Complex of Lascoux in Partition (3,3,2)

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Abstract

In this paper, the complex of Lascoux in the case of partition (3,3,2) has been studied by using diagrams ,divided power of the place polarization $\partial_{ij}^{(k)}$,Capelli identites and the idea of mapping Cone .

Key words: Divided power algebra**,** Resolution of Weyl module**,** Place polarization**,** Mapping Cone.

1.Introduction

Let R be commutative ring with 1, F be a free module and $D_S F$ be the divided power of degree s . In [1] Buchsbaum used another type of maps whose images define Schur and Weyl modules which sends an element $a \otimes b$ of $D_{p+k} \otimes D_{q-k}$ to $\sum a_p \otimes a'_k b$, where $\sum a_p \otimes a'_k$ is the component of the diagonal of a in $D_p \otimes D_k$, the generalization of this map to ones, where there more factors were called in the 'box map'.

In [3] ,[4] and [5] the author studied the complex of characteristic zero in the partition (2,2,2) ,(3,3,3) and (4,4,3), using this modified and the letter place methods [3] , In this paper we study the complex of Lascoux in the case of partition (3,3,2) as a diagram by using the idea of the mapping Cone [6] , and the map $\partial_{ij}^{(k)}$ which means the k^{th} divided power of the place polarization ∂_{ij} where j must be less than i with it's Capelli identities [1] , specificly in this work we used only the following identities

$$\partial_{32}^{(l)}\partial_{21}^{(k)} = \sum_{\alpha \geq 0} \partial_{21}^{(k-\alpha)}\partial_{32}^{(l-\alpha)}\partial_{31}^{(\alpha)} \quad (1.1)$$

$$\partial_{21}^{(k)}\partial_{32}^{(l)} = \sum_{\alpha \geq 0} (-1)^{\alpha}\partial_{32}^{(l-\alpha)}\partial_{21}^{(k-\alpha)}\partial_{31}^{(\alpha)} \quad (1.2)$$

$$\partial_{21}^{(1)} \circ \partial_{31}^{(1)} = \partial_{31}^{(1)} \circ \partial_{21}^{(1)} \quad and \quad \partial_{32}^{(1)} \circ \partial_{31}^{(1)} = \partial_{31}^{(1)} \circ \partial_{32}^{(1)} \quad (1.3)$$

Where ∂_{ij} is the place polarization from place j to place i .

2.The terms of Lascoux complex in the case of partition (3, 3, 2)

The terms of the Lascaux complex are obtained from the determinantal expansion of the Jocobi-Trudi matrix of the partition. The positions of the terms of the complex are determined by the length of the permutation to which they correspond [2], [3]. Now in the case of the partition $\lambda = (3,3,2)$, we have the following matrix:

$$\begin{bmatrix} D_3 F & D_2 F & D_0 F \\ D_4 F & D_3 F & D_1 F \\ D_5 F & D_4 F & D_2 F \end{bmatrix}$$

Then the Lascoux complex has the correspondence between it's terms as follows:

$D_3 F \otimes D_3 F \otimes D_2 F \leftrightarrow identity$

$D_3 F \otimes D_1 F \otimes D_4 F \leftrightarrow (2\,3)$

$D_2 F \otimes D_1 F \otimes D_5 F \leftrightarrow (1\,2\,3)$

$D_2 F \otimes D_4 F \otimes D_2 F \leftrightarrow (2\,3)$

$D_0 F \otimes D_3 F \otimes D_5 F \leftrightarrow (1\,3)$

$D_0 F \otimes D_4 F \otimes D_4 F \leftrightarrow (1\,3\,2)$

So, the complex of Lascoux in the case of the partition $\lambda = (3,3,2)$ has the form:-

$$D_5 F \otimes D_3 F \otimes D_0 F \to \begin{matrix} D_4 F \otimes D_4 F \otimes D_0 F \\ \oplus \\ D_5 F \otimes D_2 F \otimes D_1 F \end{matrix} \to \begin{matrix} D_3 F \otimes D_4 F \otimes D_1 F \\ \oplus \\ D_4 F \otimes D_2 F \otimes D_2 F \end{matrix} \to D_3 F \otimes D_3 F \otimes D_2 F$$

3.The complex of Lascoux as a diagram

Consider the following diagram :

$$\begin{array}{ccccc} D_5 F \otimes D_3 F \otimes D_0 F & \xrightarrow{p_1} & D_4 F \otimes D_4 F \otimes D_0 F & \xrightarrow{p_2} & D_3 F \otimes D_4 F \otimes D_1 F \\ k_1 \downarrow & \text{S} & \downarrow k_2 & \text{H} & \downarrow k_3 \\ D_5 F \otimes D_2 F \otimes D_1 F & \xrightarrow{q_1} & D_4 F \otimes D_2 F \otimes D_2 F & \xrightarrow{q_2} & D_3 F \otimes D_3 F \otimes D_2 F \end{array}$$

So, if we define

$p_1 : D_5 F \otimes D_3 F \otimes D_0 F \to D_4 F \otimes D_4 F \otimes D_0 F$

as $p_1(v) = \partial_{21}(v)$ where ; $v \in D_5 F \otimes D_3 F \otimes D_0 F$

$k_1 : D_5 F \otimes D_3 F \otimes D_0 F \to D_5 F \otimes D_2 F \otimes D_1 F$

as $k_1(v) = \partial_{32}(v)$ where ; $v \in D_5 F \otimes D_3 F \otimes D_0 F$

$k_2 : D_4 F \otimes D_4 F \otimes D_0 F \to D_4 F \otimes D_2 F \otimes D_2 F$

as $k_2(v) = \partial_{32}^{(2)}(v)$ where ; $v \in D_4 F \otimes D_4 F \otimes D_0 F$.

Now, we have to define the following map which makes the diagram S commutative:

$q_1 : D_5 F \otimes D_2 F \otimes D_1 F \to D_4 F \otimes D_2 F \otimes D_2 F$

so we have:

$q_1 \circ k_1 = k_2 \circ p_1$

which implies that

$q_1 \circ \partial_{32} = \partial_{32}^{(2)} \circ \partial_{21}$

Now we use Capelli identities from (1.1), (1.2):

$$\partial_{32}^{(2)} \circ \partial_{21} = \partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{31}^{(2)} \circ \partial_{32}^{(1)}$$
$$= (\partial_{21}^{(1)} \circ \tfrac{1}{2}\partial_{32}^{(1)} + \partial_{31}^{(1)}) \circ \partial_{32}^{(1)}$$

Thus , $q_1 = \frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)}$.

On the other hand, if we define

$q_2 : D_4 F \otimes D_2 F \otimes D_2 F \to D_3 F \otimes D_3 F \otimes D_2 F$

as $q_2(v) = \partial_{21}(v)$ where ; $v \in D_4 F \otimes D_2 F \otimes D_2 F$

and $k_3 : D_3 F \otimes D_4 F \otimes D_1 F \to D_3 F \otimes D_3 F \otimes D_2 F$

as $k_3(v) = \partial_{32}(v)$ where ; $v \in D_3 F \otimes D_4 F \otimes D_1 F$.

Now we need to define p_2 to make the diagram H commute:

$p_2 : D_4 F \otimes D_4 F \otimes D_0 F \to D_3 F \otimes D_4 F \otimes D_1 F$.

Such that $k_3 \circ p_2 = q_2 \circ k_2$ i.e. $\partial_{32} \circ p_2 = \partial_{21} \circ \partial_{32}^{(2)}$

again by using Caplli identities we get

$$\partial_{21}^{(1)} \circ \partial_{32}^{(2)} = \partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$$
$$= \partial_{32} \circ (\tfrac{1}{2}\partial_{32}^{(1)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)})$$

then $p_2 = \frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}$.

Now consider the following diagram:

$$\begin{array}{ccccc}
D_5 F \otimes D_3 F \otimes D_0 F & \xrightarrow{p_1} & D_4 F \otimes D_4 F \otimes D_0 F & \xrightarrow{p_2} & D_3 F \otimes D_4 F \otimes D_1 F \\
\downarrow k \quad \text{E} & & \nearrow \omega & & \text{F} \quad \downarrow k_3 \\
D_5 F \otimes D_2 F \otimes D_1 F & \xrightarrow{q_1} & D_4 F \otimes D_2 F \otimes D_2 F & \xrightarrow{q_2} & D_3 F \otimes D_3 F \otimes D_2 F
\end{array}$$

Define $\omega : D_5 F \otimes D_2 F \otimes D_1 F \to D_2 F \otimes D_4 F \otimes D_1 F$

by $\omega(v) = \partial_{21}^{(2)}(v)$ where ; $v \in D_5 F \otimes D_2 F \otimes D_1 F$.

Proposition 3.1: The diagram E is commutative.

proof:- To prove E is commutative, we need to prove $p_2 \circ p_1 = \omega \circ k_1$

$$\begin{aligned} p_2 \circ p_1 &= \left(\tfrac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}\right) \circ \partial_{21}^{(1)} \\ &= \partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{31}^{(1)} \circ \partial_{21}^{(1)} \\ &= \partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{31}^{(1)} \circ \partial_{21}^{(1)} \\ &= \partial_{21}^{(2)} \circ \partial_{32}^{(1)} \\ &= \omega \circ k_1 \cdot \end{aligned}$$

□

Proposition 3.2: The diagram F is commutative .

proof:-

$$\begin{aligned} q_2 \circ q_1 &= \partial_{21}^{(1)} \circ \left(\tfrac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)}\right) \\ &= \partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} \\ &= \partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} \\ &= \partial_{32}^{(1)} \circ \partial_{21}^{(2)} \\ &= k_3 \circ \omega \cdot \end{aligned}$$

□

Finally by using the mapping Cone we can define the maps σ_1, σ_2 and σ_3 where:

$$\sigma_3 : D_5 F \otimes D_3 F \otimes D_0 F \longrightarrow \begin{matrix} D_4 F \otimes D_4 F \otimes D_0 F \\ \oplus \\ D_5 F \otimes D_2 F \otimes D_1 F \end{matrix}$$

$$\sigma_2 : \begin{matrix} D_4 F \otimes D_4 F \otimes D_0 F \\ \oplus \\ D_5 F \otimes D_2 F \otimes D_1 F \end{matrix} \longrightarrow \begin{matrix} D_3 F \otimes D_4 F \otimes D_1 F \\ \oplus \\ D_4 F \otimes D_2 F \otimes D_2 F \end{matrix}$$

and

$$\sigma_1 : \begin{matrix} D_3 F \otimes D_4 F \otimes D_1 F \\ \oplus \\ D_4 F \otimes D_2 F \otimes D_2 F \end{matrix} \longrightarrow D_3 F \otimes D_3 F \otimes D_2 F$$

$$0 \to D_5 F \otimes D_3 F \otimes D_0 F \xrightarrow{\sigma_3} \begin{matrix} D_4 F \otimes D_4 F \otimes D_0 F \\ \oplus \\ D_5 F \otimes D_2 F \otimes D_1 F \end{matrix} \xrightarrow{\sigma_2} \begin{matrix} D_3 F \otimes D_4 F \otimes D_1 F \\ \oplus \\ D_4 F \otimes D_2 F \otimes D_2 F \end{matrix} \xrightarrow{\sigma_1} D_3 F \otimes D_3 F \otimes D_2 F$$

by

- $\sigma_3(x) = (p_1(x), k_1(x));\ \ \forall x \in D_5 F \otimes D_3 F \otimes D_0 F$
- $\sigma_2((x_1, x_2)) = (p_2(x_1) - \omega(x_2)),\ k_1(x_2) - k_2(x_1);$

$$\forall (x_1, x_2) \in \begin{array}{c} D_4 F \otimes D_4 F \otimes D_0 F \\ \oplus \\ D_5 F \otimes D_2 F \otimes D_1 F \end{array}$$

- $\sigma_1((x_1, x_2)) = (k_3(x_1) + q_2(x_2));$

$$\forall (x_1, x_2) \in \begin{array}{c} D_3 F \otimes D_4 F \otimes D_1 F \\ \oplus \\ D_4 F \otimes D_2 F \otimes D_2 F \end{array}$$

Proposition 3.3 :

$$0 \to D_5 F \otimes D_3 F \otimes D_0 F \xrightarrow{\sigma_3} \begin{array}{c} D_4 F \otimes D_4 F \otimes D_0 F \\ \oplus \\ D_5 F \otimes D_2 F \otimes D_1 F \end{array} \xrightarrow{\sigma_2} \begin{array}{c} D_3 F \otimes D_4 F \otimes D_1 F \\ \oplus \\ D_4 F \otimes D_2 F \otimes D_2 F \end{array} \xrightarrow{\sigma_1} D_3 F \otimes D_3 F \otimes D_2 F$$

is complex.

proof:-

Since $\partial_{32}^{(1)}$ and $\partial_{21}^{(1)}$ are injective from it is definition (see [1]), then we get σ_3 is injective.

Now

$$\begin{aligned} \sigma_2 \circ \sigma_3(x) &= \sigma_2(p_1(x), k_1(x)) \\ &= \sigma_2(\partial_{21}(x), \partial_{32}(x)) \\ &= (p_2(\partial_{21}(x) - \omega(\partial_{32}(x)), q_1(\partial_{32}(x)) - k_2(\partial_{21}(x))) \cdot \end{aligned}$$

Now

$$\begin{aligned} p_2(\partial_{21}(x)) - \omega(\partial_{32}(x)) &= (\tfrac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}) \circ \partial_{21}(x) - \partial_{21}^{(2)} \circ \partial_{32}(x) \\ &= (\partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{31}^{(1)} \circ \partial_{21} - \partial_{21}^{(2)} \circ \partial_{32})(x) \\ &= (\partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(1)} \circ \partial_{21}^{(2)} + \partial_{21} \circ \partial_{31})(x) \\ &= 0 \cdot \end{aligned}$$

$$\begin{aligned} k_1(\partial_{32}(x)) - g_2(\partial_{21}(x)) &= (\partial_{21}^{(1)} \circ \tfrac{1}{2}\partial_{32}^{(1)} + \partial_{31}^{(1)}) \circ \partial_{31}(x) - \partial_{32}^{(2)} \circ \partial_{21}(x) \\ &= (\partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{32}^{(1)} \circ \partial_{31} - \partial_{32}^{(2)} \circ \partial_{21})(x) \\ &= (\partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32} \circ \partial_{31} + \partial_{32}^{(1)} \circ \partial_{31} - \partial_{32}^{(2)} \circ \partial_{21})(x) \\ &= 0 \cdot \end{aligned}$$

so we get $(\sigma_2 \circ \sigma_3)(x) = 0 \cdot$

and

$$
\begin{aligned}
(\sigma_1 \circ \sigma_2)(x_1, x_2) &= \sigma_1(p_2(x_1) - \omega(x_2)),\, q_2(x_2) - k_2(x_1) \\
&= \sigma_1((\tfrac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)})(x_1) - \partial_{21}^{(2)}(x_2)\,,\, (\tfrac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)})(x_2) - \partial_{32}^{(2)}(x_1)) \\
&= \partial_{32}^{(1)}(\tfrac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)})(x_1) - \partial_{32}^{(1)} \circ \partial_{21}^{(2)}(x_2) \\
&\quad + \partial_{21}^{(1)}(\partial_{21}^{(1)} \circ \tfrac{1}{2}\partial_{32}^{(1)} + \partial_{31}^{(1)})(x_2) - \partial_{21}^{(1)} \circ \partial_{32}^{(2)}(x_1) \\
&= (\partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)} - \partial_{21}^{(1)} \circ \partial_{32}^{(2)})(x_1) \\
&\quad + (\partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(1)} \circ \partial_{21}^{(2)})(x_2) \cdot
\end{aligned}
$$

from then (1.1) and (1.2) we get

$$
\begin{aligned}
(\sigma_1, \sigma_2)(x_1, x_2) &= (\partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{32}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)} - \partial_{21}^{(1)} \circ \partial_{32}^{(2)})(x_1) \\
&\quad + (\partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(1)} \circ \partial_{21}^{(2)})(x_2) \\
&= 0 \cdot
\end{aligned}
$$

□

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سلسلة لاسكو في التجزئة (3,3,2)

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الخلاصة
درست في هذا البحث سلسلة لاسكو في حالة التجزئة (3,3,2) بأعتماد المخططات ، والقوى المقسمة لاستقطاب مكان $\partial_{ij}^{(k)}$ مع مشخصات كابلي وتطبيقات كون .

الكلمات المفتاحية: القوى المقسمة الجبرية ،تحلل مقاس وايل ،مكان الاستقطاب ،تطبيق كون .