The Construction of Minimal (b,t)-Blocking Sets Containing Conics in PG(2,5) with the Complete Arcs and Projective Codes Related with Them

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Received in : 28 September 2014 , Accepted in : 21 December 2014

Abstract

A (b,t)-blocking set B in PG(2,q) is set of b points such that every line of PG(2,q) intersects B in at least t points and there is a line intersecting B in exactly t points.
In this paper we construct a minimal (b,t)-blocking sets, t = 1,2,3,4,5 in PG(2,5) by using conics to obtain complete arcs and projective codes related with them.

Keywords: Blocking set, complete arc, projective code.

1- Introduction

Let GF(q) denotes the Galois field of q elements and V(3,q) be the vector space of row vectors of length three with entries in GF(q). Let PG(2,q) be the corresponding projective plane. The points of PG(2,q) are the non zero vectors of V(3,q) with the rule that $X = (x_1,x_2,x_3)$ and $y = (\lambda x_1,\lambda x_2,\lambda x_3)$ represent the same point, where $\lambda \in GF(q)\backslash\{0\}$. The number of points of PG(2,q) is $q^2 + q + 1$.

If the point P(X) is the equivalence class of the vector X, then we will say that X is a vector representing P(X). A subspace of dimension one is a set of points all of whose representing vectors form a subspace of dimension two of V(3,q), such subspaces are called lines.

The number of lines in PG(3,q) is $q^2 + q + 1$. There are q + 1 points on every line and q + 1 lines through every point. The point $X(x_1,x_2,x_3)$ is on the line $Y[y_1,y_2,y_3]$ if and only if $x_1y_1 + x_2y_2 + x_3y_3 = 0$.

Definition (1.1): [1]

A (*k*,*n*)–arc is a set of *k* points of a projective plane such that some n but no *n* + 1 of them are collinear, $n \geq 2$.

Definition (1.2): [2]

A (*k*,*n*)–arc is complete if it is not contained in a (*k* + 1,*n*)-arc.

Definition (1.3): [2]

A line l in PG(2,q) is an *i*-secant on a (k,n)-arc K if $|\ell \cap K| = i$.

Definition (1.4): [2]

A point N which is not on a (*k*,*n*)-arc has index *i* if there are exactly *i* (*n*-secants) of the arc through N, we denote the number of points N of index *i* by N_i.

Remark (1.5): [3]

The (*k*,*n*)-arc is complete iff $N_0 = 0$. Thus the arc is complete iff every point of PG(2,q) lies on some *n*-secant of the arc.

Definition (1.6): [3]

An (b,t)-blocking set B in PG(2,q) is a set of b points such that every line of PG(2,q) intersects B in at least t points, and there is a line intersecting B in exactly t points. If B contains a line, it is called trivial, thus B is a subset of PG(2,q) which meets every line ℓ in PG(2,q), but contains no line completely; that is $t \leq |B \cap \ell| \leq q$ for every line ℓ in PG(2,q). So B is a blocking set iff PG(2,q)\B is a blocking set. A blocking set is minimal if B\{P} is not blocking set for every p in B.

Lemma (1.7): [4]

A (b,1)-blocking set B is minimal in PG(2,q) iff there is a line ℓ in PG(2,q) such that $B \cap \ell = \{Q\}$ for every Q in B.

Definition (1.8): [3]

A variety V(F) of PG(2,q) is a subset of PG(2,q) such that:
$V(F) = \{P(A) \in PG(2,q) \mid F(A) = 0\}$.

Definition (1.9): [5]

Let Q(2,q) be the set of quadrics in PG(2,q); that is the varieties V(F), where:

$$F = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3 \quad ...(1)$$

If V(F) is non-singular, then the quadric is a **conic**.

That is, if $A = \begin{bmatrix} a_{11} & \frac{a_{12}}{2} & \frac{a_{13}}{2} \\ \frac{a_{12}}{2} & a_{22} & \frac{a_{23}}{2} \\ \frac{a_{13}}{2} & \frac{a_{23}}{2} & a_{33} \end{bmatrix}$ is nonsingular, then the quadric (1) is a conic.

1.10 The Relation Between The Blocking (b,t)-Set and The (k,n)-arc [5]

The (k,n)-arc and the (b,t)-blocking set are each complement to the other in the projective plane PG(2,q), that is, $n + t = q + 1$ and $k + b = q^2 + q + 1$. Thus the complement of the (b,t)-blocking set is the set of points that intersects every line in at most n points which represents the (k,n)-arc. Also finding minimal (b,t)-blocking set is equivalent to finding maximal (k,n)-arc in PG(2,q).

Lemma (1.11): [4]

Let $\beta = C \cup \ell \cup \{P\} \setminus \{P_1,P_2\}$, where C is a conic, ℓ is a (2-secant) of C such that $C \cap \ell = \{P_1,P_2\}$, P is the point of intersection of the two tangents to C at P_1 and P_2, then β is a minimal $(2p - 1,1)$-blocking set.

Definition (1.12): [5]

Let V(n,q) denote the vector space of all ordered n-tuples over GF(q). A linear code C over GF(q) of length n and dimension k is a k-dimensional subspace of V(n,q). The vectors of C are called code words. The Hamming distance between two codewords is defined to be the number of coordinate places in which they differ. The minimum distance of a code is the smallest distances between distinct codewords. Such a code is called an $[n,k,d]_q$ code if its minimum hamming distance is d.

There exists a relationship between complete (n,r)-arcs in PG(2,q) and $[n,3,d]_q$ codes given by the next theorem.

Theorem (1.13): [5]

There exists a projective $[n,3,d]_q$ code if and only if there exists an $(n,n - d)$-arc in PG(2,q).

Theorem (1.14): [6]

Let β_2 be a double blocking set in PG(2,q):

(1) If $q < 9$, then β_2 has at least 3q points.

(2) If $q = 11, 13, 17$ or 19, then $|\beta_2| \geq (5q + 7)/2$.

Theorem (1.15): [6]

Let β_3 be a trible blocking set in PG(2,q):

(1) If $q = 5, 7,9$, then β_3 has at least 4q points and if $q = 8$, then β_3 has at least 31 points.

(2) If $q = 11, 13$ or 17, then $|\beta_3| \geq (7q + 9)/2$.

Now, we prove the following theorem:

Theorem (1.16):

A (b,t)-blocking set B is minimal in PG(2,q) then every point P in B there is a t-secant of B containing P.

Proof:

Suppose B is minimal blocking set, let P be any point in B. Let K be the complement of B, then K is complete (k,n)-arc in PG(2,q) and P is not K., then P is an (n-secant) of K, but $q + 1 = t + n$ and so $t = q + 1 - n$. Thus P is on an (t-secant) of B.

2- The Projective Plane PG(2,5)

In this paper we consider the case q = 5 and the elements of GF(5) are denoted by 0,1,2,3,4.

A projective plane π = PG(2,5) over GF(5) consists of 31 points, 31 lines each line contains 6 points and through every point there is 6 lines.

Let P_i and ℓ_i be the points and lines of PG(2,5) respectively. Let i stands for the point P_i, i = 1,2,…,31. The points and lines of PG(2,5) are given in the table (1).

2.1 The Conic in PG(2,5) Through The Reference and Unit Points

The general equation of the conic is:

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3 = 0 \qquad \dots(1)$$

By substituting the reference points:
1(1,0,0), 2(0,1,0), 7(0,0,1) and the unit point 13 (1,1,1), which are four points no three of them are collinear, in (1), we get:
$a_{12} + a_{13} + a_{23} = 0$ and $a_{11} = a_{22} = a_{33} = 0$, so (1) becomes:

$$a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3 = 0 \qquad \dots(2)$$

If $a_{12} = 0$, then the conic is degenerated, therefore $a_{12} \neq 0$, similarly, $a_{13} \neq 0$ and $a_{23} \neq 0$.
Dividing equation (2) by a_{12}, we get:

$x_1x_2 + \alpha x_1x_3 + \beta x_2x_3 = 0$,where $\alpha = \frac{a_{13}}{a_{12}}$, $\beta = \frac{a_{23}}{a_{12}}$, then $\beta = -(1 + \alpha)$ since $1 + \alpha + \beta = 0$ (mod 5).

Then $x_1 x_2+\alpha x_1x_3-(1+\alpha) x_2x_3 =0$, where $\alpha \neq 0$ and $\alpha \neq 4$, for if $\alpha = 0$ or $\alpha = 4$ we get a degenerated conic, that is, $\alpha = 1,2,3$.

2.2 The Equations and the Points of the Conics in PG(2,5) Through the Reference and Unit Points

For any value of α, there is a unique conic contains 6 points, 4 of them are the reference and unit points

1. If $\alpha = 1$, then the equation of the conic C_1 is
$$x_1x_2 + x_1x_3 + 3x_2x_3 = 0$$
The points of C_1 are : 1,2,7,13,20,26.

2. If $\alpha = 2$, then the equation of the conic C_2 is
$$x_1x_2 + 2x_1x_3 + 2x_2x_3 = 0$$
The points of C_2 are : 1,2,7,13,21,29.

3. If $\alpha = 3$, then the equation of the conic C_3 is
$$x_1x_2 + 3x_1x_3 + x_2x_3 = 0$$
The points of C_3 are : 1,2,7,13,24,30.

Thus we found five conics two of them are degenerated and the remaining three conics C_1, C_3, C_3 are non-degenerated.

Table (1)

i	*P_i*			*L_i*					
1	1	0	0	2	7	12	17	22	27
2	0	1	0	1	7	8	9	10	11
3	1	1	0	6	7	16	20	24	28
4	2	1	0	4	7	14	21	23	30
5	3	1	0	5	7	15	18	26	29
6	4	1	0	3	7	13	19	25	31
7	0	0	1	1	2	3	4	5	6
8	1	0	1	2	11	16	21	26	31

9	2	0	1	2	9	14	19	24	29
10	3	0	1	2	10	15	20	25	30
11	4	0	1	2	8	13	18	23	28
12	0	1	1	1	27	28	29	30	31
13	1	1	1	6	11	15	19	23	27
14	2	1	1	4	9	16	18	25	27
15	3	1	1	5	10	13	21	24	27
16	4	1	1	3	8	14	20	26	27
17	0	2	1	1	17	18	19	20	21
18	1	2	1	5	11	14	17	25	28
19	2	2	1	6	9	13	17	26	30
20	3	2	1	3	10	16	17	23	29
21	4	2	1	4	8	15	17	24	31
22	0	3	1	1	22	23	24	25	26
23	1	3	1	4	11	13	20	22	29
24	2	3	1	3	9	15	21	22	28
25	3	3	1	6	10	14	18	22	31
26	4	3	1	5	8	16	19	22	30
27	0	4	1	1	12	13	14	15	16
28	1	4	1	3	11	12	18	24	30
29	2	4	1	5	9	12	20	23	31
30	3	4	1	4	10	12	19	26	28
31	4	4	1	6	8	12	21	25	29

2.3 The Construction of Minimal (b,t)-Blocking Sets By Using Conic-Type Blocking Sets

We construct minimal (b,t)-blocking set in PG(2,5) from the minimal blocking (9,1)-sets of lemma (1.15) by using conic.

2.3.1 The Construction of Minimal (9,1)-Blocking Set by Lemma (1.11)

We take the conic C_1 in section 2.

Let $\beta_1 = C_1 \cup L_1 \setminus \{P_1,P_2\} \cup \{P\}$, $C_1 = \{1,2,7,13,20,26\}$, $L_1 = \{2,7,12,17,22,27\}$, $C_1 \cap L_1 = \{2,7\}$, L_4 and L_9 are the two tangents to C_1 at the points 7 and 2 respectively. $L_4 \cap L_9 = \{14\}$, then

$\beta_1 = \{1,12,13,14,17,20,22,26,27\}$, β_1 is a (9,1)-blocking set in PG(2,5). Since each point of β_1 is on line ℓ in PG(2,9) such that $\beta_1 \cap \ell = \{P\}$ (lemma 1.7), β_1 satisfies the following conditions:

(a) β_1 intersects every line in PG(2,5) in at least one point.

(b) Every point in β_1, there is a line ℓ in PG(2,5) such that $\beta_1 \cap \ell = \{P\}$.

The complement of β_1 is the complete (22,5)-arc K_5, by theorem (1.13) there exists a projective [22,3,17] code.

2.3.2 The Construction of Minimal (b,2)-Blocking Set In PG(2,5)

We construct two (9,1)-blocking sets.

Let $\beta_1 = \{1,12,13,14,17,20,22,26,27\}$ be the minimal (9,1)-blocking set of section (2.3.1). We construct another (9,1)-blocking set

$\alpha_1 = C_2 \cup L_8 \setminus \{C_2 \cap L_8\} \cup \{15\}$, where $C_2 = \{1,2,7,13,21,29\}$, $L_8 = \{2,11,16,21,26,31\}$, $C_2 \cap L_8 = \{2,21\}$, $L_{10} \cap L_{24} = \{15\}$ and L_{10} and L_{24} are tangents to C_2 at the points 2 and 21 respectively.

$\alpha_1 = \{1,7,11,13,15,16,26,29,31\}$ is (9,1)-blocking set.

Now, we construct (b,2)-blocking set as follows:

Let A = $\alpha_1 \cup \beta_1$ = {1,7,11,12,13,14,15,16,17,20,22,26,27,29,31}.
A must satisfies the following conditions:
(a) A intersects every line of PG(2,5) in at least two points.
(b) Every point in A is on at least one 2-secant of A.
We add three points 3,10 and 18 to A and eliminate the points 15 and 26 from A to satisfy these conditions, then:
β_2 = A∪{3,10,18}\ {15,26} = {1,3,7,10,11,12,13,14,16,17,18,20,22,27,29,31} is a minimal (16,2)-blocking set. The complement of β_2 is the complete (15,4)-arc K_4. By theorem (1.13) there exists a projective [15,3,11] code.

2.3.3 The Construction of Minimal (b,3)-Blocking Set In PG(2,5)

We take the (9,1)-blocking sets in section (2.3.2)
α_1 = {1,7,11,13,15,16,26,29,31}, β_1 = {1,12,13,14,17,20,22,26,27}, Let $\gamma_1 = C_3 \cup L_{28} \cup \{8\} \setminus \{C_3 \cap L_{28}\}$, C_3 ={1,2,7,13,24,30}, L_{28} = {3,11,12,18,24,30}, $C_3 \cap L_{28}$ = {24,30} and $L_{21} \cap L_{26}$ = {8}, where L_{21} and L_{26} are tangents to C_3 at the points 24 and 30 respectively.
γ_1 = {1,2,3,7,8,11,12,13,18} is a minimal (9,1)-blocking set.
We must construct a minimal (b,3)-blocking set from α_1, β_1 and γ_1 as follows:.
Let B= $\alpha_1 \cup \beta_1 \cup \gamma_1$ ={1,2,3,7,8,11,12,13,14,15,16,17,18,20,22,26,27,29,31}.
B must satisfy the following conditions:
(a) B intersects every line in PG(2,5) in at least three points.
(b) Every point in B is on at least one 3-secant of B.
We add two points 4 and 5 to B and eliminate the point 31 from B to satisfy these conditions, then:
β_3 = B∪{4,5}\ {31} = {1,2,3,4,5,7,8,11,12,13,14,15,16,17,18,20,22,26,27,29} is a minimal (20,3)-blocking set which is trivial since β_3 contains some lines completely. The complement of β_3 is the complete (11,3)-arc K_3. By theorem (1.13) there exists a projective [11,3,8] code in PG(2,5).

2.3.4 The Construction of Minimal (b,4)-Blocking Set In PG(2,5)

We take three minimal (9,1)-blocking sets in section (2.3.3) which are:
α_1 = {1,7,11,13,15,16,26,29,31}, β_1 = {1,12,13,14,17,20,22,26,27},
γ_1 = {1,2,3,7,8,11,12,13,18}.
Let $\omega_1 = C_1 \cup L_2 \cup \{30\} \setminus \{C_1 \cap L_2\}$, where C_1 is the conic C_1 = {1,2,7,13,20,26}, L_2 = {1,7,8,9,10,11}, $C_1 \cap L_2$ = {1,7}, $L_4 \cap L_{12}$ = {30}, L_4 and L_{12} are tangents to C_1 at the points 7 and 1 respectively, then.
ω_1 = {2,8,9,10,11,13,20,26,30} is a minimal (9,1)-blocking set.
We construct a minimal (b,4)-blocking set from α_1, β_1, γ_1 and ω_1 as follows:.
Let C = $\alpha_1 \cup \beta_1 \cup \gamma_1 \cup \omega_1$ = {1,2,3,7,…,14,15,16,17,18,20,22, 26,27,29,30,31}. C must satisfy the following conditions:
(a) C intersects every line in at least four points.
(b) Every point in C is on at least one 4-secant of C.
We add the points 6,45,21,24,28 to C, and eliminate one point 29 from C to satisfy these conditions, then:
β_4=C∪{6,21,24,28}\{29}={1,2,3,6,7,…,18,20,21,22,24,26,27,28,30,31} is a minimal (25,4)-blocking set which is trivial since β_4 contains some lines completely. The complement of β_4 is the complete (6,2)-arc K_2. By theorem (1.13) there exists a projective [6,3,4] code.

2.3.5 The Construction of Minimal (b,5)-Blocking Set In PG(2,5)

We take four minimal (9,1)-blocking sets of section (2.3.4) which are
α_1 = {1,7,11,13,15,16,26,29,31}, β_1 = {1,12,13,14,17,20,22,26,27},
γ_1 = {1,2,3,7,8,11,12,13,18}, ω_1 = {2,8,9,10,11,13,20,26,30}.
We construct another minimal (9,1)-blocking set.

Let $\delta_1 = C_2 \cup L_6 \setminus \{7,13\} \cup \{24\}$, where C_2 is a conic, $C_2 = \{1,2,7,13,21,29\}$, $L_6 = \{3,7,13,19,25,31\}$, $C_2 \cap L_6 = \{7,13\}$, $L_3 \cap L_{22} = \{24\}$, where L_3 and L_{22} are tangents to C_2 at the points 7 and 13 respectively, then.

$\delta_1 = \{1,2,3,19,21,24,25,29,31\}$ is a minimal (9,1)-blocking set.

Now, we must construct a minimal (b,5)-blocking set from α_1, β_1, γ_1, ω_1 and δ_1 as follows:.

Let $D=\alpha_1\cup\beta_1\cup\gamma_1 \cup\omega_1 \cup\delta_1=\{1,2,3,7,\ldots,22, 24,\ldots,27,29,30,31\}$. D must satisfy the following conditions:

(a) D intersects every line in at least five points.

(b) Every point of D is on at least one 5-secant of D.

We add four points 5,6,23,28 to D to satisfy these conditions, then:

$\beta_5 = D \cup \{5,6,23,28\} = \{1,2,3,5,\ldots,31\}$ is a minimal (30,5)-blocking set which is trivial since β_5 contains some lines completely. The complement of β_5 is not arc since every (k,n) cannot exist when $n < 2$.

Conclusion

1. We construct a minimal (9,1)-blocking set, which is containing a conic as in lemma (1.12). Also we construct minimal (16,2)-blocking by taking the union of two blocking (9,1)-sets of type in lemma (1.12). We construct minimal (20,3)-blocking set, by taking the union of three (9,1)- blocking sets of type in lemma (1.12). We construct minimal (25,4)-blocking set by taking the union of four (9,1)-blocking sets of type in lemma (1.12) and finally we construct minimal (30,5)-blocking set B_5 by taking the union five (9,1)-blocking sets of type in lemma (1.12).
2. The minimal (9,1)-blocking set B_1 and the minimal (16,2)-blocking set B_2 are non-trivial, but the minimal (20,3)-blocking set B_3, the minimal (25,4)-blocking set B_4 and the minimal (30,5)-blocking set B_5 are trivial

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بناء مجموعات قالبية –(b,t) صغرى تحتوي على مخروطيات في PG(2,5) والاقواس الكاملة والشفرات الاسقاطية المرتبطة بها

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أستلم البحث في : 28ايلول 2014, قبل البحث في : 21كانون الاول 2014

الخلاصه

المجموعة القالبية - B(b,t) في PG(2,q) هي مجموعة من b من النقاط بحيث ان كل مستقيم في PG(2,q) يقطع B في t من النقاط في الاقل ويوجد مستقيم يقطع B في t من النقاط فقط.
في هذا البحث قمنا ببناء مجموعات قالبية – (b,t) صغرى في PG(2,5)، t = 1,2,3,4,5 ، باعتماد مخروطيات وحصلنا على أقواس كاملة وشفرات إسقاطية مرتبطة بها.

الكلمات المفتاحية : مجموعة قالبية ، قوس كامل ، شفرة إسقاطية.