The Approximation Solution of a Nonlinear Parabolic Boundary Value Problem Via Galerkin Finite Elements Method with Crank-Nicolson

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Abstract

   This paper deals with finding the approximation solution of a nonlinear parabolic boundary value problem (NLPBVP) by using the Galekin finite element method (GFEM) in space and Crank Nicolson (CN) scheme in time, the problem then reduces to solve a Galerkin nonlinear algebraic system(GNLAS). The predictor and the corrector technique (PCT) is applied here to solve the GNLAS, by transforms it to a Galerkin linear algebraic system (GLAS). This GLAS is solved once using the Cholesky method (CHM) as it appears in the matlab package and once again using the Cholesky reduction order technique (CHROT) which we employ it here to save a massive time. The results, for CHROT are given by tables and figures and show the efficiency of this method, from other sides we conclude that the both methods are given the same results, but the CHROT is very fast than the CHM.

Keywords: nonlinear parabolic boundary value problem, Galerkin finite element methods, Crank-Nicolson.

1. Introduction

   In the last decades many researchers interested to study the solution of boundary value problems (bvps) in general and the solution of NLPBVP in particular, there are many different methods for solving the NLPBVP, f.g. in 2000, Karlsen and Riserbo used a corrected operator Splitting method [1], *Pao* in 2001 used the time period solutions [2], in 2006, Alam and etc used the simultaneous space–time adaptive wavelet method [3]. Timothy in 2010 studied the explicit and implicit difference method [4], in 2011 Ghoreishi and Ismail are used the Homotopy Perturbation Method (HPM) [5], and many others.

      The study of the solution for the parabolic bvp using the finite element method (FEM) back to the beginning of the 17th century, and are studied from many researchers so as Douglas and Dupont [6], in 1993 Reddy introduced in his book an introduction to the FEM was applied to linear, one and two-dimensional problems of engineering and applied sciences [7]. In 1997-2006 Thomee [8] studied the GFEM with backward Euler method for nonlinear parabolic bvp. According to these studied it was important in this paper to study the approximate solution for NLPBVP using the GFEM method for the space variable and the CN scheme for the time variable.

This paper starts with give a description of proposed NLPBVP and its weak form (wf). The approximation solution of the problem is obtained by discretize the wf by using the GFEM for the space variable and the CN Scheme for the time variable, the problem then reduces to solve a GNLAS which transforms it to a LAS which is solved once using the CHM and once again

using that we give it the name CHROT to save a massive time, which is explained in a two steps formula. Finally, illustrative examples are given to solve different problems using MATLAB R2013a software, the results are given by tables and by figures and are show the efficiency of this method, and are show that the CHROT is very fast to solve the linear algebraic system than the CHM.

Definition 1, [9]: A point $x^* \in S \subset \mathbb{R}^2$ is said to be fixed point of a given function $q: S \to \mathbb{R}^2$, if $q(x^*) = x^*$.

Definition 2, [9]: A function $q: S \subset \mathbb{R}^2 \to \mathbb{R}^2$ is said to be contractive on S, if for each $x, y \in S$: $\|q(x) - q(y)\| \le \beta \|x - y\|$, where $0 < \beta < 1$ is a constant.

Theorem 3, [9]: A contractive function q on a complete normed space S has a unique fixed point x^* in .

Theorem 4, [9]: Let $\|\cdot\|$ is a norm in $\mathbb{R}^2$ and $S \subset \mathbb{R}^2$. If $q: S \to \mathbb{R}^2$ is contractive function on S, and one of the following is satisfied:

(1) $q(x) \in S, \forall x \in S$.

(2) $S = \{x | \|x - y\| \le \mu\}$ and $\|q(y) - y\| \le (1 - \beta)\mu$.

(3) $S = \{x | \|x - x^*\| \le \mu\}$, where x^* is a fixed point of q Then $\{x^{(l)}\} \in S$, where $x^{(l)}$ is the $l - $th iterative value of x .

Theorem 5, [9]: Let $\|\cdot\|$ is a norm in $\mathbb{R}^2$ and S be a closed subset of $\mathbb{R}^2$. If $q: S \to \mathbb{R}^2$ is contractive function on, and $\{x^{(l)}\} \in S$, then

(1) $\{x^{(l)}\}$ is converge to a fixed point $x^* \in S$

(2) x^* is a unique in S

3. Description of the (NLPBVPCC)

Let $W = \{\vec{x} = (x_1, x_2) \in \mathbb{R}^2: 0 < x_1, x_2 < 1\}$, with Lipischitz boundary ∂W, and let $I = (0, T)$, $0 < T < \infty$, and $P = W \times I$.

The nonlinear parabolic equation is given by: -

$$u_t - \Delta u = H(\vec{x}, t, u), \text{ in } P \tag{1}$$

with the boundary condition (b.c)

$$u(\vec{x}, t) = 0 \text{ , on } \partial W \times I \tag{2}$$

and the initial condition (i.c)

$$u(\vec{x}, 0) = u^0(\vec{x}), \text{ in } W \tag{3}$$

where $u = u(\vec{x}, t)$, $\Delta u = \sum_{i=1}^2 \frac{\partial^2 u}{\partial x_i^2}$ is the Laplace operator and $H \in C(W)$.

In this work the inner product and norm in $L^2(W)$ will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|_0$, the inner product and norm in Sobolev space $V = H_0^1(W)$ will be denoted by $(\cdot, \cdot)_1$ and $\|\cdot\|_1$, the duality bracket between V and its dual V^* will be denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_P$ be the norm in $L^2(P)$.

Now, the wf of (1-3) is given by:

$$\langle u_t, \xi \rangle + \langle \nabla u, \nabla \xi \rangle = (H(u), \xi), \forall \xi \in V \quad \text{a.e on I} \tag{4}$$

$$(u(0), \xi) = (u^0, \xi), \text{ in } W \tag{5}$$

with u^0 belongs to V and to $L^2(W)$ since $V \subset L^2(W)$.

Assumptions:

(1) for some positive constants γ_1, γ_2 and for each $\eta_1, \eta_2 \in V \& t \in \bar{I}$, the following inequality are satisfies

$$|(\nabla \eta_1, \nabla \eta_2)| \le \gamma_1 \|\nabla \eta_1\|_1 \|\nabla \eta_2\|_1$$

$$(\nabla \eta, \nabla \eta) \ge \gamma_2 \|\nabla \eta\|_1^2$$

(2) the function H is of a Carathéodory type on $P \times \mathbb{R}$, satisfies the following sub linearity and Lipischitz conditions

$|H(\vec{x}, t, u)| \leq \delta(\vec{x}, t) + c_1|u|$, where $c_1 > 0$, $(\vec{x}, t) \in P$, , $u \in \mathbb{R}$ and $\delta \in L^2(P, \mathbb{R})$

$|H(\vec{x}, t, u_1) - H(\vec{x}, t, u_2)| \leq L|u_1 - u_2|$, where $(\vec{x}, t) \in P, u_1, u_2 \in \mathbb{R}$, L is a Lipischitz constant.

4. Discretization of the Continuous Equation:

The wf of (4-5) is discretized by using the GFEM, as follows, let the domain W is a polyhedron. For every integer n, let $\{W_i^n\}_{i=1}^{N(n)}$ be an admissible regular traingulation of $\bar{W}$ into closed disimplices [8], $\{I_j^n\}_{j=0}$ be a subdivision of the interval $\bar{I}$ into $NT(n)$ intervals, where $t_j^n = [t_j^n, t_{j+1}^n]$, of equal length $\Delta t = T/NT$, let $O_{ij} = W_i^n \times I_j^n$ and $V_n \subset V = H_0^1(W)$ be the space of continuous piecewise affine in W. The Discrete state equation (DSEq) of the wf (4-5) is obtained after using the CN formula and is given by

$$\left(u_{j+1}^n - u_j^n, \xi\right) + \Delta t \left(\nabla u_{\frac{1}{2}j}^n, \nabla \xi\right) = \Delta t \left(H\left(t_{\frac{1}{2}j}^n, u_{\frac{1}{2}j}^n\right), \xi\right), \quad j=0,1,...,NT\text{-}1 \tag{6}$$

$$(u_0^n, \xi) = (u^0, \xi) \tag{7}$$

where $\xi \in V_n$, $u_{\frac{1}{2}j}^n = \frac{1}{2}\left(u_{j+1}^n + u_j^n\right)$, $t_{\frac{1}{2}j}^n = \frac{1}{2}(t_{j+1}^n + t_j^n$, $j=0,1...,NT, u^0 \in V$ and $u_j^n = u(t_j^n) \in V$, $\forall j = 0,1,...,NT$.

5. The Approximation Solution of the Nonlinear Parabolic Equation:

To find the approximation solution (app.sol) $u^n = (u_0^n, u_1^n, ..., u_{NT}^n)$ of (6-7) by using the GFEM, the following procedure can be used:

(1) For fixed any j , $(0 \leq j \leq NT - 1)$, let $\{\xi_i, i = 1,2,...,N, with \xi_i(\vec{x}) = 0, on \partial W\}$ be a continuous piecewise affine finite basis of V_n in W , then for any $i=1,2,...,N$ and $u_j^n, u_{j+1}^n \in V_n$ (6-7) can be rewritten as:

$$\left(u_{j+1}^n - u_j^n, \xi_i\right) + \Delta t \left(\nabla u_{\frac{1}{2}j}^n, \nabla \xi_i\right) = \Delta t \left(H\left(t_{\frac{1}{2}j}^n, \frac{1}{2}u_{j+1}^n + \frac{1}{2}u_j^n\right), \xi_i\right) \tag{8}$$

$, \xi_i \in V_n$, $(u_o^n, \xi_i) = (u^0, \xi_i)$

$$\tag{9}$$

$u_0^n = $ **(2)** Using the Galerkin method [8], with the basis $(\xi_1, \xi_2, ..., \xi_N)$ of V_n , one has

, , $u_j^n = \sum_{k=1}^N X_k^j \xi_k$, and $u_{j+1}^n = \sum_{k=1}^N X_k^{j+1} \xi_k \sum_{k=1}^N X_k^0 \xi_k$

where , $X_k^j = X_k(t_j^n)$, for each $j = 0,1,...,NT$ are unknown constants to be determine.

(3) Substitute u_j^n and u_{j+1}^n in (8) to get the following nonlinear algebraic system

$$\left(Y + \frac{1}{2}\Delta t Z\right)X^{j+1} = \left(Y - \frac{1}{2}\Delta t Z\right)X^j + \vec{b}\left(t_{\frac{1}{2}j}^n\right), \quad j = 0,1,...,NT - 1 \tag{10}$$

and substituting u_0^n in (9) to get the following linear algebraic system

$$YX^0 = b^0 \tag{11}$$

where $Y = (y_{ik})_{N \times N}$, $y_{ik} = (\xi_k, \xi_i)$, $Z = (z_{ik})_{N \times N}$, $z_{ik} = (\nabla\xi_k, \nabla\xi_i)$, $X_{N \times 1}^j = (x_1^j, x_2^j, ..., x_N^j)^T$, $\vec{V}_{N \times 1} = (\xi_1, \xi_2, ..., \xi_N)^T$, $\vec{b} = (b_i)_{N \times 1}$, $b_i = \Delta t\left(H\left(\frac{1}{2}\vec{V}^T X^{j+1} + \frac{1}{2}\vec{V}^T X^j\right), \xi_i\right)$ and $b^0 = (b_i^0)_{N \times 1}$, $b_i^0 = (u^0, \xi_i)$, $\forall i, k = 1,2,...,N$.

System (10-11) has a unique solution [10]. To solve it, the linear algebraic system (11) is solved at first to get X^0, then to solve the nonlinear system (10) the PCT is used here [8], as follows : For each j $(0 \leq j \leq NT - 1)$ we predict at first the value X^{j+1} by using the

explicit form (just the value of X^j) in the component of $\vec{b}$ in the right hand side (RHS) of (10) , then by substitute $\bar{X}^{j+1} = X^{j+1}$, in the component of $\vec{b}$ in the RHS of (10), makes system (10) linear, solving it w.r.t X^{j+1} to get the corrector solution , (this procedure can be repeated (more than one time if we need) by substitute the corrector solution $\bar{X}^{j+1} = X^{j+1}$ in the RHS of the linear system (10) and solve it again w.r.t X^{j+1} , for fixed j , to get a new corrector solution). Hence the corrector equation described as follows:

$$\left(u_{j+1}^{(l+1)} - u_j, \xi\right) + \tfrac{1}{2}\Delta t\left(\nabla u_{j+1}^{(l+1)} + \nabla u_j, \nabla \xi\right) = \Delta t \left(H\left(\tfrac{1}{2}u_{j+1}^{(l)} + \tfrac{1}{2}u_j\right), \xi\right) \tag{12}$$

where $u_{j+1}^{(l)} := u_{j+1}^n$ is the predictor solution at the iteration $l+1$, $u_{j+1}^{(l+1)} := u_{j+1}^n$ is its corresponding corrector solution at the iteration l and $u_j = u_j^n$ is the known corrector solution for the previous step j, i.e. (12) can be written as :

$$u^{(l+1)} = q(u^{(l)}) \tag{13}$$

Theorem 6: The discrete state Equation (6-7) with fixed point and for Δt sufficiently small has a unique solution $u^n = (u_0^n, u_1^n, \ldots, u_N^n)$, and the sequence of corrector solutions is convergence in $\mathbb{R}$.

Proof: Let $u^{(l+1)} = \left(u_0^{(l+1)}, \ldots, u_j^{(l+1)}, \ldots, u_N^{(l+1)}\right)$ and $v^{(l+1)} = (v_0^{(l+1)}, \ldots, v_j^{(l+1)}, \ldots, v_N^{(l+1)})$ are two solutions of (12) , i.e.

$$(u_{j+1}^{(l+1)} - u_j, \xi) + \tfrac{1}{2}\Delta t(\nabla u_{j+1}^{(l+1)} + \nabla u_j, \nabla \xi) = \Delta t(H\left(\tfrac{1}{2}u_{j+1}^{(l)} + \tfrac{1}{2}u_j\right), \xi) \tag{14}$$

$$(v_{j+1}^{(l+1)} - u_j, \xi) + \tfrac{1}{2}\Delta t(\nabla v_{j+1}^{(l+1)} + \nabla u_j, \nabla \xi) = \Delta t(H\left(\tfrac{1}{2}v_{j+1}^{(l)} + \tfrac{1}{2}u_j\right), \xi) \tag{15}$$

By subtracting (15) from (14), setting $\xi = u_{j+1}^{(l+1)} - v_{j+1}^{(l+1)}$ in the obtained equation and using Lipschitz condition on H with respect to for u, once get that

$$\left\|u_{j+1}^{(l+1)} - v_{j+1}^{(l+1)}\right\|_0^2 + \tfrac{1}{2}\Delta t\left\|\nabla u_{j+1}^{(l+1)} - \nabla v_{j+1}^{(l+1)}\right\|_0^2 \leq \tfrac{1}{2}\Delta t L \left(\left|u_{j+1}^{(l)} - v_{j+1}^{(l)}\right|, \left|u_{j+1}^{(l+1)} - v_{j+1}^{(l+1)}\right|\right).$$

Keep in mind that the 2nd term in the left hand side (LHS) is positive and then using Cauchy Schwarz (CS) inequality on the RHS of above inequality, once get that

$$\left\|u_{j+1}^{(l+1)} - v_{j+1}^{(l+1)}\right\|_0 \leq \beta \left\|u_{j+1}^{(l)} - v_{j+1}^{(l)}\right\|_0 \quad , \text{where } \beta = \tfrac{1}{2}\Delta t L ,$$

using (13), to get that

$$\left\|q\left(u_{j+1}^{(l)}\right) - q(v_{j+1}^{(l)})\right\|_0 \leq \beta \left\|u_{j+1}^{(l)} - v_{j+1}^{(l)}\right\|_0$$

Since Δt is sufficiently small and $\beta < 1$, then q is contractive, and by theorem (1) we get $u^{(l+1)} = v^{(l+1)}$, hence the DSEq has a unique solution, also since $\{u^{(l)}\} \in \mathbb{R}, \forall l$ then

$q(u^{(l)}) = u^{(l+1)} \in \mathbb{R}$, $\forall l$ implies that $q(u) \in R$, $\forall u \in \mathbb{R}$, and by using Theorem (3) with S=$\mathbb{R}$, we get that $\{u^{(l)}\}$ is converged to a point in $\mathbb{R}$.

6. Cholesky Reduction Order Technique

This technique in fact is based on an idea which is introduce first in [11] about reducing the diagonal elements of the Galerkin matrix into columns, we formulate it by the following steps hence we called it by the Cholesky reduction order technique (CHROT)**:**

First, the $N \times N$ obtained matrix is reduced to $N \times M1$ matrix A by transform the lower diagonals ($M1$) of $N \times N$ matrix to columns, **second** the reduction of matrix A is a new $N \times M1$ matrix R which is computed by using the following formula :

for $= 1,2,\dots,N$, $j = i + 1,\dots,\min(i + M, N)$
- if $i = 1$, then $R_{iM1} = \sqrt{A_{iM1}}$ and $R_{jl} = \dfrac{A_{jl}}{R_{iM1}}$, $l = i - j + M1$

- if $i > 1$, then $R_{iM1} = \sqrt{A_{iM1} - \sum_{r=K-i+M1} R_{ir}^2}$, $K = \max(i - M, 1) : i - 1$

$\quad R_{jl} = \dfrac{1}{R_{iM1}}\left(A_{jl} - \sum_{r=K-i+M1} R_{ir}R_{js}\right)$, $s = r + i - j$, with $j - K \leq M$.

Example 7: Consider the following nonlinear parabolic b.v.p.:

$u_t - \Delta u = H(\vec{x}, t, u)$, where $\vec{x} = (x_1, x_2)$
$u(\vec{x}, t) = 0$, on $\partial W \times I$
$u(\vec{x}, 0) = x_1 x_2 (1 - x_1)(1 - x_2)$, on W
where, $H(\vec{x}, t, u) = e^t[(x_1 x_2 - x_1 x_2^2 - x_2 x_1^2 + x_1^2 x_2^2)(1 - \sin(e^t(x_1 x_2 - x_1 x_2^2 - x_2 x_1^2 + x_1^2 x_2^2)))] + u \sin u$

The exact solution of this problem is: $u(\vec{x}, t) = x_1 x_2 (1 - x_1)(1 - x_2)e^t$

This problem solved using the GFEM for M=9 and NT=20, the results are shown in **Table 1.** and **Figure 1.** at time $\hat{t} = 0.5$, the table shows the approximate solution $\bar{u}(x_1, x_2, t)$, the exact solution $u(x_1, x_2, t)$ and the absolute error at x_1 & x_2. The Mat lap Software is used to solve this problem, it takes **5-hours** when we use the CHM, while takes **1-hour and 7-minutes** when we use the CHROT.

Table 1. Comparison between exact and approximation solutions

x_1	x_2	$u(x_1,x_2,t)$	$\bar{u}(x_1,x_2,t)$	absolute error	x_1	x_2	$u(x_1,x_2,t)$	$\bar{u}(x_1,x_2,t)$	absolute error
0.1	0.1	0.0137	0.0137	0.0000	0.6	0.5	0.1014	0.1029	0.0015
0.2	0.1	0.0243	0.0247	0.0004	0.7	0.5	0.0887	0.0901	0.0014
0.3	0.1	0.0319	0.0323	0.0004	0.8	0.5	0.0676	0.0687	0.0011
0.4	0.1	0.0365	0.0370	0.0005	0.9	0.5	0.0380	0.0386	0.0006
0.5	0.1	0.0380	0.0386	0.0006	0.1	0.6	0.0365	0.0370	0.0005
0.6	0.1	0.0365	0.0370	0.0005	0.2	0.6	0.0649	0.0659	0.0010
0.7	0.1	0.0319	0.0324	0.0005	0.3	0.6	0.0852	0.0865	0.0013
0.8	0.1	0.0243	0.0247	0.0004	0.4	0.6	0.0974	0.0988	0.0014
0.9	0.1	0.0137	0.0140	0.0003	0.5	0.6	0.1014	0.1029	0.0015
0.1	0.2	0.0243	0.0247	0.0004	0.6	0.6	0.0974	0.0988	0.0014
0.2	0.2	0.0433	0.0440	0.0007	0.7	0.6	0.0852	0.0865	0.0013
0.3	0.2	0.0568	0.0577	0.0009	0.8	0.6	0.0649	0.0659	0.0010
0.4	0.2	0.0649	0.0659	0.0010	0.9	0.6	0.0365	0.0370	0.0005
0.5	0.2	0.0676	0.0687	0.0011	0.1	0.7	0.0319	0.0324	0.0005
0.6	0.2	0.0649	0.0659	0.0010	0.2	0.7	0.0568	0.0576	0.0008

| 0.7	0.2	0.0568	0.0576	0.0008	0.3	0.7	0.0745	0.0757	0.0012
0.8	0.2	0.0433	0.0438	0.0005	0.4	0.7	0.0852	0.0865	0.0013
0.9	0.2	0.0243	0.0247	0.0004	0.5	0.7	0.0887	0.0901	0.0014
0.1	0.3	0.0319	0.0323	0.0004	0.6	0.7	0.0852	0.0865	0.0013
0.2	0.3	0.0568	0.0577	0.0009	0.7	0.7	0.0745	0.0756	0.0011
0.3	0.3	0.0745	0.0756	0.0011	0.8	0.7	0.0568	0.0577	0.0009
0.4	0.3	0.0852	0.0865	0.0013	0.9	0.7	0.0319	0.0323	0.0004
0.5	0.3	0.0887	0.0901	0.0014	0.1	0.8	0.0243	0.0247	0.0004
0.6	0.3	0.0852	0.0865	0.0013	0.2	0.8	0.0433	0.0438	0.0005
0.7	0.3	0.0745	0.0757	0.0012	0.3	0.8	0.0568	0.0576	0.0008
0.8	0.3	0.0568	0.0576	0.0008	0.4	0.8	0.0649	0.0659	0.0010
0.9	0.3	0.0319	0.0324	0.0005	0.5	0.8	0.0676	0.0687	0.0011
0.1	0.4	0.0365	0.0370	0.0005	0.6	0.8	0.0649	0.0659	0.0010
0.2	0.4	0.0649	0.0659	0.0010	0.7	0.8	0.0568	0.0577	0.0009
0.3	0.4	0.0852	0.0865	0.0013	0.8	0.8	0.0433	0.0440	0.0007
0.4	0.4	0.0974	0.0988	0.0014	0.9	0.8	0.0243	0.0247	0.0004
0.5	0.4	0.1014	0.1029	0.0015	0.1	0.9	0.0137	0.0140	0.0003
0.6	0.4	0.0974	0.0988	0.0014	0.2	0.9	0.0243	0.0247	0.0004
0.7	0.4	0.0852	0.0865	0.0013	0.3	0.9	0.0319	0.0324	0.0005
0.8	0.4	0.0649	0.0659	0.0010	0.4	0.9	0.0365	0.0370	0.0005
0.9	0.4	0.0365	0.0370	0.0005	0.5	0.9	0.0380	0.0386	0.0006
0.1	0.5	0.0380	0.0386	0.0006	0.6	0.9	0.0365	0.0370	0.0005
0.2	0.5	0.0676	0.0687	0.0011	0.7	0.9	0.0319	0.0323	0.0004
0.3	0.5	0.0887	0.0901	0.0014	0.8	0.9	0.0243	0.0247	0.0004
0.4	0.5	0.1014	0.1029	0.0015	0.9	0.9	0.0137	0.0137	0.0000
0.5	0.5	0.1057	0.1072	0.0015					
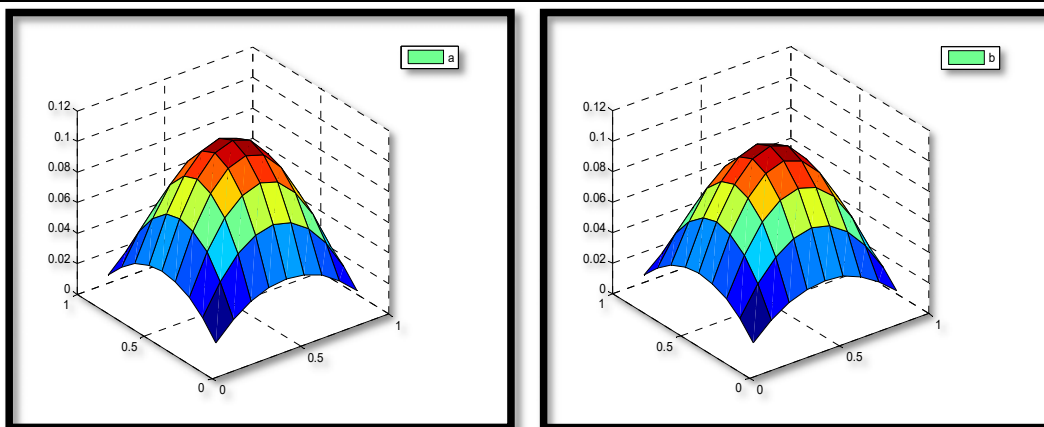
Figure 1. (a) shows the approximation solution and (b) shows the exact solution

Example 8: Consider the following nonlinear b.v.p.:

$u_t - \Delta u = H(\vec{x}, t, u)$, where $\vec{x} = (x_1, x_2)$

Associated with the i.c and b.c

$u(\vec{x}, t) = 0$, on $\partial W \times I$

$u(\vec{x}, 0) = 0$, on W

where $H(\vec{x}, t, u) = (x_1^2 x_2 + x_2^2 x_1 - x_1^2 x_2^2 - x_1 x_2)[1 + t \sin(t(x_1 x_2 - x_1 x_2^2 - x_2 x_1^2 + x_1^2 x_2^2))] -$

$$2t(x_1 + x_2 - x_1^2 - x_2^2) + u \sin u$$

The exact solution of this problem is: $u(\vec{x}, t) = -x_1 x_2 t (1 - x_1)(1 - x_2)$

This problem is solved using the GFEM for M=9 and NT=20, the results are shown in **Table 2.** and **Figure 2.** at $\hat{t} = 0.5$, the table shows the approximate solution $\bar{u}(x_1, x_2, t)$, the exact solution $u(x_1, x_2, t)$ and the absolute error at x_1 & x_2 . This problem is take *13 hours* when we use the CHM to solve the PCT, while takes *3 hours and 27 minute* when we use the CHROT.

Table 2. Comparison between exact and approximation solutions

x_1	x_2	$u(x_1,x_2,t)$	$\bar{u}(x_1,x_2,t)$	absolute error	x_1	x_2	$u(x_1,x_2,t)$	$\bar{u}(x_1,x_2,t)$	absolute error
0.1	0.1	-0.0043	-0.0044	0.0001	0.6	0.5	-0.0315	-0.0328	0.0013
0.2	0.1	-0.0076	-0.0079	0.0003	0.7	0.5	-0.0276	-0.0287	0.0011
0.3	0.1	-0.0099	-0.0103	0.0004	0.8	0.5	-0.0210	-0.0219	0.0009
0.4	0.1	-0.0113	-0.0118	0.0005	0.9	0.5	-0.0118	-0.0123	0.0005
0.5	0.1	-0.0118	-0.0123	0.0005	0.1	0.6	-0.0113	-0.0118	0.0005
0.6	0.1	-0.0113	-0.0118	0.0005	0.2	0.6	-0.0202	-0.0210	0.0008
0.7	0.1	-0.0099	-0.0103	0.0004	0.3	0.6	-0.0265	-0.0276	0.0011
0.8	0.1	-0.0076	-0.0079	0.0003	0.4	0.6	-0.0302	-0.0315	0.0013
0.9	0.1	-0.0043	-0.0044	0.0001	0.5	0.6	-0.0315	-0.0328	0.0013
0.1	0.2	-0.0076	-0.0079	0.0003	0.6	0.6	-0.0302	-0.0315	0.0013
0.2	0.2	-0.0134	-0.0140	0.0006	0.7	0.6	-0.0265	-0.0276	0.0011
0.3	0.2	-0.0176	-0.0184	0.0008	0.8	0.6	-0.0202	-0.0210	0.0008
0.4	0.2	-0.0202	-0.0210	0.0008	0.9	0.6	-0.0113	-0.0118	0.0005
0.5	0.2	-0.0210	-0.0219	0.0009	0.1	0.7	-0.0099	-0.0103	0.0004
0.6	0.2	-0.0202	-0.0210	0.0008	0.2	0.7	-0.0176	-0.0184	0.0008
0.7	0.2	-0.0176	-0.0184	0.0008	0.3	0.7	-0.0232	-0.0241	0.0009
0.8	0.2	-0.0134	-0.0140	0.0006	0.4	0.7	-0.0265	-0.0276	0.0011
0.9	0.2	-0.0076	-0.0079	0.0003	0.5	0.7	-0.0276	-0.0287	0.0011
0.1	0.3	-0.0099	-0.0103	0.0004	0.6	0.7	-0.0265	-0.0276	0.0011
0.2	0.3	-0.0176	-0.0184	0.0008	0.7	0.7	-0.0232	-0.0241	0.0009
0.3	0.3	-0.0232	-0.0241	0.0009	0.8	0.7	-0.0176	-0.0184	0.0008
0.4	0.3	-0.0265	-0.0276	0.0011	0.9	0.7	-0.0099	-0.0103	0.0004
0.5	0.3	-0.0276	-0.0287	0.0011	0.1	0.8	-0.0076	-0.0079	0.0003
0.6	0.3	-0.0265	-0.0276	0.0011	0.2	0.8	-0.0134	-0.0140	0.0006
0.7	0.3	-0.0232	-0.0241	0.0009	0.3	0.8	-0.0176	-0.0184	0.0008
0.8	0.3	-0.0176	-0.0184	0.0008	0.4	0.8	-0.0202	-0.0210	0.0008
0.9	0.3	-0.0099	-0.0103	0.0004	0.5	0.8	-0.0210	-0.0219	0.0009
0.1	0.4	-0.0113	-0.0118	0.0005	0.6	0.8	-0.0202	-0.0210	0.0008
0.2	0.4	-0.0202	-0.0210	0.0008	0.7	0.8	-0.0176	-0.0184	0.0008
0.3	0.4	-0.0265	-0.0276	0.0011	0.8	0.8	-0.0134	-0.0140	0.0006
0.4	0.4	-0.0302	-0.0315	0.0013	0.9	0.8	-0.0076	-0.0079	0.0003
0.5	0.4	-0.0315	-0.0328	0.0013	0.1	0.9	-0.0043	-0.0044	0.0001
0.6	0.4	-0.0302	-0.0315	0.0013	0.2	0.9	-0.0076	-0.0079	0.0003
0.7	0.4	-0.0265	-0.0276	0.0011	0.3	0.9	-0.0099	-0.0103	0.0004
0.8	0.4	-0.0202	-0.0210	0.0008	0.4	0.9	-0.0113	-0.0118	0.0005
0.9	0.4	-0.0113	-0.0118	0.0005	0.5	0.9	-0.0118	-0.0123	0.0005
0.1	0.5	-0.0118	-0.0123	0.0005	0.6	0.9	-0.0113	-0.0118	0.0005
0.2	0.5	-0.0210	-0.0219	0.0009	0.7	0.9	-0.0099	-0.0103	0.0004
0.3	0.5	-0.0276	-0.0287	0.0011	0.8	0.9	-0.0076	-0.0079	0.0003
0.4	0.5	-0.0315	-0.0328	0.0013	0.9	0.9	-0.0043	-0.0044	0.0001
0.5	0.5	-0.0328	-0.0342	0.0014					

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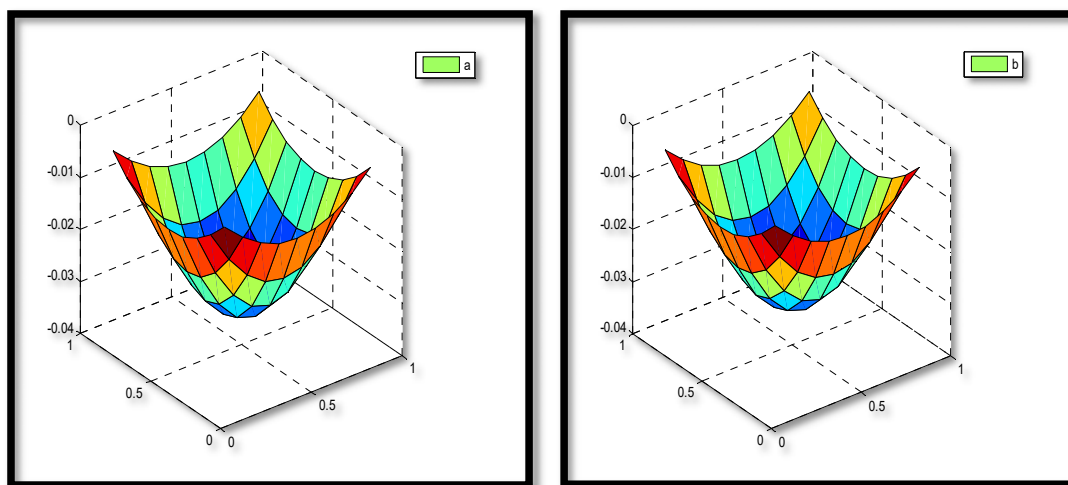
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Figure 2. (a) shows the approximation solution and (b) shows the exact solution

7. Conclusion

- The GFEM associated with the PCT is suitable, efficient and very fast to solve the nonlinear parabolic boundary value problems.
- The CHROT is very fast than the CHM with same results and this is important when we have problems gives very large algebraic systems which take a long time in the classical CHM.
- The value of $\hat{t}$ is chose arbitral in the interval I , same results with same accuracy will obtained if we can take any other value of $\hat{t}$ provided this value belong to I.

References

1. Karlsen K. H.; Riserbo, N. H. Corrected Operator Splitting for Nonlinear Parabolic Equations. *Siam J. Numr. Anal*. **2000,** *37, 3*, 980–1003.
2. Pao**,** C. V. Numerical Methods for Time-Periodic Solutions of Nonlinear Parabolic Boundary Value Problems. *SIAM Journal on Numerical Analysis*. **2001**, *39, 2,* 647-667.
3. Alam, J. M.; Kevlahan, N. K.-R.; Vasilyev**,** O. V. Simultaneous space–time Adaptive Wavelet Solution of Nonlinear Parabolic Differential Equations**.** *Journal of Computational Physics.* **2006**, *214*, 829–857.
4. Timothy, B. *Explicit and Implicit Methods In Solving Differential Equations*. Honors Scholarheses. University of Connecticut – Storrs. 2010**.**
5. Ghoreishi, M. Ismail, A. I. B, The Homotopy Perturbation Method (HPM) for Nonlinear Parabolic Equation with Nonlocal Boundary Conditions. *Applied Mathematical Sciences*, **2011**, *5, 3,* 113-123.
6. Douglas, J.; Dupont, T. Galerkin Methods for Parabolic Problems, *SIAM J. Numerical Analysis*. **1970,** *7, 4,* 575-626.
7. Reddy, J.N. *An Introduction to the Finite Element Method*, and Ed., McGraw-Hill, 1993.

8. Thomee, V. *Galerkin Finite Element Methods For Parabolic Problems*, Springer Verlag Berlin Heidelberg, 2006.

9. Suhas, P.; Uttam, D. Random Fixed Point Theorems for Contraction Mappings in Metric Space. *International Journal of Science and Research* (IJSR). **2015,** *23*,19-7064.

10. Burden, R. L.; Douglas Faires, J. *Numerical Analysis,* seven ed. Unite state, 2001.

11. Bacopoulos, A.; Chryssoverghi, I. *Numerical Solutions of Partial Differential Equations by Finite Elements Methods*. Symeon Publishing Co. Athens. 2003.