we take the following coding

$$f(t_0+)=\lim_{t\to t_0+}f(t)\quad and\quad f(t_0-)=\lim_{t\to t_0-}f(t)$$

Definition 1 (**Limit**)

The limit of a function $f$at a point $t_0$exists if and only if $f(t_0+)$ and $f(t_0-)$ exists and are equal, that is:

$$\lim_{t\to t_0}f(t)=f(t_0+)=f(t_0-)$$

Definition 2 (**Continuous**)

The function f is continuous at a point t_0 in an open interval (a,b) if and only if

$$\lim_{t\to t_0}f(t)=f(t_0)$$

and equivalent to

$$f(t_0) = f(t_0+) = f(t_0-)$$

Definition 3 (**Jump discontinuity**)

If $f(t_0+) \neq f(t_0-)$then we say that f has a jump discontinuity at t_0, and the value $f(t_0+) - f(t_0-)$ is called the jump in f at t_0.

Definition 4 (**Removable discontinuity**)

If the limit of a function f at a point $t_0$exists, but either f is not defined at t_0 or it's defined but

$$\lim_{t\to t_0} f(t) \neq f(t_0)$$

Then, we say that f has a removable discontinuity at t_0.

Definition 5 (**Piecewise continuous**)

- A function f is said to be piecewise continuous on a closed interval [0,T]if $f(0+)$ and $f(T-)$ are finite and f is continuous on the open interval (0,T) except possibly at finitely manypoints, where f may have jump discontinuities or removable discontinuities.
- A function f is said to be piecewise continuous on the interval $[0,\infty)$ if it's piecewisecontinuous on [0,T] for every T > 0.

Definition 6(**Functions of exponential order**)

A function f is said to be of exponential order s_0 if there exist constants M such that:

$$|f(t)| < Me^{s_0 t} \quad for\ all\ t \in [0,\infty)$$

Theorem 1(**Sufficient conditions**).

If a function f is piecewise continuous on$[0,\infty)$and of exponential order s_0, then the transform of f exists for all $u > s_0$.

Proof

To proof the existences of the transform of f, we must show $\bar{f}(u)$is well-defined, that is, finite, by bounding its absolute value by a finite number.

$$|\bar{f}(u)| = \left|\int_0^\infty e^{-t} f\left(\frac{t}{u}\right) dt\right| = \left|u\int_0^\infty e^{-ux} f(x) dx\right| \qquad (4)$$

We got (4) by putting $x=\frac{t}{u}$.

Since $f$of exponential order s_0, then there exist $M > 0$ such that

$|f(x)| < Me^{s_0 x}\ \ for\ all\ x \in [0,\infty)$, then (4) becomes

$$\left|u\int_0^\infty e^{-ux} f(x) dx\right| \le \int_0^\infty e^{-ux} |uf(x)| dx \le |u| \int_0^\infty e^{-ux}\, M\, e^{s_0 x}\, dx = |u|M \int_0^\infty e^{-(u-s_0)x}\, dx$$

$$= \frac{|u|M}{u-s_0}$$

Then by *lemma*(1),the condition to this result is u-s_0>0, this implies u>s_0. Then we have that

$$|\bar{f}(u)| \le \frac{|u|M}{u-s_0} \square$$

The General Properties of the New Transform.

Theorem 2(Linear combination)

If the transforms $\mathbb{T}\{f\}$and $\mathbb{T}\{g\}$ of thefunctions f and g are well-defined and a, b are constants, then the following equation holds:

$$\mathbb{T}\{af(t)+bg(t)\}=a\mathbb{T}\{f(t)\}+b\mathbb{T}\{g(t)\}$$

Proof

$$\mathbb{T}\{af(t)+bg(t)\}=\int_0^\infty e^{-t}\left(af\left(\frac{t}{u}\right)+bg\left(\frac{t}{u}\right)\right)dt=\int_0^\infty e^{-t}af\left(\frac{t}{u}\right)dt+\int_0^\infty e^{-t}bg\left(\frac{t}{u}\right)dt$$

$$=a\int_0^\infty e^{-t}f\left(\frac{t}{u}\right)dt+b\int_0^\infty e^{-t}g\left(\frac{t}{u}\right)dt=\ a\mathbb{T}\{f(t)\}+b\mathbb{T}\{g(t)\}\square$$

Theorem 3

If the functions $\mathbb{T}\{f\}$ and $\mathbb{T}\{tf\}$ are well-defined then:

$$\mathbb{T}\{tf\}=-u\frac{d}{du}\left(\frac{1}{u}\mathbb{T}\{f\}\right)$$

Proof

$$\mathbb{T}\{tf\}=\int_0^\infty e^{-t}\left(\frac{t}{u}\right)f\left(\frac{t}{u}\right)dt$$

Using the integral by part we have

$$\mathbb{T}\{tf\}=\frac{-t}{u}e^{-t}f\left(\frac{t}{u}\right)\Big|_0^\infty+\int_0^\infty e^{-t}\left[\frac{1}{u}f\left(\frac{t}{u}\right)+\frac{t}{u^2}f'\left(\frac{t}{u}\right)\right]dt$$

$$=0-0+\int_0^\infty e^{-t}\left[-u\frac{d}{du}\left(\frac{1}{u}f\left(\frac{t}{u}\right)\right)\right]dt=-u\frac{d}{du}\left[\frac{1}{u}\int_0^\infty e^{-t}f\left(\frac{t}{u}\right)dt\right]$$

$$=-u\frac{d}{du}\left[\frac{1}{u}\mathbb{T}\{f\}\right]$$

$$\therefore\quad \mathbb{T}\{tf\}=-u\frac{d}{du}\left(\frac{1}{u}\mathbb{T}\{f\}\right)\square$$

Corollary 1

If the functions $\mathbb{T}\{f\}$ and $\mathbb{T}\{t^n f\}$ are well-defined then:

$$\mathbb{T}\{t^n f\}=(-1)^n u\frac{d^n}{u^n}\left(\frac{1}{u}\mathbb{T}\{f\}\right)$$

Proof

The proof is clear by using the Mathematical induction. $\square$

Theorem 4(shift)

If the functions $\bar{f}(u)=\mathbb{T}\{f\}$is the transform of the function *f*(*t*) then:

$$\mathbb{T}\{e^{at}f\}=\frac{u}{u-a}\bar{f}(u-a)$$

Proof

$$\mathbb{T}\{e^{at}f\}=\int_0^\infty e^{-t}e^{\frac{at}{u}}f\left(\frac{t}{u}\right)dt=\int_0^\infty e^{-(1-\frac{a}{u})t}f\left(\frac{t}{u}\right)dt\ (5)$$

If we set $x=(1-\frac{a}{u})t$then ($t=\frac{x}{1-\frac{a}{u}}$and $dt=\frac{dx}{1-\frac{a}{u}}$), substituted it in (5) we have

$$\therefore \quad \mathbb{T}\{e^{at}f\} = \frac{u}{u-a}\int_0^\infty e^{-x} f\left(\frac{x}{u-a}\right)dx = \frac{u}{u-a}\bar{f}(u-a)\square$$

The Convolution of Two Functions

Definition7

Assume that f and g are piecewise continuous functions, or one of them is a Dirac's delta generalized function. The convolution of f and g is a function denotedby $f*g$ and given by the following expression.

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau \qquad (6)$$

In [20] Gabriel Nagy summarized the main properties of the convolution illustrated in the following lemma.

Lemma 2 (Properties)

For every piecewise continuous functions f, g, and h, the following properties hold:

1. Commutativity: $f * g = g * f$.
2. Associativity: $f * (g * h) = (f * g) * h$.
3. Distributivity: $f * (g + h) = f * g + f * h$.
4. Neutral element: $f * 0 = 0$.
5. Identity element: $f * \delta = f$.

Theorem 5

If the functions f and g have well-defined transforms $\mathbb{T}\{f\}$and $\mathbb{T}\{g\}$, then

$$\mathbb{T}\{f*g\} = \frac{1}{u}\mathbb{T}\{f\}\mathbb{T}\{g\}$$

Proof

$$\mathbb{T}\{f*g\} = \int_0^\infty e^{-t}(f*g)\left(\frac{t}{u}\right)dt = \int_0^\infty e^{-t}\left[\int_0^{\frac{t}{u}} f(\tau)g\left(\frac{t}{u}-\tau\right)d\tau\right]dt \qquad (7)$$

Now let $\lambda=\frac{t}{u}$ then $d\lambda=\frac{dt}{u}$, then we get

$$\mathbb{T}\{f*g\} = \int_0^\infty e^{-u\lambda}\left[\int_0^\lambda f(\tau)g(\lambda-\tau)d\tau\right]ud\lambda = u\int_0^\infty\int_0^\lambda e^{-u\lambda}f(\tau)g(\lambda-\tau)d\tau\, d\lambda \qquad (8)$$

From Figure1 and Figure2, It is clear that the regions $R_1=\{(\tau,\lambda)\in R^2: 0\le\tau\le\lambda, 0\le\lambda<\infty\}$ and $R_2=\{(\gamma,\xi)\in R^2: \tau\le\lambda<\infty, 0\le\tau\le\infty\}$ are equal.

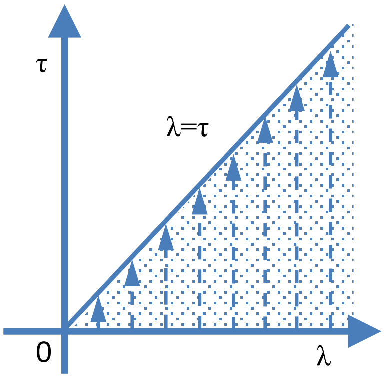

Figure (1): The region R_1

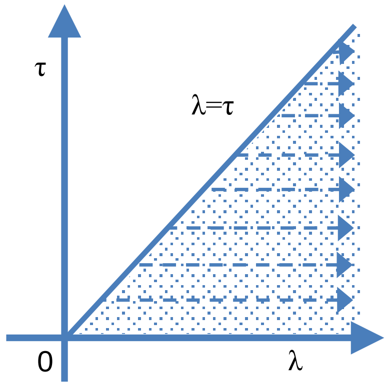

Figure (2): The region R_2

Then we can change the order of integration, i.e.,equation (8) becomes:

$$\mathbb{T}\{f*g\} = u\int_0^\infty \int_\tau^\infty e^{-u\lambda} f(\tau)g(\lambda-\tau)\,d\lambda\,d\tau \qquad (9)$$

Now,if $\gamma = \lambda\text{-}\tau$ then $d\gamma = d\lambda$, since the variable τ is constant, hence when we integrate with respect to τ, we get

$$\mathbb{T}\{f*g\} = u\int_0^\infty \int_0^\infty e^{-u(\gamma+\tau)} f(\tau)g(\gamma)\,d\lambda\,d\tau = u\int_0^\infty e^{-u\tau} f(\tau)\left[\int_0^\infty e^{-u\gamma} g(\gamma)\,d\gamma\right] d\tau$$

$$= u\left[\int_0^\infty e^{-u\tau} f(\tau) d\tau\right]\left[\int_0^\infty e^{-u\gamma} g(\gamma)\,d\gamma\right]$$

If $x = u\tau$ and $y = u\gamma$ then we have $dx = ud\tau$ and $dy = ud\gamma$, i.e.,

$$\therefore \mathbb{T}\{f*g\} = u\left[\int_0^\infty e^{-x} f\left(\frac{x}{u}\right)\frac{dx}{u}\right]\left[\int_0^\infty e^{-y} g\left(\frac{y}{u}\right)\frac{dy}{u}\right] = \frac{1}{u}\mathbb{T}\{f\}\mathbb{T}\{g\}\square$$

6. The Transform of Derivative and Integral

Theorem 6 (Derivatives)

If the functions $\mathbb{T}\{f\}$ and $\mathbb{T}\{f'\}$are well-defined then

$$\mathbb{T}\{f'\} = u(\mathbb{T}\{f\} - f(0))$$

Proof

$$\mathbb{T}\{f'(t)\} = \int_0^\infty e^{-t} f'\left(\frac{t}{u}\right) dt$$

Using the integral by part we have

$$\mathbb{T}\{f'(t)\} = u\,e^{-t} f\left(\frac{t}{u}\right)\Big|_0^\infty + u\int_0^\infty e^{-t} f\left(\frac{t}{u}\right) dt = 0 - uf(0) + u\mathbb{T}\{f\}$$

$$\therefore\ \mathbb{T}\{f'\} = u\mathbb{T}\{f\} - uf(0)\square$$

Corollary2(n^{th}Derivatives)

If the functions $\mathbb{T}\{f\}$,$\mathbb{T}\{f'\}$, …, $\mathbb{T}\{f^{(n)}\}$are well-defined, n=1,2,3,… then

$$\mathbb{T}\{f^{(n)}\} = u^n\mathbb{T}\{f\} - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0)$$

Proof

We will use the Mathematical induction

If n=1 then the corollary (2) holds by theorem (6).

Now suppose it holds for n=m, i.e.,

$$\mathbb{T}\{f^{(m)}\} = u^m\mathbb{T}\{f\} - \sum_{k=0}^{m-1} u^{m-k} f^{(k)}(0)$$

and suppose n=m+1:

$$\begin{aligned}\mathbb{T}\{f^{(m+1)}\} &= \mathbb{T}\left\{\left(f^{(m)}\right)'\right\} = u\mathbb{T}\{f^{(m)}\} - uf^{(m)}(0)\\ &= u\left[u^m\mathbb{T}\{f\} - \sum_{k=0}^{m-1} u^{m-k} f^{(k)}(0)\right] - uf^{(m)}(0)\\ &= u^{m+1}\mathbb{T}\{f\} - \sum_{k=0}^{m} u^{(m+1)-k} f^{(k)}(0)\end{aligned}$$

Then the corollary is performing for any nonnegative integer n.□

Theorem 7 (Derivatives of other variables)

If the function $\mathbb{T}\{f(t,x)\}$is well-defined then:

$$\mathbb{T}\left\{\frac{\partial^n}{\partial x^n} f(t,x)\right\} = \frac{\partial^n}{\partial x^n}\mathbb{T}\{f(t,x)\} \qquad , \quad n = 1,2,3, \ldots$$

Proof

$$\mathbb{T}\left\{\frac{\partial^n}{\partial x^n} f(t,x)\right\} = \int_0^\infty e^{-t}\frac{\partial^n}{\partial x^n} f\left(\frac{t}{u},x\right)dt = \frac{\partial^n}{\partial x^n}\left(\int_0^\infty e^{-t} f\left(\frac{t}{u},x\right)dt\right)$$

$$\therefore \quad \mathbb{T}\left\{\frac{\partial^n}{\partial x^n} f(t,x)\right\} = \frac{\partial^n}{\partial x^n}\mathbb{T}\{f\} \qquad \square$$

Theorem 8(**Integral**)

If the function $\mathbb{T}\{f(t)\}$ is well-defined then

$$\mathbb{T}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{u}\mathbb{T}\{f\}$$

Proof

Suppose that $g(t\text{-}\tau)$=1, then from theorem (5) and example (1) we have

$$\mathbb{T}\left\{\int_0^t f(\tau)d\tau\right\} = \mathbb{T}\left\{\int_0^t g(t-\tau)f(\tau)d\tau\right\} = \mathbb{T}\{g * f\} = \frac{1}{u}\mathbb{T}\{f\}\mathbb{T}\{g\} = \frac{1}{u}\mathbb{T}\{f\}$$

The Inverse of New Transform

Definition8

Let the functions $\bar{f}(u) = \mathbb{T}\{f\}$ is the transform of the function *f*(*t*), then *f*(*t*) is called the inverse transform of the function $\bar{f}(u)$ and we will write it as :

$$f(t) = \mathbb{T}^{-1}\{\bar{f}(u)\}$$

Remark: The inverse transform has the linear combination property, i.e.,

$$\mathbb{T}^{-1}\left\{\sum_{k=1}^{n} a_k \bar{f}_k(u)\right\} = \sum_{k=1}^{n} a_k \mathbb{T}^{-1}\{\bar{f}_k(u)\}$$

The Duality with Laplace Transform

Theorem 9

Let *f*satisfy the conditions in theorem1, and has Laplace transform $L\{f(t)\} = \hat{f}(s)$. Then the transform $\bar{f}(u)$ of*f*(*t*) is given by

$$\bar{f}(u) = u\hat{f}(u)$$

Proof

$$\bar{f}(u) = \int_0^\infty e^{-t} f\left(\frac{t}{u}\right) dt \qquad (10)$$

If we set $x=\frac{t}{u}$then ($t = ux$and $dt=udx$), substituted it in equation (10) to get

$$\bar{f}(u) = \int_0^\infty e^{-t} f\left(\frac{t}{u}\right) dt = u\int_0^\infty e^{-ux} f(x)dx = u\hat{f}(u) \qquad (11)$$

□

Corollary3

Let f satisfy the conditions in theorem (1)and has the transform$\bar{f}(u)$. Then the Laplace transform $\hat{f}(s)$of *f*(*t*) is given by

$$\hat{f}(s) = \frac{1}{s}\bar{f}(s) \qquad (12)$$

The Advantages of the New Transform

The new transform has many interesting properties which make it rival to the Laplace transform. Some of these properties are:

1. The domain of the new transform is wider than or equal to the domain of Laplace transform as illustrated in the Table (2). This feature makes the new transform more widely used in problems.
2. From section 8 the new transform has the duality with Laplace transform, hence, that needs a wider domain.
3. The new transform can solve all the problems which would be solved by Laplace transform.
4. The unit step function in the *t*-domain is transformed to unity in the *u*-domain.
5. The differentiation and integration in the *t*-domain are equivalent to multiplication and division of the transformed function F(*u*) by *u* in the *u*-domain.
6. By Theorem 2(Linear combination) and example (1) we have that for any constant $a \in$R,$\mathbb{T}\{a\} = a\mathbb{T}\{1\} = a$, and hence, $\mathbb{T}^{-1}\{a\} = a$, that is, we don't have any problem when we deal with the constant term(the constant with respect to the parameter *u*).

Conclusion

The new transform has some strength points for feature among others transform such that:The new transform easier than the Laplace transform for beginners to understand and use. Itcan be used to solve problems without resorting to the frequency domain. Especially, with respect to applications in problems with physical dimensions.We showed it to be the theoretical dual to the Laplace transform, and hence ought to rival it in solving intricate problems in engineering mathematics and applied science.Also, the new transform is a convenient tool for solving differential equations in the time domain without the need for performing its inverse.The connection of the new transform with the Laplace transform goes much deeper.We also present many of the new transform properties that make it uniquely qualified to address and solve some applied problems, especially ones in which the units of the problem must be preserved.

Table (2): Illustrates the domain of Laplace and new transform

f(t)	Laplace		New Transform	
	[L(f)](s)	Domain	[Ŧ(f)](u)	Domain
t^n n=0,1,2,…	$n!/s^{n+1}$	$s>0$	$n!/u^n$	$u\neq 0$
t^a a>0	$\Gamma(a+1)/s^{a+1}$	$s>0$	$\Gamma(a+1)/u^a$	$u\neq 0$
e^{at}	$1/(s-a)$	$s>a$	$u/(u-a)$	$u\in R/[0,a]$ if $a\geq 0$ $u\in R/[a,0]$ if $a<0$
$\sin(at)$	$a/(s^2+a^2)$	$s>0$	$au/(u^2+a^2)$	$u\neq 0$
$\cos(at)$	$s/(s^2+a^2)$	$s>0$	$u^2/(u^2+a^2)$	$u\neq 0$
$\sinh(at)$	$a/(s^2-a^2)$	$s>\|a\|$	$au/(u^2+a^2)$	$\|u\|>\|a\|$
$\cosh(at)$	$s/(s^2-a^2)$	$s>\|a\|$	$u^2/(u^2-a^2)$	$\|u\|>\|a\|$
$u_a(t)=u(t-a)=H(t-a)$	e^{-as}/s	$s>0$	e^{-au}	$u>0$
$\delta(t-a)$	e^{-as}	$s>0$	e^{-au}/u	$u>0$
$ln(at)$ a>0	$(ln(a/s)-\gamma)\ /s$	$s>0$	$ln(a/u)-\gamma$	$u>0$
$\gamma = -\int_0^\infty e^{-t} lnt\, dt \cong 0.5772 \ldots$				

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