WN-2-Absorbing Submodules and WNS-2-Absorbing Submodules

Wissam A. Hussain
Departmentt of Mathematics ,College of Education for Pure science, Tikrit University,Iraq
Wissam.abbas1987@gmail.com
Haibt K. Mohammd ali
Department of Mathematics,College of Computer Science and Mathematics,
Tikrit University,Iraq

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Abstract

In this article, we study, the concept of WN - 2 - Absorbing submodules and WNS - 2 - Absorbing submodules as generalization of weakly 2-absorbing and weakly semi 2-absorbing submodules respectively. We investigate some of basic properties, examples and characterizations of them. Also, prove, the class of WN-2-Absorbing submodules is contained in the class of WNS-2-Absorbing submodules. Moreover, many interesting results about these concepts, were proven.

Keywords: WN-2-Absorbing submodules, WNS-2-Absorbing submodules, Weakly 2-Absorbing submodules, Weakly Semi-2-Absorbing submodules.

1. Introduction

Weakly 2 - absorbing submodules was introduced by Darani and Soheilinia, in 2011, where a proper submodule B of an R - module Y is called weakly 2-absorbing submodule, if whenever $0 \neq aby \in B$, with $a, b \in R$, $y \in Y$, implies that either $ay \in B$ or $by \in B$ or $ab \in [B:Y]$ [1] . And the concept of a weakly semi 2-absorbing submodule was introduce by Haibt and Khalaf in 2018, where a proper submodule B of an R - module Y is called a weakly semi 2- absorbing submodule , if whenever $0 \neq a^2y \in B$, with $a \in R$, $y \in Y$, implies that either $ay \in B$ or'" $a^2 \in [B : Y]$ [2].

These two concepts are generalized in this article, to WN-2-Absorbing submodules and WNS-2-Absorbing submodules, we prove that the class of WN-2-Absorbing submodules is contained in the class of WNS-2-Absorbing submodules while the converse is not true see example (3.14). Recall that a submodule A of an R - module Y is called small if for any submodule B of Y, Y = A + B, implies that A = Y [3]. Recall that an R-epimorphism $f : Y \rightarrow Y$ is called small if Kerf is a small submodule of Y, and $f(J(M)) = J(M`) = J(f(M))$ and $J(M) = f^{-1}(J(M))$[3]. A ring R is a good ring if $J(R)\,Y = J(Y)$, where Y is an R-module equivalently R is a good ring if $J(Y) \cap A = (A)$ for every submodule A of Y [3]. If Y is an R-module and A, B, C are submodules of Y with $B \subseteq C$. Then $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$ [3]. Recall that an R- module Y is regular if R / ann (x) is regular ring [4]. Recall that a subset S of a ring R is called multiplicatively closed subset of R if $1 \in S$ and $ab \in S$ for all $a, b \in S$ [5] . This note consists of two parts in the first part, we introduced the concept of WN -2 - Absorbing submodule , and in the second part we introduced the concept of WNS - 2 - Absorbing submodule .

2. WN-2-Absorbing Submodules and Related Concept

In this part of the research, we introduce and studied the concept of WN-2 - Absorbing submodules as a generalization of weakly 2 - absorbing submodules .

Definition 1

A proper submodule B of an R - module Y is said to be WN- 2 - Absorbing submodules if whenever $0 \neq aby \in B$, where $a, b \in R$, $y \in Y$, implies that either $ay \in B + J(Y)$ or $by \in B + J(Y)$ or $ab \in [B + J(Y): Y]$, where $J(Y)$ is the Jacobsen radical of Y. An ideal I of a ring R is said to be WN- 2- Absorbing ideal of R, if I is a WN- 2- Absorbing submodules of an R - module R.

Remark 2

Every weakly 2-absorbing submodule of an R-module Y is WN-2-Absorbing submodules, while the converse is not true.

Proof

Clear. For the converse consider the following example : let $Y = Z_{16}$, $R = Z$ and $B = \langle\bar{8}\rangle$ it is clear that B is a WN-2-Absorbing submodules of Y since $B + J(Y) = \langle\bar{8}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle$. But B is not weakly 2-absorbing submodule of Y since, $\bar{0} \neq 2.2.\bar{2} \in B$, but $2.\bar{2} \notin B$ and $2.2 \notin [B:Y] = 8Z$.

Proposition 3

Let Y be an R - module, and B a proper submodule of Y with $J(Y) \subseteq B$ then B is a weakly 2 - absorbing submodule of Y if and only if B is a WN-2 - Absorbing submodule of Y.

Proof

($\Longrightarrow$) By remark (2.2).

($\Longleftarrow$) since $J(Y) \subseteq B$ then $B + J(Y) = B$, hence proof is direct.

Proposition 4

Let Y be an R-module, and B a proper submodule of Y with $A \subset B$. If A is a WN-2-Absorbing submodule of Y and $J(Y) \subseteq J(B)$, then A is a WN-2- Absorbing submodule of B.

Proof

Let $0 \neq aby \in A$, where $a, b \in R, y \in B$, since A is a WN-2-Absorbing submodule of Y then either $ay \in A + J(Y)$ or $by \in A + J(Y)$ or $ab \in [A + J(Y): Y]$, but $J(Y) \subseteq J(B)$, so either $ay \in A + J(B)$ or $by \in A + J(B)$ or $ab \in [A + J(Y): Y] \subseteq [A + J(B): Y] \subseteq [A + J(B): B]$ since B is a submodule of Y. Hence A a WN-2-Absorbing submodule of B.

Proposition 5

Let Y be an R - module, and B a proper submodule of Y, if $B + J(Y)$ is a WN-2 - Absorbing submodule of Y, then B is a WN-2 - Absorbing submodule of Y.

Proof

Since $B \subseteq B + J(Y)$, hence proof is clearly.

Remark 6

The intersection of two is a WN-2 - Absorbing submodules of an R - module Y need not to be is a WN-2-Absorbing submodule. The following example explain that:
Let Y = Z, R = Z, A = 6Z, B = 7Z. Clearly A, B is a WN-2-Absorbing submodules since they are weakly 2-absorbing submodules of Y but A ∩ B = 42Z is not WN-2-Absorbing submodule of Y since, if 0 ≠ 2.3.7 ∈ A ∩ B, but 2.7 ∉ A ∩ ȷ(Y) and 3.7 ∉ A ∩ ȷ(Y) and 2.3 ∉ [A ∩ ȷ(Y): Y] = 42Z.

Proposition 7

Let Y be an R - module , and A, B are WN- 2 - Absorbing submodules of Y with A ⊆ ȷ(Y) and B ⊆ ȷ(Y), then A ∩ B is WN-2-Absorbing submodules of Y.

Proof

Let 0 ≠ aby ∈ A ∩ B , with a, b ∈ R, y ∈ Y, implies that 0 ≠ aby ∈ A and 0 ≠ aby ∈ B. it follows that either ay ∈ A + ȷ(Y) or by ∈ A + ȷ(Y) or ab ∈ [A + ȷ(Y): Y], and either ay ∈ B + ȷ(Y) or by ∈ B + ȷ(Y) or ab ∈ [B + ȷ(Y): Y]. But A ⊆ ȷ(Y) and B ⊆ ȷ(Y), then A + ȷ(Y) = ȷ(Y) and B + ȷ(Y) = ȷ(Y). Hence ay ∈ ȷ(Y) or by ∈ ȷ(Y) or ab ∈ [ȷ(Y): Y]. Thus A ∩ B ⊆ ȷ(Y), implies that A ∩ B + ȷ(Y) = ȷ(Y) thus, we have ay ∈ A ∩ B + ȷ(Y) or by ∈ A ∩ B + ȷ(Y) or ab ∈ [A ∩ B + ȷ(Y): Y]. So, A ∩ B is a WN-2-Absorbing submodule of Y.

Proposition 8

Let Y be an R - module, over a good ring and A, B are submodules of Y, A ⊈ B and ȷ(Y) ⊆ A , if B is WN-2-Absorbing submodules of Y, then A ∩ B is WN-2-Absorbing submodules of A.

Proof

Since A ⊈ B, then A ∩ B is a proper submodule of A, let 0 ≠ aby ∈ A ∩ B , with a, b ∈ R, y ∈ Y. then 0 ≠ aby ∈ A and 0 ≠ aby ∈ B. Since B WN-2-Absorbing submodules of Y, then either ay ∈ B + ȷ(Y) or by ∈ B + ȷ(Y) or ab ∈ [B + ȷ(Y): Y]. That is either either ay ∈ (B + ȷ(Y)) ∩ A or by ∈ (B + ȷ(Y)) ∩ A or abY ⊆ (B + ȷ(Y)) ∩ A , hence by moduler law we have either ay ∈ A ∩ B + ȷ(A) or by ∈ A ∩ B + ȷ(A) or ab ∈ [A ∩ B + ȷ(A): Y] ⊆ [A ∩ B + ȷ(A): A] , thus A ∩ B is WN-2-Absorbing submodules of A.
As a direct consequence of proposition 2.8, we get the following corollary

Corollary 9

Let Y be an R - module , over a good ring and A, B are submodules of Y, A ⊈ B and A is a maximal submodule of Y, if B is WN-2-Absorbing submodules of Y, then A ∩ B is WN-2-Absorbing submodules of A.

Proposition 10

Let Y be an R - module , and A proper submodule of Y. Then A is WN-2 - Absorbing submodules of Y if and only if for each submodule B of Y with [A:Y] ⊆ [A:B] and for each a, b ∈ R with 0 ≠ abB ⊆ A, implies that either aB⊆A + ȷ(Y) or bB⊆A + ȷ(Y) or ab ∈ [A+ȷ(Y):Y].

Proof

Suppose that $0 \neq abB \subseteq A$ for each submodule B of Y and a, b ∈ R. then $0 \neq aby \in A$ for each $y \in B \subseteq Y$. But A is WN-2-Absorbing submodules of Y, implies that either $ay \in A + J(Y)$ or $by \in A + J(Y)$ or $ab \in [A+J(Y):Y]$. It follows that either $aB \subseteq A + J(Y)$ or $bB \subseteq A + J(Y)$ or $ab \in [A+J(Y):Y]$.

Conversely: let $0 \neq aby \in A$ for all $y \in Y$, a, b ∈ R . That is $0 \neq abY \subseteq A$, implies that $ab \in [A:Y] \subseteq [A:B]$, it follows that $0 \neq abB \subseteq A$ hence by hypothesis either $aB \subseteq A + J(Y)$ or $bB \subseteq A + J(Y)$ or $ab \in [A+J(Y):Y]$. That is either $ay \in A + J(Y)$ or $by \in A + J(Y)$ or $ab \in [A+J(Y):Y]$. Thus A is WN-2-Absorbing submodules of Y.

Proposition 11

Let Y be an R-module and A is a proper submodule of Y. If A is WN-2 -Absorbing submodules of Y, then $S^{-1}A$ is WN-2 -Absorbing submodules of an $S^{-1}R$- module $S^{-1}Y$, where S is a multiplicatively closed subset of R.

Proof

Let $0 \neq \frac{r_1}{s_1}\frac{r_2}{s_2}\frac{y}{s_3} \in S^{-1}A$, where $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$ and $\frac{y}{s_3} \in S^{-1}Y$ with $r_1, r_2 \in R$, $s_1, s_2, s_3 \in S$, $y \in Y$. Then $0 \neq \frac{r_1 r_2 y}{t} \in S^{-1}A$, where $t = s_1 s_2 s_3 \in S$, then there exists $t_1 \in S$ such that $0 \neq t_1 r_1 r_2 y \in A$. But A is WN-2-Absorbing submodules of Y, then either $t_1 r_1 y \in A + J(Y)$ or $t_1 r_2 y \in A + J(Y)$ or $t_1 r_1 r_2 \in [A + J(Y) : Y]$. implies that $\frac{t_1 r_1 y}{t_1 s_1 s_3} \in S^{-1}(A + J(Y)) \subseteq S^{-1}A + J(S^{-1}Y)$ or $\frac{t_1 r_2 y}{t_1 s_2 s_3} \in S^{-1}(A + J(Y)) \subseteq S^{-1}A + J(S^{-1}Y)$ or $\frac{t_1 r_1 r_2}{t_1 s_1 s_2} \in S^{-1}[A + J(Y) : Y] \subseteq [S^{-1}A + J(S^{-1}Y) : S^{-1}Y]$. Thus either $\frac{r_1 y}{s_1 s_3} \in S^{-1}A + J(S^{-1}Y)$ or $\frac{r_2 y}{s_2 s_3} \in S^{-1}A + J(S^{-1}Y)$ or $\frac{r_1 r_2}{s_1 s_2} \in [S^{-1}A + J(S^{-1}Y) : S^{-1}Y]$. Hence $S^{-1}A$ is WN-2-Absorbing submodules of an $S^{-1}R$- module $S^{-1}Y$.

Proposition 12

Let $h : Y \to Y$\` be a small R-epimorphism . and A is WN-2-Absorbing submodules of Y containing Kerh. , then h(A) is WN-2-Absorbing submodules of of Y\`.

Proof

It is clear that $h(A)$ is a proper submodule of Y\` , let aby\`∈ h (A) , where a , b ∈ R, y\` ∈ Y\` , then h (y) = y\`. for some y ∈ Y. thus $0 \neq abh(y) \in h(A)$, then h(aby) = h(n) for some non-zero n ∈ A. since Kerh ⊆ A it follows that $0 \neq aby \in A$, but A is WN-2-Absorbing submodules of Y, then either $ay \in A + J(Y)$ or $by \in A + J(Y)$ or $ab \in [A + J(Y) :Y]$. Thus either $ah(y) \in h(A) + h(J(Y))$ or $bh(y) \in h(A) + h(J(Y))$ or $abh(y) \subseteq h(A) + h(J(Y))$. But h is small epimorphism then either ay\`∈ h (A) + (Y\`) or by\`∈ h (A) + (Y\`) or abY\` ⊆ h (A) + $J$(Y\`). Hence h(A) is WN-2-Absorbing submodules of of Y\`.

Proposition 13

Let $h : Y \to Y$\` be a small R-epimorphism . and A is WN-2-Absorbing submodules of Y\` then $h^{-1}(A)$ is WN-2-Absorbing submodules of Y.

Proof

Let $0 \neq aby \in h^{-1}(A)$, where a , b ∈ R, y ∈ Y, with $ay \notin h^{-1}(A) + J(Y)$and $by \notin h^{-1}(A) + J(Y)$. It follows that $ah(y) \notin h(h^{-1}(A) + J(Y)) = A + J$(Y\`) and $bh(y) \notin h(h^{-1}(A) + J(Y)) = A$

+ J(Y`) because h is a small epimorphism. We have 0 ≠ aby ∈ $h^{-1}(A)$ , implies that 0 ≠ abh(y) ∈ $A$ , but A is WN-2-Absorbing submodules of Y`, then ab ∈ [A + J(Y`): Y`] that is abY` ⊆ A + $J$(Y`), implies that abh(Y) ⊆ A + J(Y`), hence abY ⊆ $h^{-1}(A + J(Y`)) \subseteq h^{-1}(A) + J(Y)$. Thus ab ∈ [$h^{-1}(A) + J(Y)$:Y].

3. WNS-2-Absorbing Submodules and Related Concept

This section devoted to introduce and study the concept of WNS -2-Absorbing submodules as a generalization of a weakly semi 2- absorbing submodule.

Definition 14

A proper submodule B of an R - module Y is said to be a WNS - 2 - Absorbing submodule of Y , if whenever 0 ≠ a^2y ∈ B , where a ∈ R, y ∈Y, implies that either ay ∈ B + J(Y) or a^2 ∈ [B + J(Y) :Y]. An ideal I of a ring R is called a WNS - 2- Absorbing ideal if I is a WNS- 2- Absorbing R- submodule of an R - module R.

Remarks and Examples 15

1. It is clear that every weakly semi 2- absorbing submodule of an R- module Y is a WNS- 2- Absorbing submodule of Y while the converse is not true
2. In the Z- module Z_{16} , the submodule B = ⟨$\bar{8}$⟩ is a WNS-2 - Absorbing submodule of Y, but not weakly semi 2-absorbing of Y since 0 ≠ $2^2\,\bar{2} \in B$, but $\bar{2} \notin B$ and $\bar{2} \notin$ [B:Y].
3. If Y be an R-module, with J(Y) = 0, then a WNS-2-Absorbing submodule of Y, equivalent with a weakly semi 2-absorbing submodule of Y.
4. If Y is semi simple (regular) R-module, then a WNS-2-Absorbing submodule of Y and weakly semi 2-absorbing submodule of Y are equivalent.
5. If Y is a R- module , and B a proper submodule of Y, with J(Y) ⊆ B. Then B is a WNS-2 -Absorbing submodule of Y if and only if B is a weakly semi 2-absorbing submodule of Y.
6. If B is a proper submodule of Y, with B + J(Y) is a WNS-2 - Absorbing submodule of Y, then B is a WNS- 2 - Absorbing submodule of Y.

Proposition 16

Let Y be an R - module and B be a proper submodule of Y Then B + J(Y) is a WNS-2 - Absorbing submodule of Y if and only if for each non-zero a ∈ R [B + J(Y) : a^2y] = [B + J(Y) : ay] or a^2 ∈ [B + J(Y) : Y].

Proof

⟹ Suppose that $a^2 \notin$ [B + J(Y) : Y], and let c ∈ [B + J(Y) : a^2y], implies that 0 ≠ a^2cy ∈ B + J(Y), but B + J(Y) is a WNS-2-Absorbing submodule of Y and $a^2 \notin$ [B + J(Y) : Y], then acy ∈ B + J(Y), implies that c ∈ [B + J(Y) : ay]. Thus [B + J(Y) : a^2y] ⊆ [B + J(Y) : ay]. Clearly [B + J(Y) : ay] ⊆ [B + J(Y) : a^2y]. Hence [B + J(Y) : a^2y] = [B + J(Y) : ay].
⟸) let 0 ≠ a^2y ∈ B + J(Y), where a ∈ R, y ∈ Y. By hypothesis, if [B + J(Y) : a^2y] = [B + J(Y) : ay] and 0 ≠ a^2y ∈ B + J(Y), implies that [B + J(Y) : a^2y] = R implies that [B + J(Y) : ay] = R, hence ay ∈ B + J(Y) .

Proposition 17

Let Y be an R - module and A , B are submodules of Y, with A is a subset of B. If A is a WNS-2- Absorbing submodule of Y and $J(Y) \subseteq (B)$, then A is a WNS-2-Absorbing submodule of B.

Proof

Similarly as in proposition 2.4

Proposition 18

Let Y be an R - module over a good ring R and A , B are proper submodules of Y. If A is a WNS- 2- Absorbing submodule of Y then A is a WNS- 2- Absorbing submodule of B.

Proof

Let $0 \neq b^2y \in B$, for $b \in R$, $y \in B \subseteq Y$, it follows that either $by \in A + J(Y)$ or $b^2 \in [A + J(Y):Y]$, implies that $by \in (A + J(Y)) \cap B$ or $b^2y \in (A + J(Y)) \cap B$, for each $y \in B$. Thus by modular law, $by \in (A \cap B) + (J(Y) \cap B)$. But R is a good ring, then $J(Y) \cap B = J(B)$ and $A \cap B$ is a proper subset of A, hence either $by \in A + J(B)$ or $b^2y \in A + (B)$ for each $y \in B$. Thus either $by \in A + (B)$ or $b^2y \in [A + (B):B]$. Hence A is a WNS-2-Absorbing submodule of B.

Remark 19

The intersection of two WNS-2 - Absorbing submodules of an R- module Y is not necessary WNS- 2- Absorbing submodules of Y.

The following example explain that:

Let Y = Z, R = Z and A=2Z, B=25Z, clearly A, B are WNS-2-Absorbing submodules of Y, but $A \cap B = 50Z$, is not WNS-2-Absorbing submodule of Y.

Proposition 20

Let Y be an R - module , and A , B are proper submodules of Y with $J(Y) \subseteq A$, or $J(Y) \subseteq B$, if A and B are WNS -2- Absorbing submodules of Y , then $A \cap B$ is a WNS -2- Absorbing submodule of Y.

Proof

Let $0 \neq r^2y \in A \cap B$, where $r \in R$, $y \in Y$, then $0 \neq r^2y \in A$ and $0 \neq r^2y \in B$, but both A and B are WNS-2-Absorbing submodules of Y then either $ry \in A + J(Y)$ or $r^2 \in [A + J(Y): Y]$ and either $ry \in B + J(Y)$ or $r^2 \in [B + J(Y): Y]$. Implies that $ry \in A + J(Y) \cap B + J(Y)$ or $r^2Y \subseteq \big(A + J(Y)\big) \cap (B + J(Y))$ if $J(Y) \subseteq B$, then $B + J(Y) = B$, thus either $ry \in A + J(Y) \cap B$ or $r^2Y \subseteq \big(A + J(Y)\big) \cap B$, it follows that either $ry \in A \cap B + J(Y)$ or $r^2Y \subseteq \big(A \cap B + J(Y)\big)$. that is $ry \in A \cap B + J(Y)$ or $r^2 \in (A \cap B + J(Y): Y)$. Hence $A \cap B$ is a WNS-2-Absorbing submodule of Y. similarly if $J(Y) \subseteq A$, we get $A \cap B$ is a WNS-2-Absorbing submodule of Y.

Proposition 21

Let Y be an R- module , and A is a proper submodule of Y with $J(Y) \subseteq A$, if A is a WNS-2- Absorbing submodules of Y, then [A :Y] is a WNS-2- Absorbing deal of R.

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Proof

Since A is a WNS-2-Absorbing submodules of Y, and $J(Y) \subseteq A$, then by remarks and examples 15 (5), A a weakly semi 2-absorbing submodule of Y, then by [2,prop. 4], [A:Y] is a weakly semi 2- absorbing ideal of R, so by (1) [A :Y] is a WNS- 2- Absorbing ideal of R .

Proposition 22

Let Y be a cyclic R - module , and A is a proper submodule of Y if [A:Y] is a WNS -2- Absorbing ideal of R with $J(R) \subseteq [A:Y]$, then A is a WNS-2- Absorbing submodules of Y.

Proof

Follows by remarks and examples 15(5)(1) and corollary [2, coro. 2.5].

Proposition 23

Let $g : Y \to Y^{`}$ be small R-epimorphism and A proper submodules of Y, with $Kerg \subseteq A$. If A is a WNS- 2- Absorbing submodules of Y , then g(A) is a WNS- 2- Absorbing submodules of $Y^{`}$.

Proof

Similarly, as in proposition 2.12.

Proposition 24

Let $g : Y \to Y^{`}$ be small R -epimorphism and $A^{`}$ proper submodules of $Y^{`}$. If $A^{`}$ is a WNS-2 - Absorbing submodules of $Y^{`}$ , then $g^{-1}(A^{`})$ is a WNS- 2- Absorbing submodules of Y.

Proof

Similarly, as in proposition 2 .13.

Proposition 25

Let Y be an R - module , and A is a proper submodule of Y if A is a WNS-2- Absorbing submodules of Y, then $S^{-1}A$ is is a WNS -2-Absorbing submodules of $S^{-1}R -$ module $S^{-1}Y$.

Proof

Similarly as in proposition 2 .11

Proposition 26

Let Y be an R - module and A is a WN-2- Absorbing submodules of Y,then A is a WNS- 2- Absorbing submodules of Y.

Proof

Let $0 \neq a^2y \in A$, where $a \in R$, $y \in Y$ that is $0 \neq a.a\,y \in A$. Since A is a WN-2- Absorbing. Then either $ay \in A + J(Y)$ or $a^2 \in [A + J(Y): Y]$. Thus A is WNS-2-Absorbing submodules of Y.

The converse of proposition 3.13 is not true.

In general as the following examples shows that:

Example 27 let Y = Z ⊕ Z , R = R , A = 15Z ⊕ (0), A is WNS-2-Absorbing submodules of Y, but not is WN-2-Absorbing. Since 0 ≠ 3.5(1,0) ∈ A, but 3(1,0) ∉ A + J(Y) and 5(1,0) ∉ A + J(Y) and 3.5 ∉ [A + J(Y) : Y] = (0).

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