For Some Results of Semisecond Submodules

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Abstract
Let $\mathscr{R}$ be a commutative ring with unity and let $\mathscr{B}$ be a unitary R-module. Let ℵ be a proper submodule of $\mathscr{B}$, ℵ is called semisecond submodule if for any r∈$\mathscr{R}$, r≠0, n∈Z_+, either r^nℵ=0 or r^nℵ=rℵ.
In this work, we introduce the concept of semisecond submodule and confer numerous properties concerning with this notion. Also we study semisecond modules as a popularization of second modules, where an $\mathscr{R}$-module $\mathscr{B}$ is called semisecond, if $\mathscr{B}$ is semisecond submodul of $\mathscr{B}$.

Keywords: Semisecond submodules, second submodules, secondary submodules.

1. Introduction

Let $\mathscr{R}$ be a commutative ring with unity and let $\mathscr{B}$ be a unitary $\mathscr{R}$ -module. S.Yass in [1] introduced the notation of second submodule and second module where a submodule ℵ of an $\mathscr{R}$ -module $\mathscr{B}$ is called second submodule if for every r∈$\mathscr{R}$, r≠0, either rℵ =ℵ or rℵ =0 and a module $\mathscr{B}$ is called semisecond if $\mathscr{B}$ is semisecond submodule of $\mathscr{B}$. This definition leads us to introduce the notion of semisecond submodule and semisecond module as a generalization of second submodule and second module, where a submodule ℵ of an $\mathscr{R}$-module $\mathscr{B}$ is called Semisecond if for every r∈ $\mathscr{R}$, r≠0, n∈Z_+, either r^nℵ =0 or r^nℵ =rℵ and a module $\mathscr{B}$ is Semisecond if $\mathscr{B}$ is semisecond submodule of $\mathscr{B}$.
The main aim of this work is to give basic properties of Semisecond submodules. Moreover, we survey the relationships between semisecond submodules and other submodules.
Over this work we designate S.R.M. for submodule of an $\mathscr{R}$-module, for integral domain, for finitely generated, s.t. for such that and N.Z. for non-zero.

2. Semisecond Submodules

Definition (1):-let ℵ be a S.R.M. $\mathscr{B}$, ℵ is semisecond submodule if for every r∈$\mathscr{R}$, n∈Z_+, either r^nℵ=0 or r^nℵ=rℵ.
An ideal I of a ring $\mathscr{R}$ is semisecond ideal if it is semisecond submodule of the $\mathscr{R}$ -module $\mathscr{R}$.

The later result is a description of semisecond submodule.

Proposition (2):- ℵ is S.R.M. $\mathscr{B}$ is semisecond iff r^2 ℵ =0 or r^2 ℵ =r ℵ for any r∈ $\mathscr{R}$, r≠0.
<u>Proof:</u>-(⟹) Is obvious.
(⟸) if r=3, then r^3ℵ =r(r^2ℵ). Since either r^2 ℵ =0 or r^2 ℵ =r ℵ, that is either r^3ℵ =r(0)=0 or r^3ℵ =r(rℵ) =r^2ℵ =rℵ. Suppose that r^n ℵ =0 or r^n ℵ = r ℵ is whole for n=k. To evidence that the permit is whole if n=k+1. $(r)^{k+1}$ ℵ =r(r^k ℵ). But r^k ℵ =0 or r^kℵ =rℵ, that is r^{k+1}ℵ =r(0) =0 or

$(r)^{k+1}$ ℵ =r(r ℵ)=r^2 ℵ =r ℵ. Hence by the principle of mathematical induction r^n ℵ =0 or r^n ℵ =r ℵ for any r∈𝒭, r≠0, n∈Z_+. Therefore, ℵ is semisecond submodule.

Remarks and Examples (3):-

(1) Every second submodule is semisecond.

Proof: -Let ℵ be a S.R.M. ℬ such that ℵ is second submodule, that is r ℵ =0 or r ℵ = ℵ for every r∈𝒭, r≠0. If r ℵ =0, then r^2 ℵ =r(r ℵ)=r(0)=0. If r ℵ = ℵ, then r^2 ℵ =r(r ℵ)=r ℵ, that is r^2 ℵ =0 or r^2 ℵ =r ℵ so ℵ is semisecond by proposition (2.2).

The converse of this remark is not true in general for example: -

Consider the Z-module Z_8, let ℵ =<$\bar{2}$>, take r=2, r ℵ ={$\bar{0}$,$\bar{4}$}. Thus r ℵ ≠ ℵ and r ℵ ≠(0), that is ℵ is not second submodule, while for every r∈Z, r≠0, such that r is even, then r=2k for some k∈Z, so r^2 ℵ =$(2k)^2$ ℵ =0.Also if r is odd, then r=(2k+1), so r^2 ℵ =($4k^2$+4k+1) ℵ = ℵ and r ℵ =(2k+1) ℵ =2k ℵ + ℵ = ℵ. Thus r^2 ℵ =r ℵ. Thus ℵ is semisecond submodule.

(2) The submodule Z of the Z-module Q is not semisecond submodule, but Q is a semisecond submodule of Q.

(3) Any submodule of Z_{p^∞} as Z-module is not semisecond submodule.

(4) Let ℵ be a non-zero S.R.M. ℬ s.t. 𝒭 is a field, then ℵ is semisecond.

Proof: - Let r∈ 𝒭, r≠0 and suppose r^2ℵ≠0. To prove r^2ℵ=rℵ, let rn∈rℵ, then rn=r^2(r^{-1}n)∈r^2ℵ, hence rℵ⊆r^2ℵ, which implies that r^2ℵ=rℵ. Thus ℵ is a semisecond submodule.

(5) Let f: ℬ→ℬ' be an R-homorphism and ℵ is a semisecond submodule of ℬ, then f(ℵ) is a semisecond submodule of ℬ'.

Proof: - Since ℵ is semisecond, then r^2ℵ=rℵ or r^2ℵ=0. Hence either f(r^2ℵ)=f(rℵ) or f(r^2ℵ)=f(0). Thus r^2f(ℵ)=rf(ℵ) or r^2f(ℵ)=f(0). Therefore, f(ℵ) is a semisecond submodule of ℬ'.

(6) The inverse image of semisecond submodule need not to be a semisecond, for example: -

Let Π:Z →Z/<6>≅Z_6, <$\bar{2}$> is semisecond submodule in Z_6 but Π^{-1}(2)=2Z is not a semisecond.

The opposite of remark and example (2.3. (1)) is true under the class of torsion free module over an integral domain, where a module ℬ over an I.D. is called **torsion free** if τ(ℬ)=0, where τ(ℬ)= {m∈ ℬ; r∈𝒭, r≠0, rm=0}, see [2.P.45].

Proposition (4):-If ℵ is a semisecond S.R.M. ℬ such that ℬ is torsion free over an I.D. R, then ℵ is a second submodule.

Proof:- let r∈ 𝒭, r≠0. Since ℵ is semisecond submodule, then r^2ℵ=0 or r^2ℵ=rℵ. If r^2ℵ=0, then r^2=0 (since M is torsion free) and since 𝒭 is an I.D., then r=0, which is contradiction. Thus r^2ℵ=rℵ and for any n∈ℵ, hence ∃ ń∈ℵ s.t. r^2ń=rn. Thus r(n-rń)=0. Since r≠0 and M is torsion free, then (n-ŕn)=0, that is n=rń, hence ℵ⊆rℵ and so, rℵ=ℵ. Therefore, ℵ is a second submodule.

Corollary (5):- If ℬ is a torsion free over an integral domain, then ℵ is second submodule of ℬ if and only if ℵ is semisecond.

Recall that a module $\mathscr{B}$ is called **multiplication** if every submodule $\aleph$ of $\mathscr{B}$, $\exists$ an ideal I of $\mathscr{R}$ s.t. I $\mathscr{B}$ =$\aleph$, amounting to for every submodule $\aleph$ of $\mathscr{B}$, $\aleph=[\aleph \underset{\mathcal{R}}{:} \mathscr{B}]$. $\mathscr{B}$, see[3].

Proposition (6):- If $\mathscr{B}$ is a faithful F.G. multiplication R-module, $\aleph< \mathscr{B}$, then $\aleph$ is semisecond iff $[\aleph:\mathscr{B}]$ is semisecond ideal of R.

Proof:- ($\Longrightarrow$) If $\aleph$ is a semisecond submodule, then for any r$\in \mathscr{R}$, r$\neq$0, $r^2\aleph=r\aleph$ or $r^2\aleph=0$. If $r^2\aleph=r\aleph$, then $r^2[\aleph:\mathscr{B}]$. $\mathscr{B}$ =r$[\aleph:\mathscr{B}]$. $\mathscr{B}$ because $\mathscr{B}$ is a multiplication module. Since $\mathscr{B}$ is a F.G. faithful multiplication $\mathscr{R}$ -module, then by [1] $r^2[\aleph \underset{R}{:} \mathscr{B}]=r[\aleph:\mathscr{B}]$. If $r^2\aleph=0$, then $r^2[\aleph:\mathscr{B}]$. $\mathscr{B}$ =0 and hence $r^2[\aleph:\mathscr{B}]\subseteq \underset{R}{ann}\ \mathscr{B}=0$. Thus $r^2\ [\aleph:\mathscr{B}]=0$ and so $[\aleph:\mathscr{B}]$ is a semisecond ideal.

Now, to prove the opposite. Let $[\aleph \underset{\mathcal{R}}{:} \mathscr{B}]$ be a semisecond ideal, that is $[\aleph \underset{\mathcal{R}}{:} M]$ is a semisecond submodule of the $\mathscr{R}$ -module $\mathscr{R}$. Then by proposition (2.2) $\forall$ r$\in$R, r$\neq$0, $r^2[\aleph \underset{\mathcal{R}}{:} \mathscr{B}]= r[\aleph:\mathscr{B}]$ or $r^2[\aleph:\mathscr{B}]=0$, that is $r^2[\aleph \underset{\mathcal{R}}{:} \mathscr{B}]\ \mathscr{B}=r[\aleph \underset{\mathcal{R}}{:} \mathscr{B}]$. $\mathscr{B}$ or $r^2[\aleph \underset{\mathcal{R}}{:} \mathscr{B}]=0$. Since $\mathscr{B}$ is a multiplication module, we have $r^2\aleph=r\aleph$ or $r^2\aleph=0$ for every r$\in \mathscr{R}$, r$\neq$0.
Therefore, $\aleph$ is a semisecond submodule.

We notice that the provision M is faithful cannot be dropped from proposition (2.6) for instance: Consider the Z-module Z_6, Z_6 is F.G. multiplication Z-module but not faithful. However, the submodule $\aleph=<\bar{3}>$ is a semisecond submodule since for any $r^2\notin \underset{\mathcal{R}}{ann}\ \aleph=2Z$, $r^2\aleph=r\aleph$. But $[\aleph \underset{\mathcal{R}}{:} \mathscr{B}]=[(\bar{3}) \underset{Z}{:} Z_6]$ =3Z is not semisecond in Z, Since for every $r^2\notin \underset{\mathcal{R}}{ann}\ (3Z)=0$ and for each $r \neq \mp 1$ we have $r^2\ (3Z) \neq r(3Z)$.

Proposition (7):- N.Z. $\aleph$ S.R.M. $\mathscr{B}$ is a semisecond $\mathscr{R}$ -submodule iff $\aleph$ is a semisecond $\mathscr{R}$ /I-submodule, where $I\subseteq \underset{\mathcal{R}}{ann}\ \aleph$.

Proof :- $\Longrightarrow$ Let $\bar{r}$ =r+I $\in \bar{\mathcal{R}}= \mathscr{R}$ /I. $(\bar{r})^2\aleph=(r+I)^2\aleph= r\aleph$. But $r^2\aleph=0$ or $r^2\aleph=r\aleph$, since $\aleph$ is semisecond, therefore $(\bar{r})^2\aleph=0$ or $(\bar{r})^2\aleph= \bar{r}\aleph$. Thus $\aleph$ is a semisecond $\bar{\mathcal{R}}$-submodule.
Similarly, we can proof the opposite.

Hence, we have the following result.

Corollary (8):- If $\aleph$ is a N.Z. S.R.M. $\mathscr{B}$ is a semisecond submodule iff $\aleph$ is a semisecond submodules $\mathcal{R} /\underset{\mathcal{R}}{ann}\ \aleph$ – submodule.

Proposition (9):- Let $\aleph$ be N.Z. proper submodule of $\mathscr{B}$ s.t. $\underset{\mathcal{R}}{ann}\ \aleph$ is a maximal ideal, then $\aleph$ is a semisecond submodule.

Proof:- since $\underset{\mathcal{R}}{ann}\ \aleph$ is a maximal ideal, then $\mathcal{R}/\underset{\mathcal{R}}{ann}\ \aleph$ is a field and by remark and example (2.3.(4)) $\aleph$ is semisecond submodule $\mathcal{R} /\underset{\mathcal{R}}{ann}\ \aleph$-submodule. Thus by corollary (2.8), $\aleph$ is a semisecond submodule $\mathscr{R}$ -submodule.

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Remark (10):- If $\aleph=\aleph_1\oplus\aleph_2$ is semisecond submodule in $\mathscr{B}=\mathscr{B}_1\oplus\mathscr{B}_2$, then N_1 and N_2 are semiseconds in $\mathscr{B}_1$, $\mathscr{B}_2$ respectively.

Proof:- It follows directly by remark and example (2.3.(5)).

Remark (11):- Let $\mathscr{B} = \mathscr{B}_1\oplus \mathscr{B}_2$. If $\aleph_1$ and $\aleph_2$ are semisecond submodules in $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively, then it is not necessarily that $\aleph_1\oplus\aleph_2$ is semisecond submodule in M for example:-

Let $\mathscr{B}=Z_6\oplus Z_{16}$, let $\aleph=<\bar{3}>\oplus<\bar{2}>$, $<\bar{3}>$ is semisecond submodule in Z_6, $<\bar{2}>$ is semisecond submodule in Z_{16}. However $2\aleph=<\bar{0}>+<\bar{4}>$, $(2^2)\aleph=4\aleph=<\bar{0}>\oplus<\bar{8}>$, then $2^2\aleph\neq2\aleph$ and $2^2\aleph\neq<\bar{0}>\oplus<\bar{0}>$.

The following result shows the direct sum of two semisecond submodules under certain condition.

Proposition (12):- Let $\aleph_1$ and $\aleph_2$ be semisecond submodules in $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively such that $\underset{\mathcal{R}}{ann}\,\aleph_1 = \underset{\mathcal{R}}{ann}\,\aleph_2$. Then $\aleph_1\oplus\aleph_2$ is semisecond submodule in $\mathscr{B}=\mathscr{B}_1\oplus\mathscr{B}_2$.

Proof:- Let $r\in\mathcal{R}$, $r\neq0$, then ($r^2\aleph_1=r\aleph_1$ or $r^2\aleph_1=0$) and ($r^2\aleph_2=r\aleph_2$ or $r^2\aleph_2=0$). Suppose $r^2\aleph_1=0$, then $r^2\aleph_2=0$ since $\underset{\mathcal{R}1}{ann}\,\aleph_1 = \underset{\mathcal{R}2}{ann}\,\aleph_2$ and so $r^2(\aleph_1\oplus\aleph_2)=0$. If $r^2\aleph_1=r\aleph_1$ and $r^2\aleph_1\neq0$, hence $r^2\aleph_2\neq0$ so $r^2\aleph_2=r\aleph_2$. It follows that $r^2(\aleph_1\oplus\aleph_2) = r^2\aleph_1\oplus r^2\aleph_2=r\aleph_1\oplus r\aleph_2$.

Now, we survey the relationships between semisecond submodules and some kind of submodules.

A submodule $\aleph$ of a module $\mathscr{B}$ is rendering **semiprime** if $\aleph\neq\mathscr{B}$ and $r\in\mathscr{R}$, $m\in\mathscr{B}$, $k\in Z^+$ with $r^K m\in\aleph$, then $rm\in\aleph$,see[4].Equivalently $\aleph$ is semiprime if whenever $r\in\mathscr{R}$, $m\in\mathscr{B}$, $r^2m\in\aleph$, then $rm\in\aleph$,see[3,prop.(1.2)].

An R-module $\mathscr{B}$ is rendering **semiprime** if (0) is a semiprime submodule of $\mathscr{B}$.

Proposition (13):- Let $\mathscr{B}$ be a semiprime $\mathscr{R}$-module, $\aleph$ submodule of $\mathscr{B}$ if $\aleph$ is semisecond, then $\aleph$ is semiprime submodule of $\mathscr{B}$.

Proof:- Let $a^2x\in\aleph$, where $a\in\mathscr{R}$, $x\in\mathscr{B}$. to prove $ax\in\aleph$. Since $\aleph$ is semisecond, then either $a^2\aleph=0$ or $a^2\aleph=a\aleph$. Assume $a^2\aleph=(0)$. Put $a^2x=n$ for some $n\in\aleph$. Then $a^4x=a^2n\in a^2\aleph=0$, hence $ax=0\in\aleph$ (since $\mathscr{B}$ is semiprime). Assume $a^2\aleph=a\aleph$. Since $a^2x=n\in\aleph$, then $a^3x=an\in a\aleph=a^2\aleph$, so that $a^3x=a^2n_1$for some $n_1\in\aleph$. Hence $a^2(ax-n_1)=0$.As $\mathscr{B}$ is semiprime $a(ax_1-n_1)=0$ and so that $a^2x=an\ \in a\aleph=a^2\aleph$. Thus $a^2\aleph=a^2n_2$ for some $n_2\in\aleph$. This implies $a^2(x-n_2)=0$. But $\mathscr{B}$ is semiprime, so that $a(x-n_2)=0$. It follows that $ax=an_2\in\aleph$. Therefore, $\aleph$ is a semiprime submodule.

Note that the opposite of previous proposition is not hold in public for instance: -

Take $\mathscr{B}=Z$ as Z-module. $\mathscr{B}$ is prime so it is semiprime. Let $\aleph=<6>$ is semiprime, but N is not semisecond since for every $r\in Z$, $r\neq0$, $r^2\aleph\neq(0)$ and $r^2\aleph\neq r\aleph$.

Reminiscence that a module $\mathscr{B}$ is rendering **Coprime** if $\underset{\mathcal{R}}{ann}\,\mathcal{B}=\underset{\mathcal{R}}{ann}\frac{\mathcal{B}}{\aleph}$ for every proper submodule $\aleph$ of $\mathscr{B}$, see [5]. Equivalently $\mathscr{B}$ is coprime module if and only if $\mathscr{B}$ is second module, see [6, th.(2.1.6)].

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A submodule ℵ of an $\mathscr{R}$-module $\mathscr{B}$ is rendering **irreducible** if ℵ cannot be expressed as a finite intersection of proper divisors of ℵ, See [4].

Proposition (14):- Let $\mathscr{B}$ be a coprime module, let N be a submodule of $\mathscr{B}$ such that ℵ is irreducible. If ℵ is semiprime, then ℵ is second and hence semisecond.

Proof:- Let ℵ be a semiprime $\mathcal{R}$-submodule, since ℵ is irreducible, then by [3,prop.(1-10)] ℵ is prime, but $\mathscr{B}$ is coprime module, then by [6,prop(2.4.7)] ℵ is second, hence ℵ is semisecond.

Corollary (15):- Let $\mathscr{B}$ be a prime module over regular ring $\mathcal{R}$ (in sense of von Neuman), let ℵ be a submodule of $\mathscr{B}$ such that ℵ is irreducible.Then N is semisecond if and only if ℵ is semiprime.

Proof:(⟹) Since $\mathscr{B}$ is prime, so it is semiprime. Thus we have the result by proposition (2.13).

(⟸) Since $\mathscr{B}$ is prime module over regular ring, then by [6, corollary (2.4.3)] $\mathscr{B}$ is coprime, hence we have the result by proposition (2.14).

Reminiscence that a submodule ℵ of a module $\mathscr{B}$ is rendering **secondary** (dual notion of primary module) if for each r∈R, the homothety r^* on ℵ is either surjective or nilpotent, where r^* is nilpotent if there exist $k \in Z_+$, such that $(r^*)^k=0$, see[7].It is obvious that every second submodule is secondary, but the opposite is not whole in public. The next lemma explains that the opposite is whole under certain condition.

Lemma (16):- Let ℵ be an $\mathscr{R}$-submodule such that $\underset{\mathcal{R}}{ann}$ ℵ is semiprime ideal. If ℵ is secondary, then ℵ is second submodule and hence semisecond.

Proof:- Since ℵ is secondary, then for any $r \in \mathscr{R}$, $r \neq 0$, rℵ=ℵ or r^nℵ=0; $n \in Z_+$. If rℵ=ℵ, then there is nothing to prove. If r^nℵ=0, then $r^n \in \underset{\mathcal{R}}{ann}$ ℵ. But $\underset{\mathcal{R}}{ann}$ ℵ is semiprime, so $r \in \underset{\mathcal{R}}{ann}$ ℵ. Thus rℵ=0 and hence ℵ is second.

Corollary (17): - Let ℵ be a S.R.M. $\mathscr{B}$ such that $\underset{\mathcal{R}}{ann}$ ℵ is semiprime, then ℵ is secondary if and only if ℵ is second.

The opposite of corollary (17) need not to be whole in public for example: -

In Z_8 as Z-module, <$\bar{2}$> is Semisecond and not secondary.

The opposite is whole under the class of torsion free module over regular ring.

Remark (18): - If ℵ is semisecond submodules of torsion free module $\mathscr{B}$ over regular ring, then ℵ is secondary.

Proof:- The proof directly by proposition (4).

Corollary (19) :- Let $\mathscr{B}$ be torsion free over regular ring, let ℵ be submodule of $\mathscr{B}$ such that $\underset{\mathcal{R}}{ann}$ ℵ is semiprime, then ℵ is secondary if and only if ℵ is semisecond.

Now, we turn our attention to the localization of semisecond.

Proposition (20):- Let ℵ be a semisecond submodule of an $\mathscr{R}$-module $\mathscr{B}$, then $ℵ_s$ is semisecond $\mathscr{R}_s$-submodule of $\mathscr{B}_s$, s.t. S is a multiplicatively closed subset of R.

Proof:- Let $\bar{r} \in \mathscr{R}$s, $\bar{r} = \frac{r}{s}$, where $r \in \mathscr{R}$, s∈S. Assume that $(\bar{r})^2 \notin \underset{R_s}{ann}$$ℵ_s$ .To prove $(\bar{r})^2 ℵ_s = \bar{r} ℵ_s$. Since $(\bar{r})^2 \notin \underset{R_s}{ann}$$ℵ_s$, then $(\frac{r}{s})^2.(\frac{n}{a}) \neq \frac{0}{1}$ for some n∈ℵ, a∈S. $(\frac{r^2 n}{sa}) \neq \frac{0}{1}$, that is for any t∈s, $r^2 tn \neq 0$. Thus

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$r^2t \notin \underset{\mathcal{R}}{ann} \aleph$ which implies that $r^2 \notin \underset{\mathcal{R}}{ann} \aleph$, so $r^2\aleph \neq 0$.But $\aleph$ is semisecond, hence $r^2\aleph = r\aleph$. Therefore $(r^2\aleph)_s = (r\aleph)_s$. Thus $(r^2)_s \aleph_s = (r)_s \aleph_s$ and so $(\bar{r})^2\aleph_s = \bar{r}\aleph_s$.

Corollary (21):- Let $\aleph$ be a semisecond submodule of an R-module $\mathscr{B}$, then $\aleph_p$ is semisecond $\mathscr{R}_p$-submodule of $\mathscr{B}_p$ for any prome ideal P of $\mathscr{R}$.

3. Semisecond Modules

Yass in [1] introduced the notion of **second module** (where $\mathscr{B}$ is second if for every $r \in \mathscr{R}$, $r \neq 0$, $r\mathscr{B} = 0$ or $r\mathscr{B} = \mathscr{B}$). Equivalently $\mathscr{B}$ is second module if $\mathscr{B}$ is second submodule of $\mathscr{B}$. In this section we introduce the notion of semisecond module as a generalization of second module. We give some properties of semisecond module.

Definition (22):- Let $\mathscr{B}$ be an R-module, $\mathscr{B}$ is rendering semisecond if $\mathscr{B}$ is semisecond submodule, that is for any $r \in \mathscr{R}$, $r \neq 0$, $r^2\mathscr{B} = r\mathscr{B}$ or $r^2M = 0$.

Remarks and Examples (23)

(1) It is obvious that every second module is semisecond, by remark and example (2.3.(1)).The opposite is not whole in public for instance: Z_4 as Z-module is not second since $2Z_4 \neq Z_4$ and $2Z_4 \neq (0)$ but Z_4 is semisecond module.

(2) Z as Z-module is not semisecond , since for any $r \in \mathscr{R}$, $r \neq 0$, $r^2Z \neq (0)$ and $r^2Z \neq rZ$.

(3) Consider the Z-module $Z_{p\infty}$, $\underset{Z}{ann} Z_{p\infty} = 0$, that is for all $r \in Z$, $r \neq 0$, $r^2Z_{p\infty} \neq (\bar{0})$. But $Z_{p\infty}$ is divisible Z-module, so $r^2Z_{p\infty} = rZ_{p\infty}$; for all $r \in Z$, $r \neq 0$, then $Z_{p\infty}$ is semisecond.

(4) Q as Z-module is semisecond module.

(5) If n is a prime number, then Z_n is semisecond Z-module, but the opposite is not whole in public for example Z_6 is semisecond but 6 is not prime.

(6) A module $\mathscr{B}$ is semisecond $\mathscr{R}$-module iff $\mathscr{B}$ is semisecond $\mathscr{R}/I$-module, where $I \subseteq \underset{\mathcal{R}}{ann} \mathscr{B}$.

Proof :- It follows by proposition (7).

(7) A module $\mathscr{B}$ is semisecond $\mathscr{R}$-module iff $\mathscr{B}$ is semisecond $\mathscr{R}/\underset{\mathcal{R}}{ann} \mathscr{B}$-module.

Proof :- It follows by corollary (8).

(8)Let $f:\mathscr{B} \longrightarrow \mathcal{B}'$ be an R-homomorphism, if $\mathscr{B}$ is semisecond module, then $f(\mathscr{B})$ is semisecond $\mathscr{B}'$-module.

(9) Let $\mathscr{B}$ be a semisecond $\mathscr{R}$-module, then $\mathscr{B}_s$ is semisecond $\mathscr{R}_s$-module, s.t. S is a multiplicatively closed subset of $\mathscr{R}$.

Proof :- It holds by proposition (20).

(10) Let $\mathscr{B}$ be a semisecond $\mathscr{R}$-module, then $\mathscr{B}_p$ is a semisecond $\mathscr{R}_p$-module for any prime ideal P of $\mathscr{R}$.

Proof:- It follows by corollary (21).

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