Study of Two Types Finite Graphs in KU-Semigroups

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Abstract

In this research, we present the notion of the graph for a KU-semigroup X as the undirected simple graph with the vertices are the elements of X and we study the graph of equivalence classes of X which is determined by the definition equivalence relation of these vertices, and then some related properties are given. Several examples are presented and some theorems are proved. By using the definition of isomorphic graph, we show that the graph of equivalence classes and the graph of a KU-semigroup are the same , in special cases.

Key words: KU-algebra, KU-semigroup, graph, annihilator.

1.Introduction

Mathematicians Prabpayak and Leerawat [1, 2] constructed algebraic structure which is called KU-algebra and they introduced the concept of a homomorphism of KU-algebra. Kareem and Hasan [3] introduced a new class of algebras related to KU-algebras and semigroups, called a KU-semigroup. They defined some types of ideals and discussed few properties.

The study of graph theory and its properties are topics of interest in algebraic structures. Beck in [4] introduced the graph of commutative ring by studied the zerodivisor graphs of this ring. Many mathematicians studied a graph of a commutative ring by different ways; see [5-10]. In [11], Jun and Lee introduced the concept of the associated graph of BCK/BCI-algebra and they proved that: if X is a BCK-algebra, then the associated graph of X is connected but if X is a BCI-algebra, then it's not connected. Zahiri and Borzooei [12] introduced a new graph of a BCI-algebra X and they defined the concept of a-divisor of BCI-algebra X. Mostafa and Kareem [13] introduced the graph of a commutative IS-algebra X, denoted by $\Gamma(X)$ and studied the graph of equivalence classes of X.

In this research, we introduce the idea of graph for a KU-semigroup. We define the graph as the undirected graph with the vertices are the elements in KU-semigroup X and for distinct vertices x and y are adjacent if and only if $R(\{x,y\}) = L(\{x,y\}) = \{0\}$. Moreover, we study the other graph namely, graph of equivalence classes of X by definition of equivalence relation of these vertices and then some related properties are given.

2.Preliminaries

In this section, we present some definitions and background about a KU-algebra and KU-semigroup.

Definition 1[1-2]. Algebra $(X,*,0)$ is called a KU-algebra if it satisfies the following axioms:

(ku_1) $(x*y)*[(y*z))*(x*z)]=0$,

(ku_2) $x*0=0$,

(ku_3) $0*x=x$,

(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$,

(ku_5) $x*x=0$.

On a KU-algebra X, we can define a binary relation $\leq$ by putting $x\leq y \Leftrightarrow y*x=0$. Then $(X,\leq)$ is a partially ordered set and 0 is its smallest element. Thus $(X,*,0)$ satisfies the following conditions. For all $x,y,z\in X$, we that

$(ku_{1'})$ $(y*z)*(x*z)\leq(x*y)$,

$(ku_{2'})$ $0\leq x$,

$(ku_{3'})$ $x\leq y, y\leq x$ implies $x=y$,

$(ku_{4'})$ $y*x\leq x$.

Theorem 2 [14]. In a KU -algebra X. The following axioms are satisfied, for all $x,y,z\in X$

(1) $x\leq y$ imply $y*z\leq x*z$,

(2) $x*(y*z)=y*(x*z)$, for all $x,y,z\in X$,

(3) $((y*x)*x)\leq y$.

Definition 3 [1-2]. A non- empty subset I of a KU-algebra $(X,*,0)$ is called an ideal of X if for any $x,y\in X$, then

(i) $0\in I$ and

(ii) $x*y, x\in I$ imply that $y\in I$.

Definition 4 [1-2]. Let I be a nonempty subset of a KU-algebra X. Then I is said to be a KU-ideal of X, if

(I_1) $0\in I$ and

(I_2) $\forall x,y,z\in X$, $x*(y*z)\in I$ and $y\in I$ imply that $x*z\in I$.

Definition 5[15]. A KU- algebra $(X,*,0)$ is said to be a commutative if it satisfies : for all x,y in X, $(y*x)*x=(x*y)*y$, where $x\wedge y=(y*x)*x$, i.e. $x\wedge y=y\wedge x$.

Lemma 6 [15]. If X is a commutative KU-algebra, then $x\wedge(y*z)=(x\wedge y)*(x\wedge z)$.

Example 7[15]. Let $X=\{0,a,b,c,d,e\}$ be a set, with the operation $*$ defined by the following table:

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	0	b	c	b	c
b	0	a	0	b	a	d
c	0	a	0	0	a	a
d	0	0	0	b	0	b
e	0	0	0	0	0	0

Then $(X,*,0)$ is a KU-algebra and KU-commutative.

Definition 8 [3]. A KU-semigroup is a nonempty set X with two binary operations $*$, $\circ$ and constant 0 satisfying the following axioms

(I) $(X,*,0)$ is a KU-algebra,

(II) $(X,\circ)$ is a semigroup,

(III)The operation $\circ$ is distributive (on both sides) over the operation $*$, i.e.

$x\circ(y*z)=(x\circ y)*(x\circ z)$ *and* $(x*y)\circ z=(x\circ z)*(y\circ z)$, for all $x,y,z\in X$.

Example 9[3]. Let $X=\{0,1,2,3\}$ be a set. Define $*$- operation and $\circ$- operation by the following tables

$*$	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then, $(X,*,\circ,0)$ is a KU-semigroup.

Proposition 10[3]. Let $(X,*,\circ,0)$ be a KU-semigroup. The following axioms are satisfied. For all $x,y,z\in X$,

(1) $x\circ 0=0$ and $0\circ x=0$,

(2) If $x\le y$ imply $z\circ x\le z\circ y$ and $x\circ z\le y\circ z$,

(3) $x\circ(y\wedge z)=(x\circ y)\wedge(x\circ z)$ *and* $(x\wedge y)\circ z=(x\circ z)\wedge(y\circ z)$.

3.A graph of KU-semigroups

In this part, we introduce the concepts of graph KU-semigroups X and the graph of equivalence classes of X . We recall some definitions and basic facts. For a graph G , we denoted the set of vertices of G as $V(G)$ and the set of edges as $E(G)$. An edge to be

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associated with a vertex pair $x_i - x_j$; such an edge having the same vertex as end vertices are called a loop. Also if more than one edge to be associated with a given pair of vertices, then edges referred to as parallel edges.A simple graph is a graph that has neither loops nor parallel edges. A graph G is said to be complete if every two distinct vertices are joined by exactly one edge. A graph G is said to be bipartite graph if its vertex set $V(G)$ can be partitioned into disjoint subsets V_1 and V_2 such that, every edge of G joins a vertex of V_1 with a vertex of V_2. So, G is called a complete bipartite graph if every vertex in one of the bipartition subset is joined to every vertex in the other bipartition subset. If V_1 and V_2 have m and n vertices respectively, then a complete bipartite graph will be denoted by $K_{m,n}$. Consequently, a star graph is a complete bipartite graph of the form $K_{1,n}$. A graph G is said to be a connected if there is a path between any given pairs of vertices, otherwise the graph is a disconnected. The neighbors of a vertex x in a graph G, denoted by $N(x)$ are the set of vertices that are adjacent to x. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Two graphs G_1 and G_2 are said to be isomorphic if there exists a bijective mapping $f : V(G_1) \to V(G_2)$ such that $x - y \in E(G_1)$ then $f(x) - f(y) \in E(G_2)$. For more details we refer to [16].

Definition 11. Let A be a subset of a KU- semigroup $(X,*,\circ,0)$, then we define the following

$L(A) = \{x \in X : x \wedge a = 0, x \circ a = 0, \forall a \in A\}$,

$R(A) = \{x \in X : a \wedge x = 0, a \circ x = 0, \forall a \in A\}$.

If $A = \{a\}$, then we write $L(a)$ and $R(a)$ instead of $L(\{a\})$ and $R(\{a\})$, respectively.

Proposition 12. Let A and B be non-empty subsets in $(X,*,\circ,0)$, then the following statements are true:

(1) $A \subseteq R[L(A)]$ and $A \subseteq L[R(A)]$,

(2) If $A \subseteq B$, then $R(B) \subseteq R(A)$ and $L(B) \subseteq L(A)$,

(3) $R(A) = R[L[R(A)]]$ and $L(A) = L[R[L(A)]]$.

(4) $R(A \bigcup B) = R(A) \bigcap R(B)$ and $L(A \bigcup B) = L(A) \bigcap L(B)$

(5) $R(A) \bigcup R(B) \subseteq R(A \bigcap B)$ and $L(A) \bigcup L(B) \subseteq L(A \bigcap B)$

Proof. (1) Let $a \in A$ and $x \in L(A)$, then $x \wedge a = 0, x \circ a = 0$. It follows that $a \in R[L(A)]$, hence $A \subseteq R[L(A)]$. Similarly, $A \subseteq L[R(A)]$.

(2) Suppose that $A \subseteq B$ and $x \in R(B)$, then $b \wedge x = 0, b \circ x = 0$ for all $b \in B$.

but $A \subseteq B$, *therfore* $b \wedge x = 0$, $b \circ x = 0$ $\forall b \in A$. So $x \in R(A)$, hence $R(B) \subseteq R(A)$.

Similarly, $L(B) \subseteq L(A)$.

(3) by using (1) and (2) we have $A \subseteq R[L(A)]$ and $A \subseteq L[R(A)]$ implies that by (2) $L[R[L(A)]] \subseteq L(A)$ and $R[L[R(A)]] \subseteq R(A)$. If we apply (1) to $L(A)$ and $R(A)$, then

$L(A) \subseteq L[R[L(A)]]$ and $R(A) \subseteq R[L[R(A)]]$. Hence $R(A) = R[L[R(A)]]$ and $L(A) = L[R[L(A)]]$.

(4) Since $A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$, we have by part (2) of Proposition 3.2 that, $R(A \bigcup B) \subseteq R(A)$ and $R(A \bigcup B) \subseteq R(B)$, hence $R(A \bigcup B) \subseteq R(A) \bigcap R(B)$ $--(I)$

Conversely, if $x \in R(A) \bigcap R(B)$, then $x \in R(A)$ *and* $x \in R(B)$, therefore

$a \wedge x = 0, a \circ x = 0, \forall a \in A$ and $b \wedge x = 0, b \circ x = 0, \forall b \in B$. But if $c \in (A \bigcup B)$, then

$c \wedge x = 0, c \circ x = 0, \forall c \in (A \bigcup B)$

we have $x \in R(A \bigcup B)$, hence $R(A) \bigcap R(B) \subseteq R(A \bigcup B)$ $--(II)$

From (*I*) and (*II*), we have $R(A \bigcup B) = R(A) \bigcap R(B)$. Similarly, $L(A \bigcup B) = L(A) \bigcap L(B)$.

(5) we have $A \supset A \bigcap B$, $B \supset A \bigcap B$ from (2) $R(A) \subset R(A \bigcap B)$ and $R(B) \subset R(A \bigcap B)$ which implies that $R(A) \bigcup R(B) \subseteq R(A \bigcap B)$. Similarly, $L(A) \bigcup L(B) \subseteq L(A \bigcap B)$.

Example 13. Let $X = \{0,1,2,3\}$ be a set. Define $*$- operation and $\circ$- operation by the following tables

$*$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	2	0

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	0
3	0	3	0	0

Then $(X, *, \circ, 0)$ is a KU-semigroup**.** We have the following

$R(\{0,1\}) = R(\{0,2\}) = R(\{1,2\}) = \{0,3\}$**,** $R(\{0,3\}) = \{0,2\}$ and $R(\{1,3\}) = R(\{2,3\}) = \{0\}$. Also,

$L(\{0,1\}) = \{0\}, L(\{0,2\}) = \{0,3\}, L(\{1,2\}) = \{0\}, L(\{0,3\}) = \{0,1,2\}$ and $L(\{1,3\}) = L(\{2,3\}) = \{0\}$.

Remark 14. If $(X, *, \circ, 0)$ is a KU-semigroup, then

(i) $L(\{0\}) = R(\{0\}) = X$,

(ii) $L(\{X\}) = R(\{X\}) = \{0\}$.

Proof. Clear.

Lemma 15. Let $(X, *, \circ, 0)$ be a KU-semigroup and a KU-algebra be a commutative, then for any elements a and b of X. If $a * b = 0$, then $L(\{a\}) \subseteq L(\{b\})$**.**

Proof. Suppose that $a * b = 0$. Let $x \in L(\{a\}) \Rightarrow x \wedge a = 0, x \circ a = 0$, then by lemma 6

$0 = x \wedge (a * b) = (x \wedge a) * (x \wedge b) = 0 * (x \wedge b) = (x \wedge b)$ and

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$0 = x \circ (a * b) = (x \circ a) * (x \circ b) = 0 * (x \circ b) = (x \circ b)$. It follows that $x \in L(\{b\})$, hence $L(\{a\}) \subseteq L(\{b\})$.

Lemma16. If A is a subset of a KU-semigroup $(X, *, \circ, 0)$ and a KU-algebra is a commutative, then $R(A)$ is an ideal of X.

Proof. For any $a \in A$, we have $a \wedge 0 = (0 * a) * a = a * a = 0$ and $a \circ 0 = 0$. Hence $0 \in R(A)$.

Let $x * y \in R(A), x \in R(A)$, then $a \wedge (x * y) = 0$, $a \circ (x * y) = 0$ which implies that by lemma6 $(a \wedge x) * (a \wedge y) = 0$ and since $x \in R(A)$, then $0 * (a \wedge y) = 0$, hence $a \wedge y = 0$. Also, $(a \circ x) * (a \circ y) = 0$ and since $x \in R(A)$, $0 * (a \circ y) = 0$, hence $a \circ y = 0$, i.e. $y \in R(A)$. Which implies that $R(A)$ is an ideal of X. □

Definition 17. A graph of a KU-semigroup X, denoted by $\Omega(X)$ is a simple graph whose vertices are the elements of X and two distinct elements $x, y \in X$ are adjacent if and only if $R(\{x, y\}) = L(\{x, y\}) = \{0\}$.

Example 18. Let $X = \{0, a, b, c, d\}$ be a set. Define $*$- operation and $\circ$- operation by the following tables

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	a	c	d
b	0	0	0	c	d
c	0	a	b	0	d
d	0	a	b	c	0

$\circ$	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

Then $(X, *, \circ, 0)$ is a KU-semigroup. So $R(\{0, a\}) = \{0, c, d\}$, $R(\{0, b\}) = R(\{a, b\}) = \{0, c\}$ $R(\{0, c\}) = \{0, a, b\}$, $R(\{0, d\}) = R(\{a, c\}) = R(\{a, d\}) = R(\{b, c\}) = R(\{b, d\}) = R(\{c, d\}) = \{0\}$.

Also, $L(\{0, d\}) = L(\{a, c\}) = L(\{a, d\}) = L(\{b, c\}) = L(\{b, d\}) = L(\{c, d\}) = \{0\}$.

By definition17, we determine the graph of X as follows: The set of vertices is $V(X) = \{0, a, b, c, d\}$ and the set of edges is

$E(X) = \{0 - d,\ a - c,\ a - d,\ b - c,\ b - d,\ c - d\}$. The figure (1) shows the graph $\Omega(X)$.

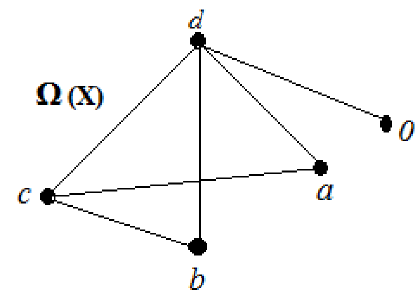

Figure 1. The Graph $\Omega(X)$

Example 19. Let $X = \{0,1,2,3\}$ be a set in example13. The set of vertices is $\{0,1,2,3\}$ and the set of edges is $\{1-3 \ and \ 2-3\}$. The figure (2) shows the graph $\Omega(X)$.

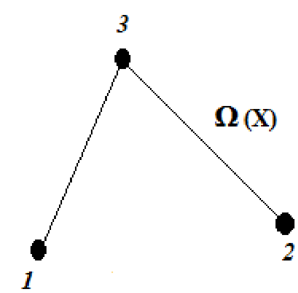

Figure 2. The Graph $\Omega(X)$

Theorem 20. A disconnected graph cannot be a graph of any KU-semigroups X.

Proof. Suppose G is a disconnected graph with components G_1 and G_2. Let $G = \Omega(X)$ be a graph of KU-semigroups$X$. Then, there exist vertices $x \in G_1$ and $y \in G_2$ such that there is no path between x and y. Let $a \in G_1$ and $b \in G_2$ be vertices adjacent to $x \in G_1$ and $y \in G_2$, respectively. Then $x \circ a = aox = 0, x \wedge a = a \wedge x = 0$ and $y \circ b = boy = 0, y \wedge b = b \wedge y = 0$. If $\circ\, b = boa = z,\ a \wedge b = b \wedge a = z$, for some$z \in X$. Then $xo\,z = zox = 0, yo\,z = zoy = 0$, $x \wedge z = z \wedge x = 0, y \wedge z = z \wedge y = 0$. Thus z is a common neighbor of x and y, that is a contradiction. Hence, a disconnected graph cannot be a graph of any KU-semigroups.

Definition21. Let X be a KU-semigroup. For any $x \in X$, we have $ann(x) = \{y \in X : R(\{x,y\}) = L(\{x,y\}) = \{0\}\}$ is called the set of annihilator of x.

Lemma 22. Let X be a KU-semigroup and $ann(x)$ be the set of annihilator of x. Then

(i) $x \notin ann(x) \quad \forall x \in X$,

(ii) There is an edge connecting x and y if and only if $x \in ann(y)$ and $y \in ann(x)$.

Proof. (i) Clear by definition 17.

(ii): Suppose that $x \notin ann(y)$ or $y \notin ann(x)$, then $R(\{x,y\}) = L(\{x,y\}) \neq \{0\}$. It implies that there is no edge connecting x and y, this is a contradiction. Thus $x \in ann(y)$ and $y \in ann(x)$. Conversely, suppose that $x \in ann(y)$ and $y \in ann(x)$, then $R(\{x,y\}) = L(\{x,y\}) = \{0\}$. It implies that there is an edge connecting x and y.

Definition 23. Define a relation $\sim$ on a KU-semigroup X as follows:

$x_1 \sim x_2$ if and only if $ann(x_1) = ann(x_2), \forall x_1, x_2 \in X$.

Lemma 24. The relation $\sim$ (from definition23) is an equivalence relation on X.

Proof. Clear.

4. A graph of Equivalence Classes of KU- Semigroups

Now, we introduce the graph of equivalence classes of a KU-semigroup X, which is constructed from classes of equivalence relation $\sim$ in definition 23. For $x, y \in X$, we say that x $\sim$ y if and only if $ann(x) = ann(y)$. As denoted in (lemma 4), $\sim$ is an equivalence relation. Furthermore, if $[x]$ denotes theclass of x, then the product $[x \circ y] = [x] \circ [y]$ and $[x \wedge y] = [x] \wedge [y]$.

Definition25. The graph of equivalence classes of a KU-semigroup X, denoted by $\Psi(X)$ is the undirected simple graph whose vertices are the set of equivalence classes $\{[x]; x \in X\}$ and two distinct classes $[x],[y]$ are adjacent in $\Psi(X)$ if and only if $[x] \circ [y] = \{0\}$ and $[x] \wedge [y] = \{0\}$.

Example 26. Let $X = \{0, a, b, c\}$ be a set. Define $*$ - operation and $\circ$ - operation by the following tables

$*$	0	a	b	c
0	0	a	b	c
a	0	0	a	c
b	0	0	0	c
c	0	a	b	0

$\circ$	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
c	0	a	b	c

Then $(X, *, \circ, 0)$ is a KU-semigroup**.** We have the set of vertices is $V(\Omega(X)) = \{0, a, b, c\}$ and the set of edges is $E(\Omega(X)) = \{0-a, 0-b, a-b, a-c, b-c\}$. So, the set of vertices of $\Psi(X)$ is $\{[0],[a],[b]\}$ since $ann(0) = \{a,b\}$, $ann(a) = \{0,b,c\}$, $ann(b) = \{0,a,c\}$ and $ann(c) = \{a,b\}$, then $E(\Psi(X)) = \{[0]-[a], [0]-[b], [a]-[b]\}$. The following Figure shows the graph $\Omega(X)$ and $\Psi(X)$.

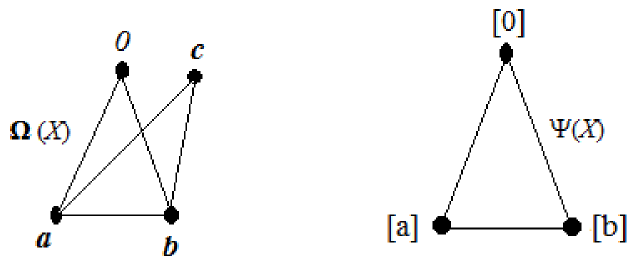

Figure 3. The Graphs $\Omega(X)$ and $\Psi(X)$

Example 27. Let $X = \{0,a,b,c,d\}$ be a set in example 18. Then the set of vertices of $\Psi(X)$ is $\{[0],[a],[c],[d]\}$ and the set of edges is $\{[0]-[d],\ [a]-[c],\ [a]-[d],\ [c]-[d],\}$

The figure (4) shows the graph of equivalence classes $\Psi(X)$.

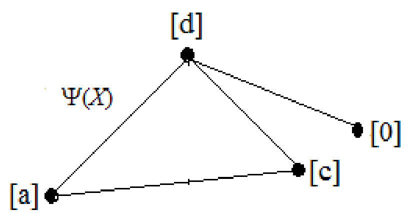

Figure 4. The Graph $\Psi(X)$

Lemma 28. With notations as before.

1) $\Psi(X)$ is a sub graph of $\Omega(X)$;

2) For all $x \in X$, we have $N(x) = ann(x)$.

Proof. Straightforward.

Theorem29. Let $\Psi(X)$ be the graph of equivalence classes of X. For any distinct vertices $[x], [y] \in \Psi(X)$, if $[x]$ *and* $[y]$ connected by an edge, then $ann(x) \neq ann(y)$.

Proof. Suppose that $ann(x) = ann(y)$, then $x \sim y$. Hence $[x] = [y]$ this is a contradiction. Therefore $ann(x) \neq ann(y)$.

The converse of this theorem is not true. In example27, we have the vertices $[0], [c]$ and $ann(0) \neq ann(c)$ but no edge joint between them.

Theorem30. Let X as mentioned above. If $\Omega(X)$ is a complete graph, then $\Omega(X) \cong \Psi(X)$. But the convers is not true.

Proof. Suppose that $V(\Omega(X)) = \{x_1, x_2, \ldots, x_n\}$. If $\Omega(X)$ is the complete graph, then every pair of its vertices are adjacent. Thus $N(x_1) = \{x_2, x_3, \ldots, x_i\}, i = 2, \ldots, n$ $N(x_2) = \{x_1, x_3, \ldots, x_i\}, i = 1,3, \ldots, n$, …, $N(x_n) = \{x_1, x_2, \ldots, x_{n-1}\}$. Then, $ann(x_1) = N(x_1),\ ann(x_2) = N(x_2), \ldots,\ ann(x_n) = N(x_n)$,Thus $ann(x_1) \neq ann(x_2) \neq \ldots \neq ann(x_n)$

therefore every vertex of $\Omega(X)$ is a equivalence class of $\Psi(X)$, thus the vertices of $\Psi(X)$ are distinct and the same number of vertices of $\Omega(X)$, then there exist an isomorphic $f:\Omega(X) \to \Psi(X)$ such that $f(x_i)=[x_i]$ for each $i \in \{1,2,...,n\}$ and the mapping of edges $f:E(\Omega(X)) \to E(\Psi(X))$, which sends the edge $x_i - x_j$ in $E(\Omega(X))$ to the edge $[x_i]-[x_j]$ in $E(\Psi(X))$ is a well-defined bijection.

The converse of this theorem is false as illustrated in example31, we have

Example 31. Let $X=\{0,a,b,c\}$ be a set. Define $*$-operation and $\circ$- operation by the following tables

$*$	0	a	b	c
0	0	a	b	c
a	0	0	a	c
b	0	0	0	c
c	0	a	b	0

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	0	c
b	0	0	b	0
c	0	c	0	c

Then $(X,*,\circ,0)$ is a KU-semigroup. We determine the graph $\Omega(X)$ as follows: $V(\Omega(X)) = \{0,a,b,c\}$ and $E(\Omega(X)) = \{0-a, a-b, a-c, b-c\}$. The set of vertices of $\Psi(X)$ is $\{[0],[a],[b],[c]\}$, since $ann(0)=\{a\}$, $ann(a)=\{0,b,c\}$, $ann(b)=\{a,c\}$ and $ann(c)=\{a,b\}$, then $E(\Psi(X)) = \{[0]-[a],[a]-[b],[a]-[c],[b]-[c]\}$. The following Figure shows the graph $\Omega(X)$ and $\Psi(X)$.

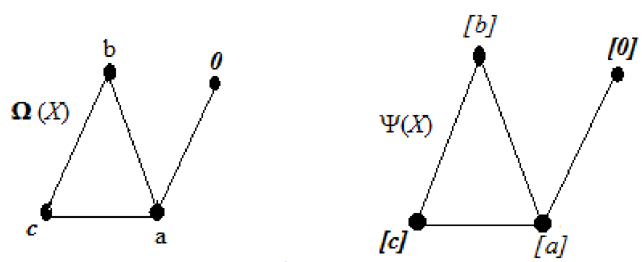

Figure 5. The graphs $\Omega(X)$ and $\Psi(X)$

In above figure $\Omega(X) \cong \Psi(X)$, but $\Omega(X)$ is not a complete graph.

Theorem 32. If $\Omega(X)$ is a star graph, then $\Psi(X)$ is an edge.

Proof. Suppose that $\Omega(X)$ is a star graph with vertex set $V(\Omega(X)) = \{x_1, x_2, \dots, x_n\}$. This set can be split into two sets $V_1=\{x_1\}$ and $V_2=\{x_2,...,x_n\}$ such that the vertex of V_1 is joined to each vertex of V_2 by exactly one edge. Thus, the set of edges is $E(\Omega(X)) = \{x_1 - x_2, x_1 - x_3, \dots, x_1 - x_n\}$, so $N(x_1)=\{x_2,x_3,...,x_n\}=V_2$ and $N(x_2)=\{x_1\}=N(x_3)=...=N(x_n)=V_1$, then $ann(x_1)=V_2$ and $ann(x_2)=ann(x_3)=...=ann(x_n)=V_1$. Then there are two distinct equivalence classes $[x_1]$ and $[x_2]$ in $\Psi(X)$, which are adjacent. Thus $\Psi(X)$ is an edge.

Lemma 33. Let G and H be two graphs and $G \cong H$. If $f(x)=y$, then $f(N(x))=N(y)$ for all $x \in V(G)$ *and* $y \in V(H)$.

Proof. Let $f:G \to H$ be a graph isomorphism, $x \in V(G)$ and $f(x)=y \in V(H)$. Then $f(N(x))=\{f(z):x-z\}=\{f(z):f(x)-f(z)\}=\{f(z):y-f(z)\}=N(y)$.

Theorem 34. Let X and Y be two KU-semigroups . If $\Omega(X)\cong\Omega(Y)$, then $\Psi(X)\cong\Psi(Y)$.
Proof. Suppose that $V(\Omega(X))=\{x_1,x_2,\dots,x_n\}$ and $V(\Omega(Y))=\{y_1,y_2,\dots,y_n\}$ such that the isomorphism $f:\Omega(X)\to\Omega(Y)$ satisfies $f(x_i)=y_i$ for each $i\in\{1,2,\dots,n\}$. By lemma 33, $f(N(x_i))=N(y_i)$ for each i, then $f(ann(x_i))=ann(y_i)$ and the mapping of edges $f:E(\Psi(X))\to E(\Psi(Y))$, which sends the edge $[x_i]-[x_j]$ in $\Psi(X)$ to the edge $[y_i]-[y_j]$ in $\Psi(Y)$ is a well-defined bijection. Thus $\Psi(X)\cong\Psi(Y)$.
The converse of this theorem is not true. In Examples 27 and 31, we have $\Psi(X)\cong\Psi(Y)$ but $\Omega(X)\ncong\Omega(Y)$.

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