WE-Prime Submodules and WE-Semi-Prime Submodules

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Abstract

In this article, we introduce the concept of a WE-Prime submodule , as a stronger form of a weakly prime submodule. And as a generalization of WE-Prime submodule, we introduce the concept of WE-Semi-Prime submodule, which is also a stronger form of a weakly semi-prime submodule. Various basic properties of these two concepts are discussed. Furthermore, the relationships between WE-Prime submodules and weakly prime submodules and studied. On the other hand, the relation between WE-Prime submodules and WE- Semi - Prime submodules are consider. Also the relation of "WE – Sime - Prime submodules and weakly semi-prime submodules are explained. Behind that, some characterizations of these concepts are investigated .

Keywords: weakly prime submodules, weakly semi-prime submodules, WE-Prime submodules, WE-Semi-Prime submodules.

1. Introduction

Weakly prime submodule have been introduced and studied by Hadi M. A in [1], where a proper submodule K of an R-module X is called a weakly prime, if wherever $0 \neq rx \in K$, where $r \in R, x \in X$, implies that either $x \in K$ or $r \in [K:X]$, where $[K:X] = \{a \in R : aX \leq K\}$. Weakly semi-prime submodule have been introduced and studied by Farzalipour F in [2], where a proper submodule K of an R-module X is called a weakly semi-prime if wherever $0 \neq r^2x \in K$, where $r \in R, x \in X$, implies that $rx \in K$. Throughout this note all rings will be commutative with identity, and all R-modules are left unitary . A proper submodule K of an R-module X is said to be fully invariant if $f(K) \leq K$ for each $f \in End(X)$ [3]. An R-module M is called X- Injective , if for every R-homomorphism $g: N \longrightarrow M$, and every R-homomorphism $f: N \longrightarrow X$, there exists an R-homomorphism $h: X \longrightarrow M$, where N is an R-module such that $hof = g$ [5] . An R-module P is called X-Projective if for every R-homomorphism $f: P \longrightarrow N$ and every R-epimorphism $g: M \longrightarrow N$, there exists an R-homomorphism $h: P \longrightarrow M$ such that $goh = f$ [5]. An R-module X is called a scalar module if for each $f \in End(X)$, there exists $r \in R$ such that $f(m) = rm$ for each $m \in X$ [6] .

2. WE-Prime Submodules

In this section, we introduce the concept WE-Prime submodule as a stronger form of a weakly prime submodule, and established some of its basic properties, examples and characterizations.

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Definition (1)

A proper submodule K of an R-module X is said to be a weakly endo-prime (for a short WE-Prime), where $E = End(X)$, if wherever, $0 \neq \psi(x) \in K$, where $\psi \in End(X)$, $x \in X$, implies that either $x \in K$ or $\psi(x) \leq K$. And an ideal I of a ring R is said to be a weakly endo-prime ideal (WE-Prime ideal), if I is a WE-Prime as an R-submodule of an R-module R .

The following proposition gives relation of WE-Prime submodules and weakly prime submodules .

Proposition (2)

Every WE-Prime submodule of an R-module X is a weakly prime submodule of X .

Proof

Assume that K is a WE-Prime submodule of X, and $0 \neq rx \in K$, where $r \in R, x \in X$, with $x \notin K$. Now, let $\psi: X \longrightarrow X$ be a mapping defined by $\psi(x) = rx$ for all $x \in X$. Clearly $\psi \in End(X)$. In fact we have $0 \neq rx = \psi(x) \in K$. But K is a WE-Prime submodule of X, and $x \notin K$, implies that $\psi(x) \leq K$, hence $rx \leq K$, so $r \in [K{:}X]$. Therefore K is a weakly prime submodule of X .

The converse of Proposition (2) is not true in general, as the following example shows .

Example (3)

Let $X = Z_3 \oplus Z$ and R=Z, $K = \langle\bar{0}\rangle \oplus 3Z$. Clearly K is a weakly prime submodule of X, but K is not WE-Prime submodule of X . Since we define $\psi: X \longrightarrow X$ by $\psi(\bar{a}, b) = (\bar{0}, b)$ for all $(\bar{a}, b) \in X$. Clearly $\psi \in End(X)$. Now $(\bar{0}, 0) \neq \psi(\bar{1}, 3) = (\bar{0}, 3) \in K$, but $(\bar{1}, 3) \notin K$ and $\psi(X) = (\bar{0}) \oplus Z \nleq K$.

The converse of Proposition (2) is true in the class of cyclic R-modules, as the following proposition shows .

Proposition (4)

Let X be a cyclic R-module, and K is a proper submodule of X such that K is a weakly prime submodule of X. Then K is a WE-Prime submodule of X .

Proof

Assume that K is a weakly prime submodule of cyclic R-module X , where $X = Rm$, $m \in X$. Suppose that $0 \neq \psi(x) \in K$, where $\psi \in End(X)$, $x \in X$ and $x \notin K$. Now, let $y \in X$, then $y = rm$ and $x = r_1 m$ for some $r, r_1 \in R$. Thus, $0 \neq \psi(x) = r_1\psi(m) \in K$, but K is a weakly prime submodule of X, then either $r_1 \in [K{:}X]$ or $\varphi(m) \in K$. But $r_1 \notin [K{:}X]$ for $x = r_1 m \notin K$. Hence $\psi(m) \in K$, hence $\psi(y) = r\psi(m) \in K$. Therefore $\psi(X) \leq K$.

Corollary (5)

Let K be a proper submodule of a cyclic R-module X . Then K is a WE-Prime if and only if K is a weakly prime submodule of X.

Proposition (6)

Let X be a faithful R-module , and K is a WE-Prime submodule of X . Then $[K{:}X]$ is a WE-Prime ideal of R.

Proof

Since K is a WE-Prime submodule of X , then by Proposition (2.2), K is a weakly prime submodule of X . Hence by [1, Prop.2.4] , we get $[K:X]$ is a weakly prime ideal of R. But R is a cyclic R-module , then by Proposition (2.4), we get $[K:X]$ is a WE-Prime ideal of R .

We need to recall the following result before we introduce the next proposition .

Lemma (7) [3]

Let N and K be two submodules of an R-module X, then

1. If $N \leq K$, then $[N:X] \leq [K:X]$.
2. If $N \leq K$, then $[N:X] \leq [N:K]$.

The following proposition is a characterization of a WE-Prime submodules .

Proposition (8)

Let K be a proper fully invariant submodule of an R-module X . Then K is a WE-Prime submodule of X if and only if $[K:\psi(X)] = [K:\psi(H)]$ for all $\psi \in End(X)$ and a non-zero submodule H of X with $K < H$.

Proof

($\Longrightarrow$) Assume that K is a WE-Prime submodule of X, and H is a non-zero submodule of X such that $K < H$. Let $\psi \in End(X)$, then by Lemma (2.7)(2) we have $[K:\psi(X)] \leq [K:\psi(H)]$, since $K < H$, then there exists $x \in H$ and $x \notin K$. Now, suppose that b is a non-zero element in $[K:\psi(H)]$, then $0 \neq b\psi(H) \leq K$, implies that $0 \neq b\psi(x) \in K$, where $x \in H \leq X$. Define $\psi: X \longrightarrow X$ by $\psi(y) = b\psi(y)$ for all $y \in X$, clearly $\psi \in End(X)$, also $0 \neq b\psi(x) = \psi(x) \in K$. But K is a WE-Prime submodule of X, and $x \notin K$, then $\psi(X) \leq K$, implies that $b\psi(X) \leq K$ and hence $b \in [K:\psi(X)]$. Thus $[K:\psi(H)] \leq [K:\psi(X)]$, and it follows that $[K:\psi(X)] = [K:\psi(H)]$.

($\Longleftarrow$) Assume that $0 \neq \psi(x) \in K$, where $x \in X$ and $\psi \in End(X)$, and suppose that $x \notin K$, we want to show that $\psi(X) \leq K$. Since $x \notin K$, then $K < K + Rx$, where $K + Rx$ is a non-zero submodule of X . Thus by our hypothesis, we get $[K:\psi(X)] = [K:\psi(K + Rx)]$. Since K is a fully invariant, then $\psi(K) \leq K$ and $\psi(Rx) \leq K$, it follows that $\psi(K + Rx) \leq K$. Hence $[K:\psi(K + Rx)] = R$, therefore $1 \in [K:\psi(K + Rx)]$, implies that $1 \in [K:\psi(X)]$, hence $\psi(X) \leq K$. Thus K is a WE-Prime submodule of X .

Proposition (9)

Let X be an R-module, and L , H are submodules of X, with H is a fully invariant submodule of X and $H \leq L$. If $\frac{L}{H}$ is a WE-Prime submodule of $\frac{X}{H}$, then L is a WE-Prime submodule of X .

Proof

Assume that $0 \neq \psi(x) \in L$, where $x \in X$ and $\psi \in End(X)$. If $x \notin L$, then we must show that $\psi(X) \leq L$. Define $\psi_1: \frac{X}{H} \longrightarrow \frac{X}{H}$ by $\psi_1(x + H) = \psi(x) + H$ for all $x \in X$. To prove that φ_1 is well define, suppose that $x_1 + H = x_2 + H$ where x_1 , $x_2 \in X$, then $x_1 - x_2 \in H$, hence $\psi(x_1 - x_2) \in \psi(H) \leq H$ because H is a fully invariant. It follows that $\psi(x_1) - \psi(x_2) \in H$. Hence $\psi(x_1) + H = \psi(x_2) + H$, implies that $\psi_1(x_1) + H = \psi_1(x_2) + H$. Since $0 \neq \psi(x) \in$

L, implies that $0 \neq \psi(x) + H = \psi_1(x + H) \in \frac{L}{H}$. But $\frac{L}{H}$ is a WE-Prime submodule of $\frac{X}{H}$, and $x + H \notin \frac{L}{H}$, implies that $\psi_1\left(\frac{X}{H}\right) \leq \frac{L}{H}$, thus, we have $\frac{\psi(X)+H}{H} \leq \frac{L}{H}$, it follows that $\psi(X) + H \leq L$. Thus $\psi(X) \leq L$. Hence L is a WE-Prime submodule of X .

Proposition (10)

Let L and K are submodules of an R-module X , with L is an X-injective, and K is a WE-Prime submodule of X . Then either $L \leq K$ or $K \cap L$ is a WE-Prime submodule of L .

Proof

Assume that $L \nleq K$, then $K \cap L$ is a proper submodule of L . Now, let $0 \neq \psi(x) \in K \cap L$, where $x \in L$ and $\psi \in End(L)$. Suppose that $x \notin K \cap L$, then $x \notin K$. Now, consider the following diagram, where i is the inclusion map. Since L is an X-injective then there exists $\phi: X \longrightarrow L$ such that $\phi o \mathrm{i} = \psi$. Clearly $\phi \in End(X)$, but $0 \neq \psi(x) = (\phi o \mathrm{i})(x) = \phi(x) \in K$, implies that $0 = \phi(x) \in K$. But K is a WE-Prime submodule of X and $x \notin K$, then $\phi(X) \leq K$. Also, we have $\psi(L) = (\phi o \mathrm{i})(L) = \phi(L) \leq L$ and $\psi\psi(L) = \phi(L) \leq \phi(X) \leq K$. Hence $\psi(L) \leq K \cap L$, it follows that $K \cap L$ is a WE-Prime submodule of L .

Proposition (11)

Let X be an R-module and K, L are non-trivial submodules of X such that L is a WE-Prime submodule of X and IK is a non-zero submodule of L for some ideal I of R. If $I \leq [L: X]$ then $K \leq L$.

Proof

Suppose that $y \in K$, since $I \nleq [L: X]$, then there exists $i \in I$ and $i \notin [L: X]$. Now, let $\psi: X \longrightarrow X$ define by $\psi(x) = ix$ for all submodule $x \in X$, clearly $\psi \in End(X)$. Since IK is a non-zero submodule of L , then iy is a non-zero element in K . That is $0 \neq \psi(y) = iy \in IK \leq L$, implies that $0 \neq iy \in L$, but L is a WE-Prime submodule of X, and $iX = \psi(X) \nleq L$, implies that $y \in L$. Thus $K \leq L$.

Proposition (12)

Let X be an R-module and $\psi: X \longrightarrow X$ be an R-homomorphism , and K be a proper fully invariant WE-Prime submodule of X with $\psi(X) \nleq K$. Then $\psi^{-1}(K)$ is a WE-Prime submodule of X .

Proof

Clearly $\psi^{-1}(K)$ is a proper submodule of X. Now, assume that $0 \neq \phi(x) \in \psi^{-1}(K)$ where $x \in X, \phi \in End(X)$. If $x \notin \psi^{-1}(K)$, then $\psi(x) \notin K$, it follows that $x \notin K$ because K is a fully invariant submodule of X . We must prove that $\phi(X) \leq \psi^{-1}(K)$. Since $0 \neq \psi o \phi(x) = \psi(\phi(x)) \in K$. That is $0 \neq \psi(\phi(x)) \in K$. But K is a WE-Prime submodule of X, and $x \notin K$, it follows that $(\psi o \phi)(X) \leq K$, implies that $\phi(X) \leq \psi^{-1}(K)$. Hence $\psi^{-1}(K)$ is a WE-Prime submodule of X .

3. WE-Semi-Prime Submodules

In this section, we introduce the concept of WE-Semi-Prime submodule as a generalization of a WE-Prime submodule and stronger form of a weakly semi-prime submodule and give some basic properties , examples and characterizations of this concept .

Definition (13)

A proper submodule K of an R-module X is said to be a weakly endo semi-prime submodule of X (for a short WE-Semi-Prime) , where $E = End(X)$, if, wherever $0 \neq \psi^2(x) \in K$, where $x \in X$ and $\psi \in End(X)$, implies that $\psi(m) \in K$. And an ideal I of a ring R is said to be a weakly endo semi- prime ideal of R, if I is a weakly endo semi- prime as an R-submodule of R-module R .

Proposition (14)

Every WE-Prime submodule of an R-module X is a WE-Semi-Prime submodule of X .

Proof

Let K be a WE-Prime submodule of X , and $0 \neq \psi^2(x) \in K$, where $x \in X$, $\psi \in End(X)$. Since K is a WE-Prime submodule, and $0 \neq \psi(\psi(x)) \in K$, then either $\psi(x) \in K$ or $\psi(X) \leq K$. Thus in any case $\psi(x) \in K$. Hence K is a WE-Semi-Prime submodule of X .

The converse of Proposition (3.2) is not true in general , as the following example shows that .

Example (15)

Let X=Z and R=Z, K=10Z as a Z-module of X. Then K is a WE-Semi-Prime but not WE-Prime submodule of X, since if we defined $\psi: Z \longrightarrow Z$ by $\psi(x) = x$, $\psi \in End(X)$ and $0 \neq 2\psi(5) = 10 \in K$, but $5 \notin K$ and $\psi(Z) = Z \nleq K = 10Z$, hence K is not WE-Prime submodule of X. But K is a WE-Semi-Prime, since $0 \neq \psi^2(10) = \psi(\psi(10)) = 10 \in K$, implies that $\psi(10) = 10 \in K$.

Proposition (16)

Every WE-Semi-Prime submodule of an R-module X is a weakly semi-prime submodule of X .

Proof

Let K be a WE-Semi-Prime submodule of X, and $0 \neq r^2x \in K$, where $r \in R, x \in K$. Now, let $\psi: X \longrightarrow X$ defined by $\psi(x) = rx$ for all $x \in X$, clearly $\psi \in End(X)$. Now, $0 \neq r^2x = \psi^2(x) \in K$, but K is a WE-Semi-Prime submodule of X, implies that $\psi(x) = rx \in K$. Thus K is a weakly semi-prime submodule of X .

The converse of Proposition (3.4) is not true in general, as the following example shows .

Example (17)

Let $X = Z \oplus Z$, R =Z, $K = Z \oplus 10Z$, K is a weakly semi-prime submodule of X but not WE-Semi-Prime : Let $r = 2 \in Z$ and $x = (3,5) \in X$, then $0 \neq 2^2(3,5) = (12,20) \in K$, implies that $2(3,5) = (6,10) \in K$. To show that K is not WE-Semi-Prime : Let $\psi: X \longrightarrow X$ defined by $\psi(x,y) = (y,x)$ for all $x, y \in Z$. Clearly $\psi \in End(X)$. Now, take $\psi(0,5) = (5,0) \notin K$ but $\psi^2(0,5) = \psi(\psi(0,5)) = \psi(5,0) = (0,5) \in K$. Hence K is not WE-Semi-Prime submodule of X .

Proposition (18)

Let K be a submodule of an R-module X with $K = \cap_{\propto \in \Lambda} L_{\propto}$, where each $L_{\propto}$ is a WE-Prime submodule of X. Then K is a WE-Semi-Prime submodule of X.

Proof

Suppose that $0 \neq \psi^2(x) \in K$, where $x \in X, \psi \in End(X)$, then $0 \neq \psi^2(x) \in L_{\propto}$ for each $\propto \in \Lambda$. But $L_{\propto}$ is a WE-Prime submodule of X, hence by Proposition (3.2) $L_{\propto}$ is a WE-Semi-

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Prime. Thus $\psi(x) \in L_{\propto}$ for each $\propto \in \Lambda$. Therefore $\psi(x) \in \cap_{\propto\in\Lambda} L_{\propto}$. Hence K is a WE-Semi-Prime submodule of X.

The following proposition shows that in the class of scalar modules, weakly semi-prime submodule and WE-Semi-Prime submodules are coinciding.

Proposition (19)

Let X be a scalar module, and L is a proper submodule of X. Then L is a WE-Semi-Prime submodule of X, if and only if L is a weakly semi-prime submodule of X.

Proof

($\Longrightarrow$) Follows from Proposition (3.4).

($\Longleftarrow$) Suppose that L is a weakly semi-prime submodule of X , and $0 \neq \phi^2(x) \in L$, where $x \in X$ and $\phi \in End(X)$. Since X is a scalar module, then there exists $r \in R$ such that $\phi(x) = rx$ for each $x \in X$. Now , $0 \neq \phi^2(x) = \phi\big(\phi(x)\big) = \phi(rx) = r^2x \in L$. But L is a weakly semi-prime submodule of X , implies that $rx \in L$. Hence $\phi(x) \in L$. Thus L is a WE-Semi-Prime submodule of X .

The following propositions are characterizations of WE-Semi-Prime submodules .

Proposition (20)

Let X be an R-module, and L is a proper submodule of X . Then L is a WE-Semi-Prime submodule if and only if $0 \neq \phi^2(K) \leq L$, where K is a submodule of X and $\phi \in End(X)$, implies that $\phi(K) \leq L$.

Proof

($\Longrightarrow$) Assume that $0 \neq \phi^2(K) \leq L$, where K is a submodule of X, $\phi \in End(X)$, implies that $0 \neq \phi^2(x) \in L$ for all $x \in K \leq X$. Since L is a WE-Semi-Prime submodule of X, then $\phi(x) \in L$ for all $x \in X$. Thus $\phi(K) \leq L$.

($\Longleftarrow$) Suppose that $0 \neq \phi^2(x) \in L$, where $x \in X$, and $\phi \in End(X)$, then by hypothesis, we have $K = (x)$ is a submodule of X, and $0 \neq \phi^2(K) \in L$, implies that $\phi(K) \leq L$, it follows that $\phi(x) \in L$. Hence L is a WE-Semi-Prime submodule of X.

Proposition (21)

Let X be an R-module, and L is a proper submodule of X . Then L is a WE-Semi-Prime submodule of X, if and only if , wherever $0 \neq \phi^n(x) \in L, x \in X, \phi \in End(X)$, and for $n \geq 2$, implies that $\phi(x) \in L$.

Proof

($\Longrightarrow$) Follows by inducation on $n \in Z_+$.

($\Longleftarrow$) Direct from definition of WE-Semi-Prime submodule .

In the class of scalar module, we get the following characterizations of WE-Semi-Prime submodules.

Proposition (22)

Let X be a scalar R-module, and L be a proper submodule of X. Then the following statements are equivalent:

1. L is a WE-Semi-Prime submodule of X .
2. $[L: r^2] = [(0): r^2] \cup [L: r]$ for non-zero r in R .
3. $[L: r^2] = [(0): r^2]$ or $[(0): r^2] = [L: r^2]$ for non-zero r in R .

Proof

(1) ⟹ (2) Since L is a WE-Semi-Prime submodule of X, then by Proposition (3.4) L is a weakly semi-prime submodule of X. Now, let $x \in [L: r^2]$, implies that $r^2x \in L$, either $0 \neq r^2x \in L$ or $r^2x = 0$. If $0 \neq r^2x \in L$, implies that $rx \in L$, hence $x \in [L: r]$. If $r^2x = 0$, implies that $x \in [(0): r^2]$, hence, we get $[L: r^2] \leq [L: r] \cup [(0): r^2]$. Clearly we have by Lemma (2.7), $[L: r] \leq [L: r^2]$, and $[(0): r^2] \leq [L: r^2]$, hence $[L: r] \cup [(0): r^2] \leq [L: r^2]$. Thus the equality holds .

(2) ⟹ (3) Direct .

(3) ⟹ (1) To prove first L is a weakly semi-prime submodule of X . Suppose that $0 \neq r^2x \in L$, where $x \in X$, $r \in R$, implies that $x \in [L: r^2]$ and $x \notin [(0): r^2]$. Thus by hypothesis, we get $x \in [L: r]$, implies that $rx \in L$, hence L is a weakly semi-prime submodule of X . Thus by Proposition (3.7), we have L is a WE-Semi-Prime submodule of X .

Recall that an element x in R-module X is called torsion if $0 \neq ann(x) = \{r \in R : rx = 0\}$. The set of all torsion elements denoted by T(X), which is a submodule of X. If T(X)=(0), then X is called torsion free [3].

Proposition (23)

Let X is a torsion free scalar R-module, and L be a proper submodule of X, such that L is a WE-Semi-Prime submodule of X. Then $[L: I]$ is a WE-Semi- Prime submodule of X for any non-zero ideal I of R .

Proof

Since L is a WE-Semi-Prime submodule of X , then by Proposition (3.4) L is a weakly semi-prime submodule of X . Thus by [2, Prop.27] we get $[L: I]$ is a weakly semi-prime submodule of X. But X is a scalar module, hence by Proposition (3.7), we have $[L: I]$ is a WE-Semi-Prime submodule of X .

Proposition (24)

Let $\phi: X \longrightarrow X'$ be an R-epimorphism, and L is a WE-Semi-Prime submodule of X with $Ker\phi \leq L$. Then $\phi(L)$ is a WE-Semi-Prime submodule of X', where X' is an X-projective R-module.

Proof

Clearly $\phi(L)$ is a proper submodule of X'. Assume that $0 \neq f^2(x') \in \phi(L)$ where $x' \in X'$, and $f \in End(X')$, we prove that $f(x') \in \phi(L)$, since ϕ is an epimorphism, and $x' \in X'$, then there exists $x \in X$ such that $\phi(x) = x'$. Consider the following diagram since X' is X-projective , then there exists a homomorphism h such that ϕoh = f. Now, $0 \neq f'(x') = f\big(f(x')\big) \in \phi(L)$, implies that $0 \neq \phi \circ \mathrm{h} \circ \phi \circ \mathrm{h}(x') \in \phi(L)$, and hence $0 \neq \phi(\mathrm{h} \circ \phi)^2(x) \in \phi(L)$. But $Ker\phi \leq L$, then $0 \neq (\mathrm{h} \circ \phi)^2(x) \in L$. Since L is a WE-Semi-Prime submodule of X, then $(\phi \circ \mathrm{h})(x)$, implies that $\phi(\mathrm{h} \circ \phi)(x) \in \phi(L)$ hence $(\phi \circ \mathrm{h})\big(\phi(x)\big) \in \phi(L)$ implies that $f(x') \in \phi(L)$. Therefore $\phi(L)$ is a WE-Semi-Prime submodule of X'.

As a direct consequence of Proposition (3.12) we get the following corollary.

Corollary (25)

Let L and K be a submodule of an R-module X with $K \leq L$, and L is a WE-Semi-Prime submodule of X. Then $\frac{L}{K}$ is a WE-Semi-Prime submodule of $\frac{X}{K}$, where $\frac{X}{K}$ is an X-projective R-module.

Recall that an R-module X is multiplication if every submodule K of X is of the form K=IX for some ideal I of R [7].

Proposition (26)

Let X be a multiplication R-module and L is a weakly semi-prime submodule of X , then L is a WE-Semi-Prime submodule of X .

Proof

Suppose that $0 \neq f^2(x) \in L$, where $x \in X, f \in End(X)$. Since X is a multiplication, then by [8, Coro.1.2] there exists $s \in R$ such that $f(x) = sx$ for all $x \in X$. Hence $0 \neq f\big(f(x)\big) = s^2x \in L$. But L is a weakly semi-prime, implies that $sx \in L$. Thus $f(x) \in L$, so L is a WE-Semi-Prime submodule of X.

It is well-known every cyclic R-module is a multiplication [7], we get the following result.

Corollary (27)

Let X be a cyclic R-module, and L is a proper submodule of X. Then L is a WE-Semi-Prime submodule if and only if L is a weakly semi-prime.

We end this section by the following result.

Proposition (28)

Let X be a faithful multiplication R-module, and L is a proper submodule of X. Then L is a WE-Semi-Prime submodule of X if and only if $[L: X]$ is a WE-Semi-Prime ideal of R .

Proof

($\Longrightarrow$) Since L is a WE-Semi-Prime submodule of X, then by Proposition (3.4) L is a weakly semi-prime submodule of X . Hence by [2, Prop.29], we have $[L: X]$ is a weakly semi-prime ideal of R. Therefore $[L: X]$ is a weakly semi-prime as R-submodule of R-module R . But R is cyclic R-module, implies that by Corollary (27) $[L: X]$ is a WE-Semi-Prime R-submodule of R-module R. Hence $[L: X]$ is a WE-Semi-Prime ideal of R.

($\Longleftarrow$) Since $[L: X]$ is a WE-Semi-Prime ideal of R, implies that $[L: X]$ is a weakly semi-prime ideal of R . Hence by [2, Theo.30] we have L is a weakly semi-prime submodule of X. But X is a multiplication, then by Proposition (26) L is a WE-Semi-Prime submodule of X.

As a direct consequence of Proposition (27), we get the following result .

Corollary (3.17)

Let X be a faithful cyclic R-module , and L is a proper submodule of X. Then L is a WE-Semi-Prime submodule of X if and only if $[L: X]$ is a WE-Semi-Prime ideal of R.

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