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On a New Kind of Collection of Subsets Noted by δ−field and Some Concepts Defined on δ−field

Hassan H. Ebrahim **Ibrahim S. Ahmed**

Department of Mathematics, College of Computer science and Mathematics, University of Tikrit, Tikrit, Iraq.

hassan1962pl@tu.edu.iq ibrahimsalhahmed69@gmail.com

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Abstract

The objective of this paper is, first, study a new collection of sets such as δ−field and we discuss the properties of this collection. Second, introduce a new concepts related to the δ−field such as measure on δ−field, outer measure on δ−field and we obtain some important results deals with these concepts. Third, introduce the concept of null-additive on δ−field as a generalization of the concept of measure on δ−field. Furthermore, we establish new concept related to δ- field noted by weakly null-additive on δ−field as a generalizations of the concepts of measure on and null-additive. Finally, we introduce the restriction of a set function Ψ on δ−field and many of its properties and characterizations are given.

Keywords: σ−field, measure on σ−field, monotone measure, null-additive.

1. Introduction

The theory of measure is an important subject in mathematics. In 1972, Robret [1], discusses many details about measure and proves some important results in measure theory. The notion of σ−field was studied by Robret and Dietmar, where $\aleph$ be a nonempty set. A collection $\wp$ is said to σ−field iff $\aleph \in \wp$ and $\wp$ is closed under complementation and countable union [1, 2]. Zhenyuan and George in 2009 and Junhi, Radko and Endre in 2014 are used the concept of null-additive on σ−field, where $\wp$ be a σ−field, then a set function $\Psi: \wp \to [-\infty, \infty]$ is called null-additive on $\wp$ if A, B are disjoint sets in $\wp$ and $\Psi(B) = 0$, then $\Psi(A \cup B) = \Psi(A)$ [3,4]. In 2016, Juha used the concept of σ−field to define measure, where $\wp$ be a σ−field, then a measure on $\wp$ is a set function $\Psi: \wp \to [0, \infty]$ such that $\Psi(\Phi) = 0$ and if $A_1, A_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\wp$, then $\Psi(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Psi(A_n)$ [5]. and also used power set to define outer measure, where $\aleph$ be a non-empty set, then a set function $\Psi: P(\aleph) \to [0, \infty]$ is called outer measure, if $\Psi(\Phi) = 0$ and if $A, B \subseteq \aleph$ such that $A \subset B$, then $\Psi(A) \le \Psi(B)$ and if $A_1, A_2, \ldots$ are subsets of $\aleph$, then $\Psi(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \Psi(A_n)$ [5]. The concept of monotone measure was studied by Peipe, Minhao and Jun in 2018, where $\wp$ be a σ−field, then a set function $\Psi: \wp \to [0, \infty]$ is called monotone measure, if $\Psi(\Phi) = 0$ and if $A, B \in \wp$ such that $A \subset B$, then $\Psi(A) \le \Psi(B)$ [6].

The main aim of this paper is to introduce and study new concepts such as δ−field, measure on δ−field, outer measure on δ−field and null-additive on δ−field and we give basic properties, characterizations and examples of these concepts.

2. The Main Results

Let $\aleph$ be a nonempty set. Then a collection of all subsets of a set $\aleph$, denoted by $P(\aleph)$, and it's called a power set of $\aleph$.

Definition 1

Let $\aleph$ be a nonempty set. A collection $\wp \subseteq P(\aleph)$ is said to be δ− field of a set $\aleph$ if the following conditions are satisfied:

1- $\Phi \epsilon \wp$.
2- If A is a nonempty set in $\wp$ and $A \subset B \subseteq \aleph$, then $B \epsilon \wp$.
3- If $A_1, A_2, \ldots \epsilon \wp$, then $\bigcap_{i=1}^{\infty} A_i \epsilon \wp$.

Proposition 2

For any δ− field $\wp$ of a set $\aleph$, the following hold:

1- $\aleph \epsilon \wp$.
2- If $A, B \epsilon \wp$, then $A \bigcap B \epsilon \wp$.
3- If $A_1, A_2, \ldots, A_n \epsilon \wp$, then $\bigcap_{i=1}^{n} A_i \epsilon \wp$.
4- If $A_1, A_2, \ldots, A_n \epsilon \wp$, then $\bigcup_{i=1}^{n} A_i \epsilon \wp$.
5- $A_1, A_2, \ldots \epsilon \wp$, then $\bigcup_{i=1}^{\infty} A_i \epsilon \wp$.

Proof

It is easy, so we omitted.

Example 3

Let $\aleph$ ={1, 2, 3, 4} and $\wp$ ={Φ, {1,2},{1,2,3}, {1,2,4}, $\aleph$ }. Then $\wp$ is a δ− field of a set $\aleph$.

Definition 4

Let $\aleph$ be a nonempty set and $\wp$ is a δ− field of a set $\aleph$ .Then a pair $(\aleph, \wp)$ is called measurable space and any member of $\wp$ is called a measurable set.

Proposition 5

Let $\{\wp_i\}_{i \in I}$be a sequence of δ− field of a set $\aleph$. Then $\bigcap_{i \in I} \wp_i$ is a δ− field of a set $\aleph$.

Proof

Since $\wp_i$ is δ− field $\forall$ i $\in$ I, then $\Phi, \aleph \epsilon \wp_i$ $\forall$ i $\in$ I, hence $\wp_i \neq \Phi$ $\forall i \in I$ and $\bigcap_{i \in I} \wp_i \neq \Phi$, therefore $\Phi, \aleph \epsilon \bigcap_{i \in I} \wp_i$. Let $A \epsilon \bigcap_{i \in I} \wp_i$ such that $\Phi \neq A \subset B \subseteq \aleph$, then $A \epsilon \wp_i$ $\forall i \in I$, but $A \subset B$. So, we get $B \epsilon \wp_i$ $\forall i \in I$, hence $B \epsilon \bigcap_{i \in I} \wp_i$. Let $A_1, A_2, \ldots \epsilon \bigcap_{i \in I} \wp_i$. Then $A_1, A_2, \ldots \epsilon \wp_i$, $\forall i \in I$ and $\bigcap_{j=1}^{\infty} A_j \epsilon \wp_i$, $\forall i \in I$ which is implies that $\bigcap_{j=1}^{\infty} A_j \epsilon \bigcap_{i \in I} \wp_i$. Hence $\bigcap_{i \in I} \wp_i$ is a δ− field.

Definition 6

Let $\wp$ be a δ − field of a set $\aleph$ and let K be a non-empty subset of $\aleph$. Then the restriction of $\wp$ on K is denoted by $\wp|K$ and define as:

$\wp|K$ ={B: B=A$\bigcap$ K, for some Aϵ $\wp$}.

Proposition 7

Let $\wp$ be a δ − field of a set $\aleph$ and K be a non-empty subset of $\aleph$ such that Kϵ $\wp$. Then $\wp|K$ = {$A \subseteq K$: Aϵ $\wp$}.

Proof

Let Bϵ $\wp$|K. Then B=A$\cap$ K, for some Aϵ $\wp$, hence Bϵ $\wp$. Therefore Bϵ\{A $\subseteq$ K: Aϵ $\wp$\} and $\wp$|K $\subseteq$ \{A $\subseteq$ K: Aϵ $\wp$\}. Let Cϵ \{A $\subseteq$ K : Aϵ $\wp$\}. Then C $\subseteq$ K and C ϵ $\wp$, hence C=C$\cap$ K, but Cϵ $\wp$, then Cϵ $\wp$|K which is implies that\{A $\subseteq$ K : A ϵ $\wp$\}$\subseteq$ $\wp$|K, therefore $\wp$|K=\{A $\subseteq$ K : Aϵ $\wp$\}.

Corollary 8

Let $\wp$ be a δ – field of a set $\aleph$ and K a non-empty subset of $\aleph$ such that Kϵ $\wp$. Then $\wp$|K $\subseteq$ $\wp$.

Proof

From Proposition 7, we have $\wp$|K=\{A $\subseteq$ K: Aϵ $\wp$\}. Now, for any Bϵ $\wp$|K, then Bϵ\{A $\subseteq$ K : Aϵ $\wp$\}. Hence B $\subseteq$ K and Bϵ $\wp$, therefore $\wp$|K $\subseteq$ $\wp$.

Proposition 9

Let $\wp$ be a δ – field of a set $\aleph$ and let K be a non-empty subset of $\aleph$ such that Kϵ $\wp$. Then $\wp$|K is a δ - field of a set K .

Proof

Since $\wp$ is a δ – field of $\aleph$, then $\Phi, \aleph \epsilon \wp$. Since K $\subseteq$ $\aleph$, then K $= \aleph \cap$K and K$\epsilon\wp$|K. Since $\Phi = \Phi \cap$K, then $\Phi\epsilon\wp$|K. Let B$\epsilon\wp$|K such that $\Phi \neq$ B $\subset$ D $\subseteq$ K Then B$\epsilon\wp$. But B $\subset$ D $\subseteq$ K $\subseteq$ $\aleph$ and $\wp$ is a δ – field of a set $\aleph$, then D$\epsilon\wp$. Now, D $\subseteq$ K and D$\epsilon\wp$, then D$\epsilon\wp$|K. Let B_1, B_2 , ... $\epsilon\wp$|K. Then there exist A_1, A_2 , ... $\epsilon\wp$ such that B_i=$A_i \cap$ K where i=1,2,…, now $\bigcap_{i=1}^{\infty} B_i$=($\bigcap_{i=1}^{\infty} A_i$) $\cap$ K. But, $\wp$ is a δ– field, then $\bigcap_{i=1}^{\infty} A_i$ $\epsilon\wp$. Hence $\bigcap_{i=1}^{\infty} B_i$ $\epsilon\wp$|K .Therefore $\wp$|K is a δ – field of a set K.

If we take Example 3 and if we assume that K=\{1,2,4\}, then $\wp$|K=\{Φ,\{1, 2\}, K\} is a δ – field of a set K and $\wp$|K $\subseteq$ $\wp$.

Definition 10

Let $\wp$ be a δ– field of a set $\aleph$. A measure on $\wp$ is a set function $\Psi: \wp \rightarrow [0\,,\infty]$ such that $\Psi(\Phi) = 0$ and if $C_1, C_2, ...$ form a finite or countably infinite collection of disjoint sets in $\wp$, then $\Psi(\bigcup_{n=1}^{\infty} C_n) = \sum_{n=1}^{\infty} \Psi(C_n)$.

Example 11

Let $\wp$ be a δ– field of a set $\aleph$ and define $\Psi: \wp \rightarrow [0\,,\infty]$ by Ψ(C) = 0, for all C$\epsilon\wp$. Then Ψ is a measure on $\wp$.

A measure space is a triple ($\aleph$, $\wp$, Ψ) where $\aleph$ is a nonempty set and $\wp$ is a δ– field of a set $\aleph$ and Ψ is a measure on $\wp$.

Definition 12

Let $\wp$ be a δ– field of a set $\aleph$. A countably subadditive on $\wp$ is a set function $\Psi: \wp \rightarrow [0\,,\infty]$ such that $\Psi(C) \leq \sum_{n=1}^{\infty} \Psi(C_n)$ where $C_1, C_2, ... \epsilon\wp$ and $C = \bigcup_{n=1}^{\infty} C_n$.

If this requirement holds only for finite collection of disjoint sets in $\wp$, then Ψ is said to be finitely subadditive on a δ – field $\wp$.

Definition 13

Let $\wp$ be a δ– field of a set $\aleph$. Then a set function $\Psi: \wp \rightarrow [0\,,\infty]$ is said to be monotone measure, if it satisfies the following requirements:

1- $\Psi(\Phi) = 0$.
2- If $B\epsilon\wp$ and B $\subset$ D $\subseteq$ $\aleph$, then $\Psi(B) \leq \Psi(D)$.

Definition 14

Let $\wp$ be a δ− field of a set ℵ. Then a set function $\Psi: \wp \to [0,\infty]$ is called outer measure, if it satisfies the following requirements:

1- $\Psi(\Phi) = 0$.
2- If $B\epsilon\wp$ and B ⊂ D ⊆ ℵ, then $\Psi(B) \le \Psi(D)$.
3- If $C_1, C_2, \dots \epsilon\wp$, then $\Psi(\bigcup_{n=1}^{\infty} C_n) \le \sum_{n=1}^{\infty} \Psi(C_n)$.

Lemma 15

Let Ψ be an outer measure on δ− field $\wp$ of a set ℵ and $t \in [0,\infty)$. If $t\Psi:\wp \to [0,\infty]$ is defined by
$(t\Psi)(A) = t.\Psi(A)\ \ \forall A\epsilon\wp$, then $(t\Psi)$ is an outer measure on $\wp$.

Proof

Since Ψ is an outer measure on $\wp$ and $\Phi\epsilon\wp$, then $\Psi(\Phi)=0$ and $(t\Psi)(\Phi) = 0$.
Let $B\epsilon\wp$ and B ⊂ D ⊆ ℵ, then $D\epsilon\wp$ and $\Psi(B) \le \Psi(D)$. Since
$(t\Psi)(\mathrm{B}) = t.\Psi(\mathrm{B}) \le \mathrm{t}.\Psi(D) = (t\Psi)(\mathrm{D})$. Let $C_1, C_2, \dots \epsilon\wp$, then $\bigcup_{n=1}^{\infty} C_n\, \epsilon\wp$
So, we have $(t\Psi)(\bigcup_{n=1}^{\infty} C_n) = t.\ \Psi(\bigcup_{n=1}^{\infty} C_n) \le t.\sum_{n=1}^{\infty} \Psi(C_n)$
But, $t.\sum_{n=1}^{\infty} \Psi(C_n) = \sum_{n=1}^{\infty} t.\Psi(C_n) = \sum_{n=1}^{\infty}(t\Psi)(C_n)$. Therefore $t\Psi$ is an outer measure on $\wp$.

Lemma 16

Let Ψ_1 and Ψ_2 be two outer measures on a δ− field $\wp$ of a set ℵ. If $\Psi_1 + \Psi_2:\ \wp \to [0,\infty]$ is defined by
$(\Psi_1 + \Psi_2)(C) = \Psi_1(C) + \Psi_2(C),\ \forall C\epsilon\wp$, then $\Psi_1 + \Psi_2$ is an outer measure on $\wp$.

Proof

Since Ψ_1 and Ψ_2 are outer measure on δ− field $\wp$ and $\Phi\epsilon\wp$, then $\Psi_1(\Phi) = \Psi_2(\Phi) = 0$ and $(\Psi_1 + \Psi_2)(\Phi) = 0$. Let $B\epsilon\wp$ and B ⊂ D ⊆ ℵ, then $D\epsilon\wp$ and $\Psi_1(B) \le \Psi_1(D)$ and $\Psi_2(B) \le \Psi_2(D)$. So we have,
$(\Psi_1 + \Psi_2)(\mathrm{B}) = \Psi_1(\mathrm{B}) + \Psi_2(\mathrm{B}) \le \Psi_1(D) + \Psi_2(D) = (\Psi_1 + \Psi_2)(\mathrm{D})$
Let $C_1, C_2, \dots \epsilon\wp$, then $\bigcup_{n=1}^{\infty} C_n\, \epsilon\wp$. So, we have

$$\begin{aligned}(\Psi_1 + \Psi_2)(\bigcup_{n=1}^{\infty} C_n) &= \Psi_1(\bigcup_{n=1}^{\infty} C_n) + \Psi_2(\bigcup_{n=1}^{\infty} C_n)\\ &\le \sum_{n=1}^{\infty} \Psi_1(C_n) + \sum_{n=1}^{\infty} \Psi_2(C_n) = \sum_{n=1}^{\infty} [\Psi_1(C_n) + \Psi_2(C_n)]\\ &= \sum_{n=1}^{\infty} (\Psi_1 + \Psi_1)(C_n).\end{aligned}$$

Therefore $\Psi_1 + \Psi_2$ is an outer measure on $\wp$.

The proof of the following proposition consequence from Lemma (15 and 16) with mathematical induction.

Proposition 17

Let $\Psi_1, \Psi_2, \dots, \Psi_n$ be outer measure on a δ− field $\wp$ of a set ℵ and $t_i \in [0,\infty)$ for all $i = 1,2,\dots,n$. If a set function $\sum_{i=1}^{n} t_i\Psi_i: \wp \to [0,\infty]$ is defined by:
$(\sum_{i=1}^{n} t_i\Psi_i)(C) = \sum_{i=1}^{n} t_i.\Psi_i(C)\ \forall C\epsilon\wp$, then $\sum_{i=1}^{n} t_i\Psi_i$ is an outer measure on δ− field $\wp$.

Proof

Since $t_i \in [0,\infty)$ and Ψ_i is an outer measure on a δ− field $\wp$ for all $i = 1,2,\dots,n$.
Then by Lemma15 we get $t_i\Psi_i$ is an outer measure on a δ− field $\wp$ $\forall i$=1,2, …, n.

Let $\psi_i = t_i\Psi_i$ $\forall i = 1,2,\dots,n$. Then we prove that $(\sum_{i=1}^{n}\psi_i)$ is an outer measure on $\wp$ by mathematical induction. If $n = 2$, then $\psi_1 + \psi_2$ is an outer measure on $\wp$ by Lemma16. Suppose that $(\sum_{i=1}^{k}\psi_i)$ is an outer measure on $\wp$, then we must prove that $(\sum_{i=1}^{k+1}\psi_i)$ is an outer measure on $\wp$, whenever ψ_i is an outer measure on $\wp$ $\forall i = 1,2,\dots,k,k+1$. $(\sum_{i=1}^{k+1}\psi_i)(\Phi) = (\sum_{i=1}^{k}\psi_i + \psi_{k+1})(\Phi)$

$= (\sum_{i=1}^{k}\psi_i)(\Phi) + \psi_{k+1}(\Phi)$

$= 0$ since $(\sum_{i=1}^{k}\psi_i)$ and ψ_{k+1} are outer measure on $\wp$

Let $B, D\epsilon\wp$ and $B \subset D$. Then $(\sum_{i=1}^{k}\psi_i)(B) \le (\sum_{i=1}^{k}\psi_i)(D)$ and $\psi_{k+1}(B) \le \psi_{k+1}(D)$.

$(\sum_{i=1}^{k+1}\psi_i)(B) = (\sum_{i=1}^{k}\psi_i)(B) + \psi_{k+1}(B)$

$\le (\sum_{i=1}^{k}\psi_i)(D) + \psi_{k+1}(D)$ since $(\sum_{i=1}^{k}\psi_i)$ and ψ_{k+1} are outer measure

$= (\sum_{i=1}^{k}\psi_i + \psi_{k+1})(D)$

$= \left(\sum_{i=1}^{k+1}\psi_i\right)(D)$.

Let $C_1, C_2, \dots \epsilon\wp$. Then $(\sum_{i=1}^{k+1}\psi_i)(\bigcup_{n=1}^{\infty}C_n) = (\sum_{i=1}^{k}\psi_i + \psi_{k+1})(\bigcup_{n=1}^{\infty}C_n)$

$= (\sum_{i=1}^{k}\psi_i)(\bigcup_{n=1}^{\infty}C_n) + \psi_{k+1}(\bigcup_{n=1}^{\infty}C_n)$

$\le \sum_{n=1}^{\infty}(\sum_{i=1}^{k}\psi_i)(C_n) + \sum_{n=1}^{\infty}\psi_{k+1}(C_n)$

$= \sum_{n=1}^{\infty}[(\sum_{i=1}^{k}\psi_i)(C_n) + \psi_{k+1}(C_n)]$

$= \sum_{n=1}^{\infty}(\sum_{i=1}^{k}\psi_i + \psi_{k+1})(C_n)$

$= \sum_{n=1}^{\infty}(\sum_{i=1}^{k+1}\psi_i)(C_n)$.

Therefore, $\sum_{i=1}^{k+1}t_i\Psi_i$ is an outer measure on $\wp$.

Definition 18

Let $\wp$ be a $\delta-$field of a set $\aleph$. Then a set function $\Psi: \wp \to [0,\infty]$ is called null-additive on $\wp$ iff C, D are disjoint sets in $\wp$ and $\Psi(D) = 0$, then $\Psi(C\cup D) = \Psi(C)$.

Example 19

Let $\aleph = \{1,2\}$ and $\wp = \{\Phi, \{1\}, \{2\}, \aleph\}$ and define $\Psi: \wp \to [0,\infty]$ by:

$\Psi(C) = \begin{cases} 0 & C = \Phi \\ 1 & C \neq \Phi \end{cases}$. Then Ψ is a null-additive.

Proposition 20

Let $\wp$ be a $\delta-$field of a set $\aleph$. Then every measure is null-additive.

Proof

Let Ψ be a measure on $\delta-$field $\wp$ and let C, D are disjoint sets in $\wp$ and $\Psi(D) = 0$. Then $\Psi(C\cup D) = \Psi(C) + \Psi(D) = \Psi(C)$. Hence Ψ is a null-additive .

While the converse is not true and Example 19 indicate that Ψ is null-additive but not measure, because $\{1\},\{2\}$ are disjoint sets in $\wp$ but $\Psi(\{1\}\cup\{2\}) \neq \Psi(\{1\}) + \Psi(\{2\})$.

Lemma 21

Let Ψ be a null-additive on a $\delta-$field $\wp$ of a set $\aleph$ and $t \in (0,\infty)$. If $t\Psi: \wp \to [0,\infty]$ is defined by:

$(t\Psi)(C) = t.\Psi(C)$ $\forall C\epsilon\wp$, then $(t\Psi)$ is a null-additive on $\wp$.

Proof

Let C, D be disjoint sets in $\wp$ such that $(t\Psi)(D) = 0$. Then $t.\Psi(D) = 0$ and hence $\Psi(D) = 0$ since $t > 0$. Now, $(t\Psi)(C\cup D) = t.\Psi(C\cup D)$

$= t.\Psi(C) = (t.\Psi)(C)$

Therefore, $t\Psi$ is a null-additive on $\wp$.

Lemma 22

Let Ψ_1 and Ψ_2 be two null-additives on a $\delta-$field $\wp$ of a set $\aleph$. If $\Psi_1+\Psi_2: \wp \to [0,\infty]$ is defined by:

$(\Psi_1+\Psi_2)(C) = \Psi_1(C)+\Psi_2(C) \quad \forall C \epsilon \wp$, then $\Psi_1+\Psi_2$ is a null-additive on $\wp$.

Proof

Let C, D be disjoint sets in $\wp$ such that $(\Psi_1+\Psi_2)(D) = 0$. Then $\Psi_1(D)+\Psi_2(D) = 0$, hence $\Psi_1(D) = \Psi_2(D) = 0$ since Ψ_1 and Ψ_2 are null-additive on $\wp$.

Now, $(\Psi_1+\Psi_2)(C\cup D) = \Psi_1(C\cup D) + \Psi_2(C\cup D)$

$= \Psi_1(C) + \Psi_2(C)$

$= (\Psi_1 + \Psi_2)(C)$.

Therefore, $\Psi_1+\Psi_2$ is a null-additive on $\wp$.

Proposition 23

Let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be a null-additive on a $\delta-$field $\wp$ of a set $\aleph$ and $t_i \in (0,\infty)$ for all $k = 1,2,\ldots,n$. If a set function $\sum_{k=1}^{n} t_k\Psi_k: \wp \to [0,\infty]$ is defined by:

$(\sum_{k=1}^{n} t_k\Psi_k)(C) = \sum_{k=1}^{n} t_k.\Psi_k(C) \quad \forall C\epsilon\wp$, then $\sum_{k=1}^{n} t_k\Psi_k$ is a null-additive on $\wp$.

Proof

Since $t_k \in (0,\infty)$ and Ψ_k is null-additive on $\wp$ for all $k = 1,2,\ldots,n$, then by Lemma 21, we get $t_k\Psi_k$ is a null-additive on $\wp$ $\forall k = 1,2,\ldots,n$. Let $\psi_k = t_k\Psi_k$

If $n = 2$, then $\psi_1 + \psi_2$ is a null-additive on $\wp$ by Lemma 22. Let C, D are disjoint sets in $\wp$ such that $(\sum_{k=1}^{n} \psi_k)(D) = 0$. Then $\psi_k(D) = 0$ for all $k = 1,2,\ldots,n$.

$(\sum_{k=1}^{n} \psi_k)(C\cup D) = \psi_1(C\cup D) + \cdots + \psi_n(C\cup D)$

$= \psi_1(C) + \cdots + \psi_n(C)$ since ψ_k is a null-additive and $\psi_k(D) = 0, \forall k$

$= (\sum_{k=1}^{n} \psi_k)(C)$. Hence $\sum_{k=1}^{n} t_k\Psi_k$ is a null-additive on $\wp$.

Definition 24

Let $\wp$ be a $\delta-$field of a set $\aleph$ and let $\Psi: \wp \longrightarrow [0, \infty]$ be a set function and $B\epsilon\wp$. If $\Psi_B: \wp \longrightarrow [0, \infty]$ is define by $\Psi_B(C) = \Psi(C \cap B)$ for all $C\epsilon\wp$, then Ψ_B is called B – restriction of Ψ .

Proposition 25

Let $\wp$ be a $\delta-$field of a set $\aleph$ and $B \in \wp$. If Ψ is a measure on $\wp$, then:

(1) Ψ_B is a measure on $\wp$.

(2) $\Psi_B(C) = \Psi(C)$, whenever $C \subseteq B$.

(3) $\Psi_B(C) = 0$, whenever C, B are disjoint sets in $\wp$.

Proof

(1). Since $\wp$ is a $\delta-$field , then $\Phi\, \epsilon\wp$ and $\Psi(\Phi) = 0$. From definition of Ψ_B we get, $\Psi_B(\Phi)=\Psi(\Phi \cap B) = \Psi(\Phi) = 0$. Let $C_1, C_2, \ldots$ are disjoint sets in $\wp$, then $\bigcup_{n=1}^{\infty} C_n\, \epsilon\wp$. Since $B, C_n\epsilon\wp$ $\forall$ n=1,2,…, then $C_n \cap B\epsilon\, \wp$ and hence $\bigcup_{n=1}^{\infty}(C_n \cap B)\, \epsilon\wp$. So, we have $\Psi_{\boldsymbol{B}}(\bigcup_{n=1}^{\infty} C_n) = \Psi((\bigcup_{n=1}^{\infty} C_n) \cap B)$

$= \Psi(\bigcup_{n=1}^{\infty}(C_n \cap B))$

$= \sum_{n=1}^{\infty} \Psi(C_n \cap B)$

$= \sum_{n=1}^{\infty} \Psi_{\boldsymbol{B}}(C_n)$. Therefore, Ψ_B is a measure on $\wp$.

(2). Since $C \subseteq B$, then $C \cap B = C$. So, we have $\Psi_B(C) = \Psi(C \cap B) = \Psi(C)$

(3). Since C, B are disjoint sets in $\wp$, then $C \cap B = \Phi$ and $\Psi_B(C) = \Psi(C \cap B)$

$= \Psi(\Phi) = 0$.

Proposition 26

Let $\wp$ be a $\delta-$ field of a set $\aleph$ and $B\epsilon\ \wp$. If Ψ is an outer measure on $\wp$, then Ψ_B is an outer measure on $\wp$.

Proof

Since $\wp$ is a $\delta-$ field, then $\Phi\ \epsilon\wp$ and $\Psi(\Phi) = 0$. From definition of Ψ_B we get , $\Psi_B(\Phi) = \Psi(\Phi \cap B) = \Psi(\Phi) = 0$. Let $A\epsilon\wp$ and $A \subset C \subseteq \aleph$, then $A\cap B \subset C\cap B$ and each of $C, A\cap B, C\cap B\epsilon\wp$. Since Ψ is an outer measure on $\wp$, then $\Psi(A\cap B) \leq \Psi(C\cap B)$ .So, we have $\Psi_B(A) \leq \Psi_B(C)$. Let $C_1, C_2, \dots\ \epsilon\wp$. Then $\bigcup_{n=1}^{\infty} C_n\ \epsilon\wp$ and $C_n \cap B\epsilon\wp$ $\forall$n=1,2,…, hence $\bigcup_{n=1}^{\infty}(C_n \cap B)\ \epsilon\wp$. So, we have,

$$\Psi_{\boldsymbol{B}}(\bigcup_{n=1}^{\infty} C_n) = \Psi((\bigcup_{n=1}^{\infty} C_n) \cap B)$$
$$= \Psi(\bigcup_{n=1}^{\infty}(C_n \cap B)) \leq \sum_{n=1}^{\infty} \Psi(C_n \cap B) = \sum_{n=1}^{\infty} \Psi_{\boldsymbol{B}}(C_n).$$

Therefore, Ψ_B is an outer measure on $\wp$.

From Proposition 26, we conclude that if Ψ is a monotone measure on $\wp$, then Ψ_B is a monotone measure on $\wp$, where $\wp$ is a $\delta-$ field of a set $\aleph$ and $B\epsilon\ \wp$.

Proposition 27

Let $\wp$ be a $\delta-$ field of $\aleph$ and $B\epsilon\ \wp$. If Ψ is a null-additive on $\wp$, then Ψ_B is a null-additive on $\wp$.

Proof

Let A, C be disjoint sets in $\wp$ and $\Psi_B(C) = 0$. Then $\Psi(C\cap B) = 0$.

Now, $\Psi_B(A\cup C) = \Psi([A\cup C]\cap B)$

$= \Psi([A\cap B]\cup[C\cap B])$

$= \Psi(A\cap B)$ since Ψ is a null-additive on $\wp$

$= \Psi_B(A)$ by definition of Ψ_B.

Hence, Ψ_B is a null-additive on $\wp$.

Proposition 28

Let $\wp$ be a $\delta-$ field of $\aleph$ and $B\epsilon\wp$. If Ψ is a measure on $\wp$, then Ψ_B is a null-additive on $\wp$.

Proof

It is easy, so we omitted.

Definition 29

Let $\wp$ be a $\delta-$ field of a set $\aleph$ and $\Psi\colon \wp \longrightarrow [0, \infty]$ be a set function and K be a non-empty subsets of $\aleph$ such that $K\epsilon\ \wp$. If $\Psi|K : \wp|K \longrightarrow [0, \infty]$ is define by:

$\Psi|K\ (A) = \Psi(A)$ for all $A\epsilon\ \wp|K$, then $\Psi|K$ is called the restriction of Ψ on $\wp|K$

Proposition 30

Let Ψ be a measure on $\delta-$ field $\wp$ of a set $\aleph$ and $\Phi \neq K \subseteq \aleph$ such that $K\epsilon\ \wp$. Then $\Psi|K$ is a measure on a $\delta-$ field $\wp|K$ of a set K.

Proof

Since $\wp$ is a $\delta-$ fieldof a set $\aleph$, then $\Phi\ \epsilon\wp$ and $\Psi(\Phi) = 0$. Since $\Phi\ \epsilon\ \wp|K$, then by definition of $\Psi|K$, we get $\Psi|K(\Phi) = \Psi(\Phi) = 0$. Let $C_1, C_2, \dots$ be disjoint sets in $\wp|K$. Then $C_n \subseteq K$ and $C_n\epsilon\wp$ for all n=1,2,…, hence $\bigcup_{n=1}^{\infty} C_n\ \epsilon\ \wp|K$. So, we have

$$\Psi|K(\bigcup_{n=1}^{\infty} C_n) = \Psi(\bigcup_{n=1}^{\infty} C_n)$$
$$= \sum_{n=1}^{\infty} \Psi(C_n) \quad \text{since } \Psi \text{ is a measure on } \wp$$
$$= \sum_{n=1}^{\infty} \Psi|K(C_n)$$

Therefore, $\Psi|K$ is a measure on a $\delta-$ field$\wp|K$ of a set K.

If Ψ is an outer measure on $\delta-$ field $\wp$ of a set $\aleph$, then we need the following two facts to prove that $\Psi|K$ is an outer measure on a $\delta-$ field $\wp|K$ of a set K.

Lemma 31

Let Ψ be a monotone measure on $\delta-$ field $\wp$ of a set $\aleph$ and $\Phi \neq K \subseteq \aleph$ such that $K \epsilon \wp$. Then $\Psi|K$ is a monotone measure on a $\delta-$ field $\wp|K$ of a set K.

Proof

Let Ψ be a monotone measure on $\wp$, then $\Psi(\Phi) = 0$. Since $\wp|K$ is a $\delta-$ field, then $\Phi \epsilon \wp|K$. From definition of $\Psi|K$, we get $\Psi|K(\Phi) = \Psi(\Phi) = 0$.

Let $B \epsilon \wp|K$ such that $B \subset C \subseteq K$, then $B \epsilon \wp$ and $B \subset C \subseteq \aleph$. Since Ψ is a monotone measure on $\wp$, then $\Psi(B) \leq \Psi(C)$. But $B, C \epsilon \wp|K$, then $\Psi|K(B)=\Psi(B)$ and $\Psi|K(C) = \Psi(C)$, hence $\Psi|K(B) \leq \Psi|K(C)$ and $\Psi|K$ is monotone measure on $\wp|K$ of K.

Lemma 32

Let Ψ be a countably subadditive on $\delta-$ field $\wp$ of a set $\aleph$ and $\Phi \neq K \subseteq \aleph$ such that $K \epsilon \wp$, then $\Psi|K$is a countably subadditive on a $\delta-$ field $\wp|K$ of a set K.

Proof

Let $C_1, C_2, \ldots \epsilon \wp|K$ and $C = \bigcup_{n=1}^{\infty} C_n$,then $C_1, C_2, \ldots \epsilon \wp$ and $C \epsilon \wp$. Since Ψ be a countably subadditive on $\wp$, then $\Psi(c) \leq \sum_{n=1}^{\infty} \Psi(C_n)$, but $C, C_1, C_2, \ldots \epsilon \wp|K$. So, we have $\Psi(C) = \Psi|K(C)$ and $\Psi(C_n) = \Psi|K(C_n)$ for all n=1,2,…, hence $\Psi|K(C) \leq \sum_{n=1}^{\infty} \Psi|K(C_n)$ and $\Psi|K$ is a countably subadditive on $\wp|K$ of a set K.

Proposition 33

Let Ψ be an outer measure on $\delta-$ field $\wp$ of a set $\aleph$ and $\Phi \neq K \subseteq \aleph$ such that $K \epsilon \wp$. Then $\Psi|K$ is an outer measure on $\delta-$ field $\wp|K$ of a set K.

Proof

Since Ψ is an outer measure on $\wp$, then Ψ is a monotone measure and countably subadditive. By Lemma 31 and Lemma 32 we have $\Psi|K$ is a monotone measure and countably subadditive on $\wp|K$ of K. Therefore $\Psi|K$ is an outer measure on $\wp|K$ of K.

Proposition 34

Let Ψ be a null-additive on $\delta-$ field $\wp$ of a set $\aleph$ and $\Phi \neq K \subseteq \aleph$ such that $K \epsilon \wp$. Then $\Psi|K$ is a null-additive on $\delta-$ field $\wp|K$.

Proof:

Let C, D be disjoint sets in $\wp|K$ and $\Psi|K(D) = 0$. Then $\Psi(D) = 0$.

Now, $\Psi|K(C \cup D) = \Psi([C \cup D)$

$= \Psi(C)$ since Ψ is a null-additive on $\wp$

$= \Psi|K(C)$ by definition of $\Psi|K$.

Hence, $\Psi|K$ is a null-additive on $\wp$.

3. Conclusions

The main results of this paper are the following:

(1) Let $\aleph$ be a nonempty set. A collection $\wp \subseteq P(\aleph)$ is said to be $\delta-$ field of a set $\aleph$ if the following conditions are satisfied:

1. $\Phi \epsilon \wp$.
2. If A is a nonempty set in $\wp$ and $A \subset B \subseteq \aleph$, then $B \epsilon \wp$.
3. If $A_1, A_2, \ldots \epsilon \wp$, then $\bigcap_{i=1}^{\infty} A_i \epsilon \wp$.

(2) Let $\{\wp_i\}_{i \in I}$be a sequence of $\delta-$ field of a set $\aleph$. Then $\bigcap_{i \in I} \wp_i$ is a $\delta-$ field of a set $\aleph$.

(3) Let $\wp$ be a δ – field of a set $\aleph$ and let K be a non-empty subset of $\aleph$. Then the restriction of $\wp$ on K is denoted by $\wp|$K and $\wp|$K ={B: B=A$\cap$ K, for some Aϵ $\wp$}.

(4) Let $\wp$ be a $\delta-$ field of a set $\aleph$. Then every measure is null-additive.

(5) Let Ψ_1 , Ψ_2 ,…, Ψ_n be null-additive on a $\delta-$ field $\wp$ of a set $\aleph$ and $t_i \in (0,\infty)$ for all $k = 1,2, \dots, n$. If a set function $\sum_{k=1}^{n} t_k \Psi_k: \wp \to [0,\infty]$ is defined by: $(\sum_{k=1}^{n} t_k \Psi_k)\,(C) = \sum_{k=1}^{n} t_k . \Psi_k\,(C)$ $\forall C \epsilon \wp$, then $\sum_{k=1}^{n} t_k \Psi_k$ is a null-additive on $\wp$.

(6) Let $\wp$ be a $\delta-$ field of a set $\aleph$ and $B \epsilon\ \wp$. If Ψ is a measure on $\wp$, then:

1. Ψ_B is a measure on $\wp$.
2. $\Psi_B(C) = \Psi(C)$, whenever $C \subseteq B$.
3. $\Psi_B(C) = 0$, whenever C, B are disjoint sets in $\wp$.

(7) Let $\wp$ be a $\delta-$ field of a set $\aleph$ and $B \epsilon\ \wp$. If Ψ is an outer measure on $\wp$, then Ψ_B is an outer measure on $\wp$.

(8) Let $\wp$ be a $\delta-$ field of $\aleph$ and $B \epsilon\ \wp$. If Ψ is a null-additive on $\wp$, then Ψ_B is a null-additive on $\wp$.

(9) Let Ψ be a measure on $\delta-$ field $\wp$ of a set $\aleph$ and $\Phi \neq$ K $\subseteq \aleph$ such that Kϵ $\wp$. Then $\Psi|$K is a measure on a $\delta-$ field $\wp|$K of a set K.

(10) Let Ψ be a monotone measure on $\delta-$ field $\wp$ of a set $\aleph$ and $\Phi \neq K \subseteq \aleph$ such that Kϵ $\wp$. Then $\Psi|$K is a monotone measure on a $\delta-$ field $\wp|$K of a set K.

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